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F.S. Cater, A.B. Thaheem

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

On Decomposition of Hilbert Spaces

F.S. Cater* and A.B. Thaheem**

1. Introduction

Let X be a Banach space and α be a bounded operator on X. Denote by $N(\alpha)$, the null space of α and by $R(\alpha)$, the range of α. There are some situations when X admits a decomposition: $X = N(\alpha) \oplus R(\alpha)$, for instance when X is finite–dimensional and $N(\alpha) = N(\alpha^2)$ [8, pp. 271–73]. This result may not hold in general when X is infinite–dimensional (see [9] for some further information). However, it was proved in [9] that if α is a *–automorphism of a von Neumann algebra M, say, then a weaker decomposition holds in the sense that $M = \overline{N(\alpha - 1) + R(\alpha - 1)}$ and $N(\alpha - 1) \cap R(\alpha - 1) = \{0\}$ and the smallest weakly closed subalgebra M_1 generated by $R(\alpha - 1)$ is a two–sided ideal and if $M_1 = Me$ for a central projection e in M, then $p = (1 - e)$ is the largest central projection such that $\alpha(px) = px$ for all $x \in M$. These results have been of fundamental importance in the decompositional properties of von Neumann algebras relative to an operator equation: $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ for a pair of *–automorphisms α and β and a similar operator equation $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ for all $t \in R$ for two one–parameter groups of *–automorphisms $\{\alpha_t : t \in R\}$ and $\{\beta_t : t \in R\}$ (see e.g. [10, 11] which contain further references about this operator equation). Subsequently, this operator equation has been studied for C^*–algebra [1], for rings [2] and for unitaries on Hilbert spaces [12] and some of the results on decomposition of von Neumann algebras have been used in the Tomita–Takesaki theory [5]. In view of the importance of the theory of

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contractions on Hilbert spaces, it is natural to investigate decompositional properties of Hilbert spaces relative to contractions which is, in fact, the main aim of this paper. To begin with we generalize the results of [9] for contractions on a Hilbert space. As an application we are able to obtain a decomposition of a Hilbert space H relative to two one–parameter groups of commuting invertible contractions $\{\alpha_t : t \in R\}$ and $\{\beta_t : t \in R\}$ satisfying the operator equation $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ for all $t \in R$. We show (Theorem 2.6) that there exists a projection p on H such that $\alpha_t = \beta_t$ on $(1-p)H$ and $\alpha_t = \beta_{-t}$ on pH for all $t \in R$. Sz. Nagy and Foias (see e.g. [6]) proved that if $\{\alpha_t\}$ is a semigroup of contractions on a Hilbert space H, then there exists a unique decomposition $H = H_0 \oplus H_1$ such that H_0 and H_1 reduce α_t and $\alpha_t|H_0$ is unitary and $\alpha_t|H_1$ is completely nonunitary for all $t \in R$. The proof depends on technical arguments related to the corresponding generators and cogenerators. Formulating this result for one–parameter group of contractions we are able to provide an alternate and relatively simpler proof of this result using the techniques developed here.

We shall use the notations of Rudin [7] for Hilbert spaces and we refer to [4, 6] for more details on contractions on Hilbert spaces.

2. Decomposition Results

Theorem 2.1. *Let H be a Hilbert space and α be a contraction on H. Then $N(\alpha - 1) + R(\alpha - 1)$ is dense in H.*

Proof. Suppose that $N(\alpha-1) + R(\alpha-1)$ is not dense in H. Then there exists a nonzero continuous linear functional ϕ on H that vanishes on $N(\alpha-1)$ and $R(\alpha-1)$. That ϕ vanishes on $R(\alpha-1)$ implies that $\phi\ (\alpha(x) - x) = 0$ or $(\phi o \alpha)(x) = \phi(x)$ for all $x \in H$. By Riesz representation theorem, there exists a unique vector y_0 in H such that

$\phi(x) = \langle x, y_0 \rangle$ for all $x \in H$. It follows that $\langle x, y_0 \rangle = \phi(x) = (\phi o \alpha)(x) = \phi(\alpha(x)) = \langle \alpha(x), y_0 \rangle = \langle x, \alpha^*(y_0) \rangle$ for all $x \in H$. This implies that $\alpha^*(y_0) = y_0$. Since α is a contraction, therefore $\alpha(y_0) = y_0$ (see e.g. [2, p. 188], [6, p. 408]). This implies that $y_0 \in N(\alpha - 1)$. Since ϕ vanishes on $N(\alpha - 1)$, therefore, $\phi(y_0) = 0$. This implies that $\phi(y_0) = \|y_0\|^2 = 0$ and hence $y_0 = 0$. Thus we conclude that $\phi(x) = 0$ for all $x \in H$ and consequently $\phi = 0$, a contradiction. This completes the proof.

Remark 2.2.

(a) We observe that $N(\alpha - 1) \cap R(\alpha - 1) = \{0\}$. The proof is similar to the one in [9] for a *–automorphism α on a von Neumann algebra. Since this requires a simple argument we provide the proof for the sake of completeness. If $y \in N(\alpha - 1) \cap R(\alpha - 1)$, then $\alpha(y) = y$ and $\alpha(x) - x = y$ for some $x \in H$. Then $\alpha(x) = x + y$. This implies $\alpha^2(x) = \alpha(x) + \alpha(y) = x + 2y$ and by induction $\alpha^n(x) = x + ny$ for all positive integers $n \geq 1$. Then $n\|y\| = \|ny\| = \|\alpha^n(x) - x\| \leq \|\alpha^n(x)\| + \|x\| \leq 2\|x\|$ for all positive integers $n \geq 1$. This implies $\|y\| = 0$ and hence $y = 0$.

(b) $N(\alpha - 1) \perp R(\alpha - 1)$. Indeed, if $x \in N(\alpha - 1)$ and $y \in R(\alpha - 1)$, then $\alpha(x) = x$ and $y = \alpha(z) - z$ for some $z \in H$. Then $\langle x, y \rangle = \langle x, \alpha(z) - z \rangle = \langle x, \alpha(z) \rangle - \langle x, z \rangle = \langle \alpha^*(x), z \rangle - \langle x, z \rangle = \langle x, z \rangle - \langle x, z \rangle = 0$ because $\alpha(x) = x$ implies $\alpha^*(x) = x$. If $M = \overline{R(\alpha - 1)}$, the smallest closed subspace containing $R(\alpha - 1)$ then it follows from continuity that $N(\alpha - 1) \perp M$.

(c) $N(\alpha - 1) = M^{\perp}$. Then inclusion $N(\alpha - 1) \subseteq M^{\perp}$ follows from (b). For the reverse inclusion, let $x \in M^{\perp}$. Then $\langle x, y \rangle = 0$ for all $y \in M$ and in particular $\langle x, (\alpha - 1)z \rangle = 0$ for all $z \in H$ because $R(\alpha - 1) \subseteq M$. Thus we have $\langle x, \alpha(z) \rangle = \langle x, z \rangle$ or $\langle \alpha^*(x), z \rangle = \langle x, z \rangle$ for all $z \in H$. This implies $\alpha^*(x) = x$ and since

α is a contraction, therefore $\alpha(x) = x$. This implies that $x \in N(\alpha - 1)$ and consequently $H = N(\alpha - 1) \oplus M$.

The preceding remarks lead to the following:

Theorem 2.3. *Let $M = \overline{R(\alpha - 1)}$ be the smallest closed subspace containing $R(\alpha - 1)$. If f is the projection associated to M such that $fH = M$ and $p = (1 - f)$ then p is the largest projection such that $\alpha(px) = px$ for all $x \in H$.*

Proof. We first remark that $R(\alpha - 1)$ is invariant under α then so is M. Also $M^{\perp} = N(\alpha - 1)$ is invariant under α. Thus M reduces α and consequently f commutes with α ([4, p.71]) and hence α also commutes with p. Since $p(\alpha(x) - x) = 0$ for all $x \in H$, therefore, $p(\alpha(x)) = p(x)$ for all $x \in H$ and commutativity of α and p implies that $\alpha(p(x)) = p(x)$ for all $x \in H$. That p is the maximal projection amongst all projections with this property follows from the above remarks.

Theorem 2.4. *Let α and β be invertible contractions on a Hilbert space H satisfying the operator equation*

$$(*) \qquad \beta^{-1}\alpha\beta + \alpha^{-1} = \beta + \beta^{-1}.$$

Then there exists a projection p on H such that $\alpha^{-1} = \beta^{-1}$ on pH and $\alpha = \beta^{-1}$ on $(1 - P)H$.

Proof. Since $\alpha\beta$ is a contraction, therefore by the above theorem there exists a projection p on H such that $(\alpha\beta - 1)(pH) = 0$. It follows from the operator equation (*) that $(\beta^{-1} - \alpha^{-1})(\alpha\beta - 1) = \beta^{-1}\alpha\beta + \alpha^{-1} - \beta - \beta^{-1} = 0$. Therefore, $R(\alpha\beta - 1) \subseteq N(\beta^{-1} - \alpha^{-1})$ and hence $\overline{R(\alpha\beta - 1)} \subseteq N(\beta^{-1} - \alpha^{-1})$, and consequently $(\beta^{-1} - \alpha^{-1})((1 - p)H) = 0$. Thus we obtain a projection p on H such that $\alpha\beta = 1$ on pH and $\alpha^{-1} = \beta^{-1}$

on $(1-p)H$. Equivalently, $\alpha = \beta^{-1}$ on pH and $\alpha^{-1} = \beta^{-1}$ on $(1-p)H$. This completes the proof.

In case α and β commute, then the operator equation (*) reduces to the equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. It follows from the commutativity of α and β that $R(\alpha\beta - 1)$ is α, β–invariant and so is M. Therefore, $\alpha^{-1}(x) = \beta^{-1}(x)$ for all $x \in M$ implies that $x = \alpha\beta^{-1}(x)$ and hence $\beta(x) = \beta(\alpha\beta^{-1}(x)) = \alpha(x)$. Thus $\alpha(x) = \beta(x)$ for all $x \in (1-p)(H)$. It is easy to verify that pH and $(1-p)H$ reduce both α and β and hence p commutes with α and β. These remarks lead to the following corollary that generalizes a decomposition result of [12] proved for unitaries.

Corollary 2.5. *Let α and β be commuting invertible contractions on a Hilbert space H satisfying the equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then there exists a projection p that commutes with α, β and $\alpha = \beta^{-1}$ on pH and $\alpha = \beta$ on $(1-p)H$.*

The following result is an analogue of the decomposition theorems of [10, 11] and [12] for automorphisms of von Neumann algebras and unitaries on Hilbert spaces respectively.

Theorem 2.6. *Let $\{\alpha_t : t \in R\}$ and $\{\beta_t : t \in R\}$ be two strongly continuous one-parameter groups of commuting invertible contractions on a Hilbert space H satisfying the equation $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ for all $t \in R$. Then there exists a projection p on H such that $\alpha_t = \beta_t$ on $(1-p)H$ and $\alpha_t = \beta_{-t}$ on pH for all $t \in R$.*

Proof. By Corollary 2.5, for each $n \in N$, H decomposes into mutually orthogonal (closed) subspaces p_nH and $(1-p_n)H$ that reduce $\alpha_{2^{-n}}$ and $\beta_{2^{-n}}$ and $\alpha_{2^{-n}} = \beta_{2^{-n}}^{-1}$ on p_nH and $\alpha_{2^{-n}} = \beta_{2^{-n}}$ on $(1-p_n)H$ where p_n is a projection that commutes with $\alpha_{2^{-n}}$ and $\beta_{2^{-n}}$. Also, $\alpha_{2^{-(n+1)}} = \beta_{2^{-(n+1)}}^{-1}$ on $p_{n+1}H$. Squaring both sides implies $\alpha_{2^{-(n+1)}}^2 =$

$\beta^{-2}_{2^{-(n+1)}}$ and hence $\alpha_{2^{-n}} = \beta^{-1}_{2^{-n}}$ on $p_{n+1}H$ and by maximality $p_{n+1}H \subseteq p_nH$. This implies that (p_n) is a decreasing sequence of projections on H. Let $p = \lim\limits_{n\to\infty} p_n$ (in the strong operator topology) (see e.g. [13, p. 84]). Then pH is invariant under $\alpha_{2^{-n}}$ and $\beta^{-1}_{2^{-n}}$ and hence $\alpha_{k_2^{-n}} = \beta_{k_2^{-n}}$ on pH for any integer $k \in Z$. The density of the set $\{k2^{-n} : n \in N, \quad k \in Z\}$ in R and the continuity of the action $t \in R \to \alpha_t$ on H implies that $\alpha_t = \beta_{-t}$ on pH and by a similar argument $\alpha_t = \beta_t$ on $(1-p)H$ and p commutes with α_t and β_t for all $t \in R$. This completes the proof.

We conclude this note by providing an alternative proof of a result due to Sz. Nagy and Foias [6, Theorem 10.17, p. 156] about a family of contractions on H. Their proof depends on the technical arguments about infinitesimal cogenerators. We use the techniques developed in the above theorem to prove the result. We formulate the result for one–parameter group of contractions. First recall that a contraction T on a Hilbert space H is completely nonunitary if there exists no nontrivial closed reducing subspace on which T acts unitarily.

Theorem 2.7. *Let $\{\alpha_t\}$ be a strongly continuous one–parameter group of contractions on a Hilbert space H. Then there exists a unique direct sum decomposition $H = H_0 \oplus H_1$ that reduces α_t and $\alpha_t|H_0$ is unitary and $\alpha_t|H_1$ is completely nonunitary for all $t \in R$.*

Proof. By Sz. Nagy and Focias [6, Theorem 9.1, p. 130], for each $n \in N$ there exists a unique direct sum $H = H_n^{(0)} \oplus H_n^{(1)}$ where $H_n^{(0)}$ and $H_n^{(1)}$ are closed subspaces that reduce $\alpha_{2^{-n}}$ and $H_n^{(1)} = H_n^{(0)\perp}$. Also $\alpha_{2^{-n}}|H_n^{(0)}$ is unitary and $\alpha_{2^{-n}}|H_n^{(1)}$ is completely nonunitary. Let p_n be the projection associated with $H_n^{(0)}$. Then $\alpha_{2^{-n}}|p_nH$ is unitary and $\alpha_{2^{-n}}|(1-p_n)H$ is completely nonunitary. Arguing as in Theorem 2.6, we get a projection p on H such that pH reduces $\alpha_{2^{-n}}$ and $\alpha_{2^{-n}}$ is unitary on pH and completely nonunitary on $(1-p)H$. Since pH reduces $\alpha_{2^{-n}}$ then obviously pH reduces $\alpha_{k2^{-n}}$ for any integer $k \in Z$. The density of the set $\{k2^{-n} : n \in N,\ k \in Z\}$ in R and the continuity of the action $t \in R \to \alpha_t$ on H implies that pH reduces α_t and α_t is unitary on $H_0 = pH$ and completely nonunitary on $H_1 = (1-p)H$. Thus we obtain a unique decomposition $H = H_0 \oplus H_1$ such that H_0 and H_1 reduce α_t and $\alpha_t|H_0$ is unitary and $\alpha_t|H_1$ is completely nonunitary. This completes the proof.

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* Department of Mathematics
Portland State University
Portland, Oregon 97207
U.S.A.

** Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia