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Ghori, Q.k. & Naseer Ahmad

DHAHRAN 31261● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

LEVI - CIVITA'S THEOREM FOR GYROSCOPIC SYSTEMS

Q.K. Ghori* and Naseer Ahmad**

*Dept. of Math. Sciences, King Fahd Univ. of Petroleum and Minerals, Dhahran, Saudi Arabia

**Math. Dept., Quaid-i-Azam Univ., Islamabad, Pakistan

A theorem of Levi-Civeta uses invariant relations of a holonomic mechanical system to construct certain families of particular solutions of the equations of motion. A variant of this theorem is presented in the Poincaré formalism when gyroscopic forces are added to the system. As an application, a generalization of Routh's motion is studied.

1 Introduction

According to a theorem of Levi-Civita [8], to any set of m invariant relations (resp. m integrals) of an autonomous canonical system which are in involution, there corresponds a family of $\infty^m (resp. \infty^{2m})$ particular solutions of the canonical system. For non-autonomous mechanical systems which are described by Pioncaré's equations in the canonical form [5], this theorem has been generalized in [6] for holonomic systems with redundant coordinates, and in [7] for nonholonomic systems. The results of Capodanno[1, 2] in Lagrangian coordinates, and those of Pignedoli [9] in quasi-coordinates appear as special cases of this generalized theory.

The aim of this paper is to continue the investigations initiated in [6] and formulate a variant of Levi-Civita's theorem for holonomic systems subjected to gyroscopic forces. Sufficient conditions for this formulation to be possible are given. The theory is applied to study a generalization of Routh's motion.

2 Canonical equations of a gyroscopic system

We begin our derivation of the equations of motion by summarizing the facts of Poincaré's formalism [3, 10]. Throughout, the indices take the values

$$p, q, r, s = 1, 2, \ldots, n; \sigma, \rho, \mu, \nu = 1, 2, \ldots, m < n; \alpha, \beta, \gamma = m + 1, \ldots, n.$$

Sumunation over repeated indices is understood.

Let x_p be the coordinates of a conservative holonomic mechanical system with n degrees of freedom, which determine the position of the system at any time t. We assume that the virtual displacements and actual velocities of the system can be written as

$$\delta x_p = \xi_p^r(x,t)\omega_r, \quad \dot{x}_p = \xi_p^r(x,t)\eta_r + \xi_p(x,t) \tag{2.1}$$

where the parameters ω_r of virtual displacements and the Poincaré parameters η_r are independent. It is assumed that the corresponding n + 1 operators

$$X_0 = \frac{\partial}{\partial t} + \xi_p(x,t)\frac{\partial}{\partial x_p}, \quad X_p = \xi_p^r(x,t)\frac{\partial}{\partial x_r}$$

which are linearly independent, form a basis of the (n + 1) - dimensional Lie algebra

$$[X_p, X_q] = C_{pq}^r X_r, \quad [X_0, X_p] = C_{op}^r X_r.$$

Then the variation of an arbitrary function G(x, t) in a virtual displacement of the system is given by

$$\delta G = w_p X_p G \tag{2.2}$$

and in a real displacement by

$$dG = (X_0 G + \eta_p X_p G)\, dt\,. \tag{2.3}$$

For the sake of simplicity we take

$$\frac{\partial \xi_p^r}{\partial t} = 0, \ \xi_p = 0, \ C_{op}^r = 0.$$

Let $L_0(t,x,\dot{x})$ be the Lagrange function of the system and $L(x,\eta) \equiv L_0(x, \xi_1^r\eta_r, \ldots, \xi_n^r\eta_r)$; and let $g_{ps}(x)\eta_s$, satisfying the conditions $g_{ps} = -g_{sp}$, be the gyroscopic forces acting on the system. The general equation of dynamics expressing the principle of d'Alembert-Lagrange may be written in the form [4]:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \eta_p} - C_{qp}^r \eta_q \frac{\partial L}{\partial \eta_r} - X_p L - g_{ps}\,\eta_s\right)\omega_p = 0.$$

In view of the independence of ω_p, we obtain the equations of motion with gyroscopic forces

$$\frac{d}{dt}\frac{\partial L}{\partial \eta_p} - C_{qp}^r\,\eta_q \frac{\partial L}{\partial \eta_r} - X_p L = g_{ps}\,\eta_s, \tag{2.4}$$

$$\dot{x}_p = \xi_p^r\,(x)\,\eta_r. \tag{2.5}$$

With $g_{ps} = 0$, equations (2.4) become the Pioncaré equations [10]. We also remark that equations (2.4) can be obtained from the known Lagrange equations by changing them to quasi-coordinates in accordance with (2.5).

We introduce new variables $y_p = \frac{\partial L}{\partial \eta_p}$. In analytical mechanics the Lagrange function L is a non-degenerate quadratic form of the variables $\eta_1, \ldots, \eta_n$. In view of the property of Legendre's transformation, there is an inverse transformation $\eta_p = \frac{\partial H}{\partial y_p}$, generated by the Hamiltonian function $H = y_p \eta_p - L$. The equations of motion (2.4) assume the canonical form

$$\eta_p = \frac{\partial H}{\partial \eta_p}, \quad \dot{y}_p = -X_p H + C^r_{qp} \frac{\partial H}{\partial y_q} y_r + g_{ps} \frac{\partial H}{\partial y_s}. \tag{2.6}$$

These equations are considered in conjunction with equations (2.5) in the form

$$\dot{x}_p = \xi^{(r)}_p(x) \frac{\partial H}{\partial y_r}. \tag{2.7}$$

We note some properties of equations (2.6).

(i) A necessary and sufficient condition for a function $\psi(x, y)$ to be an integral of equations (2.6) is

$$(\psi, H) + g_{ps} \frac{\partial \psi}{\partial y_p} \frac{\partial H}{\partial y_s} = 0. \tag{2.8}$$

Here (f, g) denotes the generalized Poisson bracket [5] of the function f(x, y) and g(x, y), defined by

$$(f, g) = \frac{\partial g}{\partial y_p} X_p f - \frac{\partial f}{\partial y_p} X_p g + C^r_{qp} \frac{\partial f}{\partial y_p} \frac{\partial g}{\partial y_q} y_r. \tag{2.9}$$

(ii) H(x, y) is an integral of equations (2.6). This follows from (2.8) as a consequence of the anti-symmetry properties: $g_{ps} = -g_{sp}$ and $C^r_{qp} = -C^r_{pq}$.

(iii) The stationarity conditions of H(x, y) are

$$X_p H = 0, \; \frac{\partial H}{\partial y_p} = 0. \tag{2.10}$$

This follow form (2.2), giving

$$\delta H = w_p X_p H + \frac{\partial H}{\partial y_p} \partial y_p.$$

As in [12], a relation f(x, y) = 0 is an invariant relation of the canonical system (2.6) if $\frac{df}{dt} \equiv 0$ when computed in conjunction with (2.6) and f(x, y) = 0. The invariant relations include, as a particular case, integrals of system (2.6).

It is easy to see that $d(x_p H)/dt$ and $d(\frac{\partial H}{\partial y_p})/dt$ are linear combinations of $X_p H$, and $\partial H / \partial y_p$ and therefore vanish identically. This implies that the stationarity relations (2.10) are invariant relations of the canonical system of equations (2.6).

3 Invariant relations

Let the autonomous canonical system of equations (2.6) admit m independent invariant relations of the form

$$F_\mu(x_p, y_p) = 0. \tag{3.1}$$

In order to extend Levi-Civita's theorem we assume that the invariant relations satisfy the conditions

$$(F_\mu, F_\nu) + g_{ps}\frac{\partial F_\mu}{\partial y_p}\frac{\partial F_\nu}{\partial y_s} = 0 \tag{3.2}$$

These conditions are analogous to relations (2.8). Using (2.9), relations (3.2) become

$$\frac{\partial F_\nu}{\partial y_p}X_pF_\mu - \frac{\partial F_\mu}{\partial y_p}X_pF_\nu + C^r_{qp}\frac{\partial F_\mu}{\partial y_p}\frac{\partial F_\nu}{\partial y_q}y_r + g_{ps}\frac{\partial F_\mu}{\partial y_p}\frac{\partial F_\nu}{\partial y_s} = 0. \tag{3.3}$$

We assume that equations (3.1) can be solved for y_μ, so that

$$y_\mu = \phi_\mu(x_p, y_\alpha). \tag{3.4}$$

We substitute the values of y_μ from (3.4) into (3.1), obtaining an identity which leads to the relations

$$X_pF_\sigma + \frac{\partial F_\sigma}{\partial y_\mu}X_p\phi_\mu = 0, \tag{3.5}$$

and

$$\frac{\partial F_\sigma}{\partial y_\alpha} + \frac{\partial F_\sigma}{\partial y_\mu}\frac{\partial \phi_\mu}{\partial y_\alpha} = 0. \tag{3.6}$$

We substitute the expression for X_pF_σ from (3.5) into (3.3) to get

$$\frac{\partial F_\rho}{\partial y_\nu}\frac{\partial F_\sigma}{\partial y_p}X_p\phi_\nu - \frac{\partial F_\sigma}{\partial y_\mu}\frac{\partial F_\rho}{\partial y_p}X_p\phi_\mu + C^r_{qp}\frac{\partial F_\sigma}{\partial y_p}\frac{\partial F_\rho}{\partial y_q}y_r + g_{ps}\frac{\partial F_\sigma}{\partial y_p}\frac{\partial F_\rho}{\partial y_s} = 0.$$

Breaking the sums over p,q,s into sums over μ, ν and α, β; and using relations (3.6), the last result becomes

$$\frac{\partial F_\sigma}{\partial y_\nu}\frac{\partial F_\rho}{\partial y_\mu}[X_\nu\phi_\mu - X_\mu\phi_\nu - \frac{\partial \phi_\nu}{\partial y_\alpha}X_\alpha\phi_\mu + \frac{\partial \phi_\mu}{\partial y_\alpha}X_{\alpha\phi_\nu} + C^r_{\alpha\beta}\frac{\partial \phi_\mu}{\partial y_\alpha}\frac{\partial \phi_\nu}{\partial y_\beta}y_r +$$
$$(C^r_{\mu\nu} + C^r_{\alpha\nu}\frac{\partial \phi_\mu}{\partial y_\alpha} + C^r_{\mu\alpha}\frac{\partial \phi_\nu}{\partial y_\alpha})\,y_r - (g_{\mu\nu} + g_{\alpha_\mu}\frac{\partial \phi_\mu}{\partial y_\alpha} -$$
$$- g_{\beta\alpha}\frac{\partial \phi_\mu}{\partial y_\alpha}\frac{\partial \phi_\nu}{\partial y_\beta})] = 0. \tag{3.7}$$

We introduce the notation

$$\{U, V\} = \frac{\partial U}{\partial y_\alpha}X_\alpha V - \frac{\partial V}{\partial y_\alpha}X_\alpha U + C^r_{\alpha\beta}\frac{\partial U}{\partial y_\alpha}\frac{\partial V}{\partial y_\beta}y_r. \tag{3.8}$$

Then equation (3.7) becomes

$$\frac{\partial F_\sigma}{\partial y_\nu}\frac{\partial F_\rho}{\partial y_\mu}[X_\nu\phi_\mu - X_\mu\phi_\nu + \{\phi_\mu,\phi_\nu\} + (C^r_{\mu\nu} + C^r_{\mu\alpha}\frac{\partial\phi_\nu}{\partial y_\alpha} - C^r_{\mu\alpha}\frac{\partial\phi_\nu}{\partial y_\alpha})y_r -$$

$$-(g_{\mu\nu} + g_{\alpha\mu}\frac{\partial\phi_\nu}{\partial y_\alpha} - g_{\alpha\nu}\frac{\partial\phi_\mu}{\partial y_\alpha} - g_{\beta\alpha}\frac{\partial\phi_\mu}{\partial y_\alpha}\frac{\partial\phi_\nu}{\partial y_\beta})] = 0.$$

Since the Jacobian $\partial(F_1, F_2, \ldots, F_m)/\partial(y_1, y_2, \ldots, y_m)$ is different from zero, it follows that the invariant relations (3.4) under conditions (3.2) or (3.3) satisfy.

$$X_\nu\phi_\mu - X_\mu\phi_\nu + \{\phi_\mu,\phi_\nu\} + (C^r_{\mu\nu} + C^r_{\mu\alpha}\frac{\partial\phi_\nu}{\partial y_\alpha} - C^r_{\alpha\nu} - \frac{\partial\phi_\mu}{\partial\, y_\alpha})y_r -$$

$$-(g_{\mu\nu} + g_{\alpha\mu}\frac{\partial\phi_\nu}{\partial y_\alpha} - g_{\alpha\nu}\frac{\partial\phi_\mu}{\partial y_\alpha} - g_{\alpha\beta}\frac{\partial\phi_\mu}{\partial y_\alpha}\frac{\partial\phi_\nu}{\partial y_\beta}) = 0. \qquad (3.9)$$

These are sufficient conditions for generalizing the Levi-Civita theorem.

4 New invariant relations

Since the invariant relations (3.1) in the form (3.4) are admitted by equations (2.6), we have according to (2.3):

$$\dot{y}_\mu = -X_\mu H + C^r_{q\mu}\frac{\partial H}{\partial y_q}y_r + g_{\mu s}\frac{\partial H}{\partial y_s}$$
$$= \frac{\partial H}{\partial y_q}X_q\phi_\mu - \frac{\partial\phi_\mu}{\partial y_\alpha}X_\alpha H + C^r_{q\alpha}\frac{\partial H}{\partial y_q}\frac{\partial\phi_\mu}{\partial y_\alpha}y_r + g_{\alpha s}\frac{\partial\phi_\mu}{\partial y_\alpha}\frac{\partial H}{\partial y_s}$$

which is equivalent to

$$X_\mu H + \{H,\phi_\mu\} + \frac{\partial H}{\partial y_\nu}[X_\nu\phi_\mu + (C^r_{\mu\nu} + C^r_{\nu\mu}\frac{\partial\phi_\mu}{\partial y_\alpha})\, y_r - g_{\mu\nu} + g_{\alpha\nu}\frac{\partial\phi_\mu}{\partial y_\alpha}]$$

$$+\frac{\partial H}{\partial y_\alpha}[C^r_{\mu\alpha}y_r - g_{\mu\alpha} + g_{\beta\alpha}\frac{\partial\phi_\mu}{\partial y_\beta}] = 0. \qquad (4.1)$$

The least equation becomes an identity when for each of the quantities y_μ we substitute the corresponding functions ϕ_μ.

In H we replace the y_μ by their values ϕ_μ and denote the resulting function by $\tilde{H}$. Thus

$$H(x_p, y_\mu, y_\alpha) = \tilde{H}(x_p, y_\alpha). \qquad (4.2)$$

It follows that

$$X_\mu H = X_\mu\tilde{H} - \frac{\partial H}{\partial y_\nu}X_\mu\phi_\nu, \qquad (4.3)$$

$$X_\alpha H = X_\alpha \tilde{H} - \frac{\partial H}{\partial y_\mu} X_\alpha \phi_\mu, \tag{4.4}$$

and

$$\frac{\partial H}{\partial y_\alpha} = \frac{\partial \tilde{H}}{\partial y_\alpha} - \frac{\partial H}{\partial y_\mu} \frac{\partial \phi_\mu}{\partial y_\alpha}. \tag{4.5}$$

We use (4.3) and (4.4) together with (3.8) to obtain

$$X_\mu H + \{H, \phi_\mu\} = X_\mu \tilde{H} + \{\tilde{H}, \phi_\mu\} + \frac{\partial H}{\partial y_\nu} [-X_\mu \phi_\nu + \{\phi_\mu, \phi_\nu\}].$$

If we substitute in (4.1) this value of $X_\mu H + \{H, \phi_\mu\}$ and use conditions (3.9) then we arrive at the equations

$$X_\mu \tilde{H} + \{\tilde{H}, \phi_\mu\} + [C^r_{\mu\alpha} y_r - (g_{\mu\alpha} - g_{\beta\alpha} \frac{\partial \phi_\mu}{\partial y_\beta})] \frac{\partial \tilde{H}}{\partial y_\alpha} = 0. \tag{4.6}$$

A further simplification of (4.6) rersults when $g_{\beta\alpha}$ vanish or ϕ_μ do not depend on the y_α. In either case we have

$$X_\mu \tilde{H} + \{\tilde{H}, \phi_\mu\} + [C^r_{\mu\alpha} y_r - g_{\mu\alpha}] \frac{\partial \tilde{H}}{\partial y_\alpha} = 0. \tag{4.7}$$

We shall now show that the system of equations (3.4) and the equations

$$X_\alpha \tilde{H} = 0, \ \frac{\partial \tilde{H}}{\partial y_\alpha} = 0 \tag{4.8}$$

are invariant with respect to the canonical equations (2.6).

We notice that, in view of (4.6), the relations (4.8) imply that

$$X_\mu \tilde{H} = 0. \tag{4.9}$$

From equations (2.6) it follows that

$$\begin{aligned}
\frac{d}{dt} (X_\alpha \tilde{H}) &= \eta_p X_p X_\alpha \tilde{H} + \frac{\partial}{\partial y_\beta} (X_\alpha \tilde{H}) \dot{y}_\beta \\
&= \frac{\partial H}{\partial y_p} X_p X_\alpha \tilde{H} - \frac{\partial}{\partial y_\beta} (X_\alpha \tilde{H}) X_\beta H + C^r_{q\beta} \frac{\partial H}{\partial y_q} \frac{\partial}{\partial y_\beta} (X_\alpha \tilde{H}) y_r + \\
&+ g_{\beta s} \frac{\partial}{\partial y_\beta} (X_\alpha \tilde{H}) \frac{\partial H}{\partial y_s}.
\end{aligned}$$

We split sums over p,q into sums over μ, β and γ. Thus we get

$$\begin{aligned}\frac{d}{dt}(X_\alpha \tilde{H}) &= \{\tilde{H}, X_\alpha \tilde{H}\} + g_{\beta\gamma} \frac{\partial}{\partial y_\beta}(X_\alpha \tilde{H}) \frac{\partial \tilde{H}}{\partial y_\gamma} + \\ &+ \frac{\partial H}{\partial y_\mu}[C^r_{\mu\alpha} X_r \tilde{H} + X_\alpha X_\mu \tilde{H} + \{X_\alpha \tilde{H}, \phi_\mu\} \\ &+ (C^r_{\mu\beta}\, y_r - g_{\mu\beta} + g_{\gamma\beta} \frac{\partial \phi_\mu}{\partial y_\gamma}) \frac{\partial}{\partial y_\beta}(X_\alpha \tilde{H})].\end{aligned}$$

From equations (4.6) it follows that

$$X_\alpha X_\mu \tilde{H} + \{X_\alpha \tilde{H}, \phi_\mu\} + [C^r_{\mu\beta} y_r - g_{\mu\beta} + g_{\gamma\beta} \frac{\partial \phi_\mu}{\partial y_\gamma}] \frac{\partial}{\partial y_\beta}(X_\alpha \tilde{H})$$

$$= -\{\tilde{H}, X_\alpha \phi_\mu\} - \frac{\partial \tilde{H}}{\partial y_\beta} X_\alpha (C^r_{\mu\beta} y_r - g_{\mu\beta} + g_{\gamma\beta} \frac{\partial \phi_\mu}{\partial y_\gamma}).$$

Consequently

$$\frac{d}{dt}(X_\alpha \tilde{H}) = \{\tilde{H}, X_\alpha \tilde{H}\} + g_{\beta\gamma} \frac{\partial}{\partial y_\beta}(X_\alpha \tilde{H}) \frac{\partial \tilde{H}}{\partial y_\gamma} -$$

$$\frac{\partial H}{\partial y_\mu}[C^r_{\alpha\mu} X_r \tilde{H} + \{\tilde{H}, X_\alpha \phi_\mu\} + \frac{\partial \tilde{H}}{\partial y_\beta} X_\alpha (C^r_{\mu\beta} y_r - g_{\mu\beta} + g_{\gamma\beta} \frac{\partial \phi_\mu}{\partial y_\gamma})]$$

Similarly we find that

$$\frac{d}{dt}(\frac{\partial \tilde{H}}{\partial y_\alpha}) = \{\tilde{H}, \frac{\partial \tilde{H}}{\partial y_\alpha}\} + g_{\beta\gamma} \frac{\partial^2 \tilde{H}}{\partial y_\beta \partial y_\alpha} \frac{\partial \dot{H}}{\partial Y_\gamma} -$$

$$-\frac{\partial H}{\partial y_\mu}[\{H, \frac{\partial \phi_\mu}{\partial y_\alpha}\} - \frac{\partial \tilde{H}}{\partial y_\alpha} \frac{\partial}{\partial y_\alpha}(C^r_{\mu\beta} y_r - g_{\mu\beta} + g_{\gamma\beta} \frac{\partial \phi_\mu}{\partial y_\gamma})].$$

Thus, each of $d(X_\alpha \tilde{H})/dt$ and $d(\partial \tilde{H} / \partial y_\alpha)/dt$ is a linear combination of the quantities $X_p \tilde{H}$, $\partial \tilde{H} / \partial y_\alpha$ and therefore

$$\frac{d}{dt}(X_\alpha \tilde{H}) = 0, \quad \frac{d}{dt}(\frac{\partial \tilde{H}}{\partial y_\alpha}) = 0.$$

Hence the relations (4.6) are invariant with respect to the canonical equations (2.6).

From the preceding analysis we see that by means of a set of m invariant relations (3.1) of the system (2.6) which are solvable for y_μ in the form (3.4) and satisfy conditions (3.9), we can construct further 2(n-m) invariant relations (4.8) of the system of canonical equations (2.6).

We remark that relations (4.8), and (4.9) confer a stationary value upon the Hamiltonian function. In fact, the variation $\delta \tilde{H}$, computed in accordance with (2.2), is given by

$$\delta \tilde{H} = \omega_\mu X_\mu \tilde{H} + \omega_\alpha X_\alpha \tilde{H} + \frac{\partial \tilde{H}}{\partial y_\alpha} \delta y_\alpha,$$

and therefore vanishes in virtue of equations (4.8) and (4.9).

5 Generalized Levi-Civita's Theorem

Let us now consider the gyroscopic system whose motion is determined by the canonical equations (2.6). We assume that the system (2.6) admits m invariant relations (3.1) which can be solved for y_μ in the form (3.4) and satisfy conditions (3.3) or (3.9).

The second group of equations (2.6) and equations (2.7)) constitute a system of 2n first order ordinary differential equations for determining the values of the variables x_p and y_p as functions of the time t. We rewrite them in the form

$$\dot{x}_\alpha = \frac{\partial H}{\partial y_q} X_q x_\alpha, \quad \dot{y}_p = -X_p H + C^r_{qp} \frac{\partial H}{\partial y_q} y_r + g_{ps} \frac{\partial H}{\partial y_s}, \tag{5.1}$$

$$\dot{x}_\mu = \frac{\partial H}{\partial y_q} X_q x_\mu. \tag{5.2}$$

Now the new invariant relations (4.8) will furnish the values of 2(n-m) variables x_α, y_α in terms of the remaining variables x_μ. Substituting these values into equations (3.4) enables us to determine (2n-m) values of the variables x_μ and y_p in terms of x_μ. We note that these (2n-m) values satisfy equations (5.1). From the invariant character of equations (3.1), (3.4) and (4.8) it follows that on substituting these values into equations (5.2), we obtain a system of m independent equations, namely, those expressing $\dot{x}_\mu$ in terms of x_μ. The solution of this system which will contain m arbitrary constants, will give ∞^m particular solutions of the canonical equations (2.6) together with (2.7).

If the invariant relations (3.1) are integrals of the canonical equations (2.6) with (2.7), they will contain another set of m arbitrary constants, yielding ∞^{2m} particular solutions. Thus, we obtain the generalization of Levi-Civita's theorem:

<u>Theorem 1.</u> *If the system of canonical equations (2.6) together with (2.7) admits* m *independent invariant relations (resp.* m *integrals)* of the form (3.1) which are solvable with respect to y_μ *in the form* (3.4) *and satisfy conditions* (3.3) or (3.9), *then the system has*

∞^m (resp. ∞^{2m}) *particular solutions which are obtained by integrating* m *first order differential equations.*

In case when there is only one invariant relation (resp. integral), then by making use of it, we can construct further 2(n-1) invariant relations (resp. integrals). To obtain the particular solutions by Levi-Civita's technique, we shall have to integrate only one first order differential equation. Thus, we have the following theorem:

Theorem 2: *If the system of canonical equations (2.6) together with (2.7) admits one invariant relation (resp. one integral) which can be solved for one of the* $y_p's$ *then it has* ∞^1 *(resp.* ∞^2*) particular solutions which are obtained by a single quadrauture.*

Let the gyroscopic forces be derivable from a generalized potential function W, depending on the x's and $\eta's$ such that

$$g_{pq}\,\eta_q = \frac{d}{dt}\frac{\partial W}{\partial \eta_p} - C^r_{qp}\eta_q\,\frac{\partial W}{\partial \eta_r} - X_p W. \tag{5.3}$$

Putting

$$M = L - W, \tag{5.4}$$

the equations (2.4) reduce to the form

$$\frac{d}{dt}\frac{\partial M}{\partial \eta_p} - C^r_{qp}\eta_q\,\frac{\partial M}{\partial \eta_r} - X_p M = 0. \tag{5.5}$$

We set

$$y_p = \frac{\partial M}{\partial \eta_p}, \tag{5.6}$$

and define a function K by the relation

$$K = y_p\eta_p - M. \tag{5.7}$$

Then the equations (5.5) can be reduced to the canonical from

$$\eta_p = \frac{\partial K}{\partial y_p},\ \dot{y}_p = -X_p K + C^r_{qp}\eta_q y_r. \tag{5.8}$$

If the the system (5.8) possesses m time-independent invariant relations, or m integrals of the type (3.1) which are solvable in the form (3.4) and satisfy the conditions

$$(F_\sigma, F_\rho) = 0, \tag{5.9}$$

then, as discussed in [5], the Levi-Civita theorem continues to be valid for equations (5.8) as well.

The generalized Levi-Civita theorem subsumes the results given in [7] when gyroscopic forces are absent, and the results obtained in [8] and [12] when specialized to Lagrangian coordinates.

Also, if the gyroscopic forces are present but the x's are Lagrangian coordinates with $\dot{x}_p = \eta_p$, then we recover the results obtained by Rumyantsev [11] and Capodanno [1].

When the $\eta's$ are taken as quasi-velocities $\dot{\pi}'s, X_p's$ become $\partial/\partial\pi_p$, the quantities C^r_{qp} reduce to the Boltzmann three-index symbols and we obtain Pignedoli's, version [9] of Levi-Civita's theorem.

6 Generalization of Routh's motion

Let us consider the gyroscopic mechanical system with n degrees of freedom and the Hamiltonian $H(x_p, y_p)$, where the 2n quantities x_p and y_p describe the state of the system. The motion of the system is governed by the canonical equations (2.6) together with equations (2.7).

Following the viewpoint of paper [3], we assume the displacement operators $X_1, X_2, \ldots, X_m (m < n)$ to be cyclic, so that we have

(a) $X_\mu H = 0,$

(b) $(X_\mu, X_p) = 0.$

Further, we suppose that there exist certain functions $\psi_\mu(x_\alpha)$ satisfying the relations

(c) $g_{\mu\nu} = -g_{\nu\mu} = X_\nu \psi_\mu = 0;$

(d) $g_{\mu\alpha} = -g_{\alpha\mu} = X_\alpha \psi_\mu;$

and

(e) $g_{\alpha\beta}$ are functions of the variables $x_{m+1}, \ldots, x_n$.

The conditions (a) and (b) imply that $C^r_{\mu p} = 0 (p, r = 1, 2, \ldots, n;$ $\mu = 1, 2, \ldots, m < n)$. Moreover, we observe that while the first group of equations (2.6) remains unchanged under the conditions (a) - (e), the second group of equations (2.6) reduces to

$$\dot{y}_\mu = \frac{\partial H}{\partial y_\alpha} X_\alpha \psi_\mu(x_\beta) = \eta_\alpha X_\alpha \psi_\mu. \tag{6.1}$$

But in accordance with (2.3)

$$\dot{\psi}_\mu = \eta_\alpha X_\alpha \psi_\mu \ (x_\beta). \tag{6.2}$$

It follows from (6.1) and (6.2) that

$$\dot{y}_\mu = \dot{\psi}_\mu,$$

giving

$$y_\mu = \psi_\mu(x_{m+1}, \ldots, x_n) + c_\mu \tag{6.3}$$

where $c_1, c_2, \ldots c_m$ are arbitrary constants.

Let us introduce a function R, which, like the Routhian function, is obtained from H by replacing the $y'_\mu s$ by their values (6.3). Then from equation

$$H(x_\alpha, \psi_\mu + c_\mu, y_\alpha) = R(x_\alpha, y_\alpha, c_\mu)$$

we have

$$\frac{\partial H}{\partial y_\alpha} = \frac{\partial R}{\partial y_\alpha}, \tag{6.4}$$

and

$$X_\alpha H = X_\alpha R + g_{\alpha\mu} \frac{\partial H}{\partial y_\mu}. \tag{6.5}$$

Consequently equations (2.5) become

$$\eta_\alpha = \frac{\partial R}{\partial y_\alpha}, \ \dot{y}_\alpha = -X_\alpha H + C^r_{q\alpha} \frac{\partial H}{\partial y_q} y_r + g_{\alpha s} \frac{\partial H}{\partial y_s}. \tag{6.6}$$

and equations (2.7) take the form

$$\dot{x}_\alpha = \frac{\partial H}{\partial y_s} X_s x_\alpha. \tag{6.7}$$

Breaking the sum over q and s into sums over μ and β, the equations (6.6) and (6.7) assume the form

$$\eta_\alpha = \frac{\partial H}{\partial y_\alpha},$$

$$\dot{y_\alpha} = -X_\alpha H + C^r_{\mu\alpha} \frac{\partial H}{\partial y_\mu} y_r + C^r_{\beta\alpha} \frac{\partial H}{\partial y_\beta} y_r + g_{\alpha\mu} \frac{\partial H}{\partial y_\mu} + g_{\alpha\beta} \frac{\partial H}{\partial y_\beta},$$

and

$$\dot{x_\alpha} = \frac{\partial H}{\partial y_\beta} X_\beta x_\alpha.$$

By means of relations (6.3 - 6.7) and condition (b), we finally obtain

$$\left.\begin{array}{lr} \eta_\alpha = \frac{\partial R}{\partial y_\alpha}, & \text{(A)} \\ \dot{y_\alpha} = -X_\alpha R + C^r_{\beta\alpha} \frac{\partial R}{\partial y_\beta} y_r + g_{\beta\alpha} \frac{\partial R}{\partial y_\beta} & \text{(B)} \end{array}\right\} \tag{6.8}$$

which must be supplemented by the equations

$$\dot{x_\alpha} = \frac{\partial R}{\partial y_\beta} X_\beta x_\alpha. \tag{6.9}$$

If the system of equations (6.8B), in conjunction with (6.9), is integrable then we can find the values of 2(n-m) variables x_p, y_p. Substituting for x's into equations (6.3), we obtain the values of (2n-m) variables $x_{m+1}, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. Finally, we can determine the values of the remaining variables $x_1, x_2, \ldots, x_m$ by performing m quadratures of the equations

$$\dot{x_\mu} = \frac{\partial H}{\partial y_\nu} X_\nu x_\mu. \tag{6.10}$$

We have thus obtained a generalization of Routh's equatons to cover the gyroscopic system.

Let us now turn to the problem of particular solutions of the gyroscopic system described by equations (2.6) together with (2.7) which possess integrals (6.3) and satisfy conditions (a) - (e). Moreover, the integrals (6.3) satisfy conditions (3.9). Accordingly, we can apply the Levi-Civita Theorem 2 to construct particular solutions. Precisely, the new invariant relations

$$X_\alpha R(x_p, y_p, c_\mu) = 0, \quad \frac{\partial R(x_p, y_p, c_\mu)}{\partial y_\alpha} = 0,$$

furnish 2(n-m) constant values of x_α, y_α; the equations (6.3) then give y_μ as constants. Finally, we obtain $\dot{x_\mu}$ as constants and have x_μ determined as functions of time t.

The particular solutions obtained here present a generalization of Routh's motion. This generalization includes as a special case the result given in [1] by Capodanno.

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