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Jacobi's Multiplier for Poincaré's Equations

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In this paper we use Poincaré's theory of equations in group variables to describe the motion of a holonomic mechanical system and study the problem of determining Jacobi's multiplier for the equations of motion.

1. Introduction

We consider a conservative holonomic mechanical system whose position at any time t is determined by independent coordinates $x_1, x_2, \ldots, x_n$. Let the infinitesimal displacements of the system be defined by a transitive group of operators

$$X_j = \xi_i{}^j\,(x)\frac{\partial}{\partial x_i}, \qquad x = (x_1, x_2, \ldots, x_n)$$

with commutation relations

$$(X_i, X_j) = C_{ij}^k X_k. \tag{1}$$

Throughout, we use summation convention with all indices taking values from 1 to n.

According to Poincaré [3], the variation $df(\delta f)$ of an arbitrary function $f(x,t)$ in a real (virtual) displacement of the system is determined by the formula

$$df = \left(\frac{\partial f}{\partial t} + \eta_j X_j f\right) dt \qquad (\delta f = \omega_j X_j f) \tag{2}$$

where the independent parameters $\eta_j(\omega_j)$ characterize real (virtual) displacements of the system.

Setting $f = x_i$ in (2) yields

$$\dot{x}_i = \xi_i{}^j(x)\eta_j, \qquad \eta_j = \mu_j{}^i(x)\dot{x}_i. \tag{3}$$

Obviously, in terms of the Kronecker delta symbols,

$$\xi_i{}^j \mu_j{}^k = \delta_i{}^k, \quad \xi_j{}^i \mu_k{}^j = \delta_k{}^i. \tag{4}$$

By means of equations (3) we can eliminate $\dot{x}_i$'s from the Lagrangian of the system and denote it by $L(x, \eta)$. The equations of motion of the system, in the form suggested by Poincaré [3], are

$$\frac{d}{dt}\frac{\partial L}{\partial \eta_i} - C_{ji}^k \eta_j \frac{\partial L}{\partial \eta_k} - X_i L = 0. \tag{5}$$

We introduce the variables y_i and the Hamiltonian H by

$$y_i = \frac{\partial L}{\partial \eta_i}, \quad H(x, y) = y_i \eta_i - L(x, \eta).$$

The Poincaré equations (5) transform to the canonical form [1, 2]:

$$\eta_i = \frac{\partial H}{\partial y_i}, \qquad \dot{y}_i = X_i H + C_{ji}^k\, y_k \frac{\partial H}{\partial y_j}.$$

To these equations we must adjoin equations (3). Thus we obtain

$$\dot{x}_i = \xi_i{}^j \frac{\partial H}{\partial y_i}, \qquad \dot{y}_i = X_i H + C_{ji}^k\, y_k \frac{\partial H}{\partial y_j} \tag{6}$$

2. Jacobi's Multiplier

We write the equations (6), governing the motion of the conservative holonomic system, in the form

$$\frac{dx_1}{Q_1} = \cdots = \frac{dx_n}{Q_n} = \frac{dy_1}{P_1} = \cdots = \frac{dy_n}{P_n} = dt \tag{7}$$

where

$$Q_i = \xi_i{}^j \frac{\partial H}{\partial y_j}, \qquad P_i = -X_i H + C^k_{ji}\, y_k \frac{\partial H}{\partial y_j}.$$

The multiplier M for the system (7) is defined by the equation [4]:

$$\frac{d \ln M}{dt} + S = 0, \tag{8}$$

where

$$S = \frac{\partial Q_i}{\partial x_i} + \frac{\partial P_i}{\partial y_i}. \tag{9}$$

In our case, in view of the property $C^k_{ji} = -C^k_{ij}$, we have

$$S = \left(\frac{\partial}{\partial x_i} \xi_i{}^j + C^i_{ji} \right) \frac{\partial H}{\partial y_j}. \tag{10}$$

From (1) it follows that

$$C^k_{ji} = \xi_\ell{}^i\, \xi_m{}^j \left(\frac{\partial}{\partial x_\ell} \mu_k{}^m - \frac{\partial}{\partial x_m} \mu_k{}^\ell \right).$$

Consequently

$$C^i_{ji} = \xi_\ell{}^i\, \xi_m{}^j \left(\frac{\partial}{\partial x_\ell} \mu_i{}^m - \frac{\partial}{\partial x_m} \mu_i{}^\ell \right)$$

which, in view of relations (4), become

$$C^i_{ji} = -\frac{\partial}{\partial x_\ell} \xi_\ell{}^j - \xi_\ell{}^i\, \xi_m{}^j \frac{\partial}{\partial x_m} \mu_i{}^\ell.$$

Inserting this expression for C^i_{ji} in (10) and using (2), we obtain

$$S = -\xi_\ell{}^i \frac{d}{dt} \mu_i{}^\ell.$$

Let D be the determinant of the matrix $\left(\mu_i{}^\ell\right)$ and $D_{i\ell}$ the cofactor of $\mu_i{}^\ell$. Then we have

$$\xi_\ell{}^i = \frac{D_{i\ell}}{D}.$$

Consequently

$$S = -\frac{1}{D} D_{i\ell} \frac{d}{dt} \left(\mu_i{}^\ell\right) = -\frac{1}{D} \frac{dD}{dt} = -\frac{d}{dt} \ln D. \tag{11}$$

From equations (8) and (11) it follows that the multiplier for the system (6) is given by

$$M = D. \tag{12}$$

Specializing to the case when $\eta_i = \dot{x}_i$, we find that all the quantities C^k_{ji} vanish, X_i become $\partial/\partial x_i$ and the matrix $\left(\mu_i{}^\ell\right)$ reduces to the matrix $\left(\delta_i{}^\ell\right)$, so that $D = 1$. This agrees with the known result that the multiplier of a canonical system

$$\dot{x}_i = \frac{\partial H}{\partial y_i} \qquad \dot{y}_i - \frac{\partial H}{\partial x_i}$$

is unity.

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