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WHITTAKER'S REDUCTION METHOD FOR POINCARÉ'S DYNAMICAL EQUATIONS

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Whittaker's reduction method invokes the energy integral to reduce the order of Lagrange's equations of motion of a holonomic dynamical system. This paper treats the corresponding result for a nonholonomic conservative system described by Poincaré's equations which are constructed from the standpoint of the theory of Lie groups.

1 Introduction

We consider a nonholonomic conservative dynamical system whose position at any time t is determined by independent coordinates $x_1, x_2, \ldots, x_n$. There are n - k ideal stationary nonholonomic constraints, written by the aid of summation convention in the form

$$A_{\alpha p}(x)\, \dot{x}_p = 0, \qquad p = 1, 2, \ldots, n; \qquad \alpha = k+1, \ldots, n. \tag{1.1}$$

As discussed in [2, 3], along with the nonholonomic system we consider the associated holonomic system obtained by removing all the nonholonomic constraints (1.1). For the associated holonomic system, we introduce the Poincaré parameters $\eta_1, \eta_2, \ldots, \eta_n$ by the relations

$$\dot{x}_p = \xi_p^s(x)\, \eta_s, \qquad p,\, s = 1, 2, \ldots, n, \tag{1.2}$$

and express the variations δx_p of the coordinates x_p in terms of the parameters of virtual displacements $\omega_1, \omega_2, \ldots, \omega_n$:

$$\delta x_p = \xi_p^s(x) \omega_s, \qquad p, s = 1, 2, \ldots, n.$$

The variation of a function $f(x,t)$ in a real (virtual) displacement of the associated holonomic system is given by

$$df = [\frac{\partial f}{\partial t} + \eta_p X_p f]\, dt \qquad (\delta f = \omega_p X_p\, f). \tag{1.3}$$

The operators

$$X_p = \xi_p^s(x) \frac{\partial}{\partial x_s}, \qquad p, s = 1, 2, \ldots, n$$

form a local group of transformations on the configuration space. The commutators $[X_p, X_q]$ satisfy the relations

$$[X_p, X_q] = C_{pq}^r X_r, \qquad p, q, r = 1, 2, \ldots, n \tag{1.4}$$

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which serve to define the quantities C^r_{pq} with antisymmetric property $C^r_{pq} = -C^r_{qp}$.

In the presence of nonholonomic constraints (1.1) the operators X_p do not form a closed system. Yet (1.2) can be used for parametrization of constraints (1.1) by the $\eta's$. In consequence, equations (1.1) become

$$a_{\alpha p}(x)\eta_p = 0, \qquad p = 1,2,\ldots,n;\ \alpha = k+1,\ldots,n, \tag{1.5}$$

and the parameters ω_p satisfy

$$a_{\alpha p}(x)\omega_p = 0, \qquad p = 1,2,\ldots,n;\ \alpha = k+1,\ldots,n. \tag{1.6}$$

In the sequel, we consider the nonholonomic conservative system with nonholonomic constraints (1.5), derive its equations of motion and investigate the problem of reducing their order by the application of Whittaker's method [5, 7].

2 Equations of Motion

For the nonholonomic system with k degrees of freedom, the parameters η_p are connected by n - k equations (1.5). We can therefore express them in terms of certain k independent parameters $\theta_1, \theta_2, \ldots, \theta_k$,

$$\eta_p = b_{pi}(x)\,\theta_i, \qquad p = 1,2,\ldots,n;\ i = 1,2,\ldots,k. \tag{2.1}$$

Similarly, the $\omega's$ can be expressed in terms of certain independent parameters $\Omega_1, \Omega_2, \ldots, \Omega_k$,

$$\omega_p = b_{pi}(x)\,\Omega_i. \tag{2.2}$$

To obtain the equations of motion we write the fundamental equation of dynamics in the form [2]:

$$\left(\frac{d}{dt}\frac{\partial L_0}{\partial \eta_p} - C^r_{qp}\,\eta_q\,\frac{\partial L_0}{\partial \eta_r} - X_p L_0\right)\omega_p = 0, \qquad p,q,r = 1,2,\ldots,n$$

where $L_0(x,\eta)$ is the Lagrangian of the associated holonomic system. In the last equation we substitute from (2.2) to obtain a sum which, in view of the independence of parameters Ω_i, leads to the Poincaré equations of motion

$$b_{pi}\left(\frac{d}{dt}\frac{\partial L_0}{\partial \eta_p} - C^r_{qp}\,\eta_q\,\frac{\partial L_0}{\partial \eta_r} - X_p L_0\right) = 0 \tag{2.3}$$

$$(p,q,r = 1,2,\ldots,n;\ i = 1,2,\ldots,k).$$

We denote by L the function obtained from L_0 when $\eta_p's$ are replaced by $\theta_i's$ by means of (2.1). Thus

$$L(x_p,\theta_i) = L_0(x_p,\eta_p).$$

In accordance with (1.3), we obtain the relations

$$X_p L = X_p L_0 + \frac{\partial L_0}{\partial \eta_q}\theta_j X_p(b_{qj}), \qquad b_{pi} X_p L_0 = b_{pi} X_p L - \frac{\partial L_0}{\partial \eta_q}\theta_j b_{pi} X_p(b_{qj}),$$

$$\frac{\partial L}{\partial \theta_i} = \frac{\partial L_0}{\partial \eta_p} b_{pi}, \qquad \frac{d}{dt}(\frac{\partial L_0}{\partial \eta_p}) b_{pi} = \frac{d}{dt}(\frac{\partial L}{\partial \theta_i}) - \frac{\partial L_0}{\partial \eta_p}\theta_j\, b_{qj} X_q(b_{pi})$$

$$(p, q = 1, 2, \ldots, n;\ i, j = 1, 2, \ldots, k).$$

Substituting these expressions into (2.3) and defining

$$Y_i = b_{pi} X_p,$$

$$K^p_{ji} = Y_i(b_{pj}) - Y_j(b_{pi}) - C^p_{qr}\, b_{qj} b_{ri}$$

we arrive at the equations

$$\frac{d}{dt}\frac{\partial L}{\partial \theta_i} - Y_i L + K^p_{ji}\,\theta_j\,\frac{\partial L_0}{\partial \eta_p} = 0, \qquad p = 1, 2, \ldots, n;\ i, j = 1, 2, \ldots, k. \tag{2.4}$$

These are the Poincaré equations of the nonholonomic system with constraints (2.1) in terms of the independent parameters $\theta_1, \theta_2, \ldots, \theta_k$.

3 Generalized Energy Integral

To show that equations (2.3) possess the generalized energy integral

$$\frac{\partial L_0}{\partial \eta_p}\eta_p - L_0 = h = const., \qquad p = 1, 2, \ldots\ n, \tag{3.1}$$

we differentiate $L_0(x_p, \eta_p)$ according to (1.3) and use (2.1) and (2.3). Thus

$$\begin{aligned}
\frac{dL_0}{dt} &= \frac{\partial L_0}{\partial \eta_p}\ \dot{\eta}_p + \eta_p X_p L_0., \qquad p = 1, 2, \ldots, n, \\
&= \frac{\partial L_0}{\partial \eta_p}\ \dot{\eta}_p + \theta_i b_{pi} X_p L_0, \qquad i = 1, 2, \ldots, k, \\
&= \frac{\partial L_0}{\partial \eta_p}\ \dot{\eta}_p + \theta_i\, b_{pi}\,(\frac{d}{dt}\frac{\partial L_0}{\partial \eta_p} - C^r_{qp}\,\eta_q\,\frac{\partial L_0}{\partial \eta_r}), \qquad p, q, r = 1, 2, \ldots\ n, \\
&= \frac{d}{dt}(\frac{\partial L_0}{\partial \eta_p}\ \eta_p),
\end{aligned}$$

the sum $\theta_i b_{pi} C^r_{qp}\,\eta_q = C^r_{qp}\,\eta_p\,\eta_q$ vanishing by virtue of the relation $C^r_{qp} = -C^r_{pq}$. Integrating, we obtain (3.1).

Computing in a similar fashion, we find that equations (2.4) possess the generalized energy integral

$$\frac{\partial L}{\partial \theta_i}\theta_i - L = const., \qquad i = 1, 2, \ldots, k. \tag{3.2}$$

4 Whittaker's Reduction Method

We now turn to the question of applicability of Whittaker's method [7] of using the generalized energy integral (3.1) to reduce the order of the system of equations (2.3), and therefore also of equations (2.4).

Suppose that, at least, one generalized velocity $\dot{x}$ occurs in the set of parameters $\eta_1, \eta_2, \ldots, \eta_n$ as well as in the set $\theta_1, \theta_2, \ldots, \theta_k$, say $\dot{x}_1 = \eta_1 = \theta_1$, and that the corresponding operator X_1 does not appear in the commutators (X_p, X_α), so that $C^1_{p\alpha} = 0$, $p = 1, 2, \ldots, n$; $\alpha = 2, 3, \ldots, n$. Under these conditions we show that we can not only reduce the order of the system of Poincare equations but we can also preserve the form of these equations.

Noticing that $b_{1\mu} = 0$ for $\mu = 2, 3 \ldots, n$, and leaving aside the first equation in (2.3), the remaining equations (2.3) yield

$$b_{\alpha\mu}\left(\frac{d}{dt}\frac{\partial L_0}{\partial \eta_\alpha} - X_\alpha L_0 - C^\beta_{q\alpha}\,\eta_q\,\frac{\partial L_0}{\partial \eta_\beta}\right) = 0 \tag{4.1}$$

$$(q = 1, 2, \ldots, n;\ \alpha, \beta = 2, 3, \ldots, n;\ \mu = 2, 3, \ldots, k).$$

We introduce new quantities

$$\eta'_\alpha = \frac{\eta_\alpha}{\eta_1}, \quad \theta'_\mu = \frac{\theta_\mu}{\eta_1}, \quad \alpha = 2, 3, \ldots, n;\ \mu = 2, 3, \ldots, k. \tag{4.2}$$

Then (2.1) gives

$$\eta'_\alpha = b_{\alpha i}\theta'_i \quad (\eta'_1 = \theta'_1 = 1), \quad i = 1, 2, \ldots, k. \tag{4.3}$$

In the function L_0, we replace η_α by $\eta_1\eta'_\alpha$ and denote the resulting function by $K(x_p, \eta_1, \eta'_\alpha)$. Then from the equation $L_0(x_p, \eta_p) = K(x_p, \eta_1, \eta'_\alpha)$ we have

$$\frac{\partial L_0}{\partial \eta_1} = \frac{\partial K}{\partial \eta_1} - \frac{\eta_\alpha}{\eta_1^2}\frac{\partial K}{\partial \eta'_\alpha}, \qquad \frac{\partial L_0}{\partial \eta_\alpha} = \frac{1}{\eta_1}\frac{\partial K}{\partial \eta'_\alpha}, \quad X_p L_0 = X_p K \tag{4.4}$$

$$(p = 1, 2, \ldots, n;\ \alpha = 2, 3, \ldots, n)$$

The first two sets of relations (4.4) give

$$\frac{\partial K}{\partial \eta_1} = \frac{\partial L_0}{\partial \eta_1} + \frac{\eta_\alpha}{\eta_1}\frac{\partial L_0}{\partial \eta_\alpha} = \frac{\eta_p}{\eta_1}\frac{\partial L_0}{\partial \eta_p}, \quad p = 1, 2, \ldots, n;\ \alpha = 2, 3, \ldots, n. \tag{4.5}$$

Now in the energy integral (3.1), we replace η_α by $\eta_1\eta'_\alpha$ and from this equation obtain η_1 as a function of the x_p and η'_α :

$$\eta_1 = f(x_p, \eta'_\alpha). \tag{4.6}$$

We substitute from (4.6) into (4.5) and denote the resulting expression for $\frac{\partial K}{\partial \eta_1}$ by L'_0 :

$$L'_0(x_p, \eta'_\alpha) = \frac{\partial K}{\partial \eta_1} = \frac{\eta_p}{\eta_1}\frac{\partial L_0}{\partial \eta_p}. \tag{4.7}$$

From the energy integral (3.1), which by (4.7) may be written in the form

$$\eta_1 \frac{\partial K}{\partial \eta_1} - K = h,$$

we have

$$\frac{\partial K}{\partial \eta'_\alpha} = \eta_1 \frac{\partial L'_0}{\partial \eta'_\alpha}, \qquad X_p K = \eta_1 X_p L'_0. \tag{4.8}$$

From (4.4) and (4.8) it follows that

$$\frac{\partial L'_0}{\partial \eta'_\alpha} = \frac{\partial L_0}{\partial \eta_\alpha}, \quad X_p L'_0 = \frac{1}{\eta_1} X_p L_0.$$

Substituting from the foregoing relations in the Poincaré equations (4.1), we obtain

$$b_{\alpha\mu}\left(\frac{d}{dt}\frac{\partial L'_0}{\partial \eta'_\alpha} - \eta_1 X_\alpha L'_0 - C^\beta_{q\alpha}\,\eta_1\,\eta'_q \frac{\partial L'_0}{\partial \eta'_\beta}\right) = 0 \tag{4.9}$$

which, on division by $\eta_1 = \dot{x}_1$, yields

$$b_{\alpha\mu}\left(\frac{d}{dx_1}\frac{\partial L'_0}{\partial \eta'_\alpha} - X_\alpha L'_0 - C^\beta_{q\alpha}\,\eta'_q \frac{\partial L'_0}{\partial \eta'_\beta}\right) = 0 \tag{4.10}$$

$$(q = 1, 2, \ldots, n;\ \alpha,\ \beta = 2, 3, \ldots, n;\ \mu = 2, 3, \ldots, k).$$

In the function L'_0, *replace* η'_α by θ'_i by means of (4.3) and denote the function thus obtained by L'. Then the equation

$$L'(x_p, \theta'_\mu) = L'_0(x_p, \eta'_\alpha)$$

gives, in view of (1.3) and (4.3), the relations

$$X_\alpha L' = X_\alpha L'_0 + \frac{\partial L'_0}{\partial \eta'_\beta}\,\theta'_i X_\alpha(b_{\beta i}),$$

$$\frac{\partial L'}{\partial \theta'_\mu} = \frac{\partial L'_0}{\partial \eta'_\alpha}\, b_{\alpha\mu}, \quad \frac{d}{dt}\frac{\partial L'}{\partial \theta'_\mu} = \frac{d}{dt}\left(\frac{\partial L'_0}{\partial \eta'_\alpha}\right) b_{\alpha\mu} + \frac{\partial L'_0}{\partial \eta'_\alpha}\,\eta_1\, b_{qi}\,\theta'_i X_q(b_{\alpha\mu}).$$

Substituting the last relations into equations (4.9), we find that they transform into the equations

$$\frac{d}{dt}\frac{\partial L'}{\partial \theta'_\mu} - \eta_1 Y_\mu L' + K^i_{\beta\mu}\eta_1\theta'_i \frac{\partial L'_0}{\partial \eta'_\beta} = 0.$$

With $\eta_1 = \dot{x}_1$, these equations can be written in the form

$$\frac{d}{dx_1}\frac{\partial L'}{\partial \theta'_\mu} - Y_\mu L' + K^\beta_{i\mu}\,\theta'_i\,\frac{\partial L'_0}{\partial \eta'_\beta} = 0 \tag{4.11}$$

$$(i = 1,2,\ldots,k;\ \mu = 2,3,\ldots,k;\ \beta = 2,3,\ldots,n)$$

Equations (4.11) together with (4.3) constitute a system of n + k - 2 equations for the quantities $\eta'_2, \eta'_3, \ldots, \eta'_n$ and $\theta'_2, \theta'_3, \ldots, \theta'_k$. The equations (4.10) and (4.11) are referred to as Whittaker's form of the Poincaré equations (2.3) and (2.4), respectively.

If we choose $\eta_p = \dot{x}_p$ for $p = 1,2,\ldots n$, so that X_p become $\partial/\partial x_p$ and all C^r_{qp} vanish then equations (2.3) and (2.4) agree with the Maggi equations [4] and equations (4.10) and (4.11) become Whittaker's equations given by Šalaev [6]. If we specialize η_p as quasi-velocities $\dot{\pi}_p$ then X_p become $\frac{\partial}{\partial \pi_p}$, C^r_{pq} reduce to the Boltzmann three-index symbols γ_{prq} and we recover Whittaker's equations of nonholonomic systems obtained in [1] by Djukic.

5 An Example

As an example of Whittaker's method, we discuss the motion of a Chaplygin's sleigh on a horizontal plane. We use this example to illustrate the advantage which accrues from applying this method and choosing a good set of the Poincaré parameters.

Let x, y be the coordinates of the runner on the plane, ϕ the angle of rotation of the runner, a, b the coordinates of the centre of mass in a coordinate system attached to the runner, and k the radius of gyration, the body being assumed to be of unit mass. The equation of the constraint is

$$\dot{y} - \dot{x}\tan\theta = 0.$$

Taking

$$x_1 = \phi, x_2 = x, x_3 = y \text{ and } \eta_1 = \dot{\phi},\ \eta_2 = \dot{x},\ \eta_3 = \dot{y},$$

we have

$$X_1 = \frac{\partial}{\partial \phi},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},$$

so that all $C^r_{pq} = 0, p, q, r = 1,2,3$, and the constraint equation becomes

$$\eta_3 - \eta_2 \tan\phi = 0.$$

Introducing independent Poincaré parameters θ_1, θ_2 by

$$\eta_1 = \theta_1 = \dot{\phi},\ \eta_2 = \theta_2\cos\phi,\ \eta_3 = \theta_2\sin\phi, \tag{5.1}$$

we have $b_{11} = 1$, $b_{22} = \cos\phi$, $b_{32} = \sin\phi$, $b_{12} = b_{21} = b_{31} = 0$.

The kinetic energy T_0 of the body is given by

$$2T_0 = 2K = \left[\eta_2^2 - \dot{\phi}(a\sin\phi + b\cos\phi)\right]^2 + \left[\eta_3^2 - \dot{\phi}(a\cos\phi - b\sin\phi)\right]^2 + k^2\dot{\phi}^2.$$

Since $L_0 = T_0$, the energy integral is

$$T_0 = h = \text{const.} \tag{5.2}$$

Putting

$$\eta_2' = \frac{\eta_2}{\dot\phi},\ \eta_3' = \frac{\eta_3}{\dot\phi},\ \theta_2' = \frac{\theta_2}{\dot\phi}, \tag{5.3}$$

equations (5.1) give

$$\eta_2' = \theta_2' \cos\theta,\ \eta_3' = \theta_2' \sin\phi, \tag{5.4}$$

and the expression for T_0 becomes

$$2T_0 = \dot\phi^2 \left[\eta_2'^2 - 2\eta_2'(a\sin\phi + b\cos\phi) + \eta_3'^2 + 2\eta_3'(a\cos\phi - b\sin\phi) + \delta^2\right], \tag{5.5}$$

where $\delta^2 = a^2 + b^2 + k^2$. From (5.2) and (5.5) we have

$$\dot\phi = \sqrt{2h}\left[\eta_2'^2 - 2\eta_2'(a\sin\phi + b\cos\phi) + \eta_3'^2 + 2\eta_3'(a\cos\phi - b\sin\phi) + \delta^2\right]^{-1/2}. \tag{5.6}$$

The functions T_0' and T' are given by

$$T_0' = \left\{2h\left[\eta_2'^2 - 2\eta_2'(a\sin\phi + b\cos\phi) + \eta_3'^2 + 2\eta_3'(a\cos\phi - b\sin\phi) + \delta^2\right]\right\}^{1/2},$$

$$T' = \left\{2h(\theta_2'^2 - 2b\theta_2' + \delta^2)\right\}^{1/2}.$$

Substituting for T_0' and T' in equations (4.11) and noting that

$$Y_1 = \frac{\partial}{\partial\phi},\ Y_2 = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y},\ K_{12}^2 = \sin\phi,\ K_{12}^3 = -\cos\phi,\ K_{22}^2 = K_{22}^3 = 0,$$

we get only one equation which simplifies to

$$\frac{d\theta_2'}{\theta_2'^2 - 2b\theta_2' + \delta^2} = \frac{a\,d\phi}{a^2 + k^2}.$$

Integrating, we have

$$\theta_2' = b + \sqrt{a^2 + k^2}\tan\frac{a\phi + c}{\sqrt{a^2 + k^2}}, \tag{5.7}$$

c being an arbitrary constant. Using this value of θ_2', equations (5.3) and (5.4) yield η_2 and η_3 as functions of ϕ which, in turn, give $\dot x$ and $\dot y$ as functions of ϕ. Integrating these functions, we obtain

$$x = b\sin\theta + \sqrt{a^2 + k^2}\int\cos\phi\tan\frac{a\phi + c}{\sqrt{a^2 + k^2}}d\phi,$$

$$y = -b\cos\phi + \sqrt{a^2 + k^2}\int\sin\phi\tan\frac{a\phi + c}{\sqrt{a^2 + k^2}}d\phi.$$

In view of (5.4) and (5.7), the right-hand side of (5.6) is a function of ϕ which when integrated yields

$$t = \frac{1}{\sqrt{2h}}\frac{a^2 + k^2}{a}\ln\tan\left(\frac{\pi}{4} + \frac{a\phi + c}{2\sqrt{a^2 + k^2}}\right) + d, \quad d = \text{const.}$$

Our solution of the problem agrees with the solution obtained in [4] by the application of Chaplygin's theorem of reducing multiplier in conjunction with the Hamilton-Jacobi method.

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