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APPLICABILITY OF VOGEL - SCHLEICH PHASE THEORY TO QUANTUM PHASE MEASUREMENTS

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ABSTRACT

Applicability of Vogel - Schleich phase theory to the published experimental data is examined. An analytical expression for a reference parameter is obtained with the aim of facilitating analysis and comparisons.

There have been extensive discussions in the published literature as to which is the proper dynamical variable corresponding to the phase of a quantum field. Several attempts have been made to construct a satisfactory phase operator [1-5], but no clear conclusions have thus far emerged which could fully explain the experimental results.

A new method for constructing the phase distribution of a given quantum state was recently presented by Vogel and Schleich (VS) and was related to measurements based on a balanced homodyne detection scheme [6]. The aim was to avoid the use of troublesome phase operators and their accompanying eigen states. The VS approach capitalizes on the concept of area of overlap and interference in phase space. The overlap between a given state's phase distribution and a thin phase - space strip at an angle ϕ passing through the origin (thus representing zero field strength) is calculated. However, in contrast to the usual period of 2π in conventional theories involving phase operators, the VS formalism displays a period of π. But, this is not a serious deficiency [6].

According to the VS formalism, the "operational" phase distribution $p_E(\phi, |\psi\rangle)$ for a field described by the wave function $|\psi\rangle = \sum_{m=0}^{\infty} \psi_m |m\rangle$ is proportional to

$$(2\pi)^{-0.5} \left| \sum_{m=0}^{\infty} \{(2m)!/\, 2^{2m}\, (m!)^2\}^{0.5}\, \psi_{2m} \exp(-i2m\phi) \right|^2 , \qquad ....(1)$$

where the constant of proportionality is determined by the normalization condition

$$\int_0^{\pi} p_E\,(\phi,\, |\psi\rangle)\, d\phi \;=\; 1\,. \qquad .....(2)$$

Here, $|m\rangle$ denotes the mth number state and ϕ is a *c* - number angle, *not* a phase operator.

In another recent paper, Noh *et al* [7] have compared their measurements in two homodyne experiments using a single mode He - Ne laser with the well - known phase operator theories of Pegg and Barnett [1-3] and Susskind and Glogower [8], but the agreement was rather poor. Noh *et al* also identified two operators $\hat{C}_M$ and $\hat{S}_M$ corresponding to the measured cosine and sine of the phase difference based on comparisons with the equations from classical domain. They obtained the best agreement between their experimental results and this hybrid model [7].

More recently, Lynch [9] has considered the applicability of VS description of phase to the experimental results of Noh *et al* [7]. It was concluded there that even though the predictions of VS theory (as it stands now) are not fully supported by the experimental data; possible refinement of this theory might improve the situation *vis a vis* its applicability to experimental results. The purpose of the present work is to reexamine the limits of applicability of the VS description through an exact mathematical expression for a reference parameter in order to facilitate future comparisons.

For a coherent state $|\alpha\rangle$ (assumed real here for convenience) with the number state expansion described by

$$|\alpha\rangle = \exp(-\alpha^2/2) \sum_{m=0}^{\infty} \alpha^m / (m!)^{0.5} |m\rangle \ , \qquad ..... (3)$$

together with the normalization condition of equation (2) , we may write

$$p_E(\phi, |\alpha\rangle) = \{1 / I(\alpha)\}\exp(-2\alpha^2 \sin^2\phi) , \qquad ..... (4)$$

where

$$I(\alpha) = \int_0^{\pi} \exp(-2\alpha^2 \sin^2\phi)\, d\phi .$$

Noting that the integrand in I (α) is symmetric about $\phi = \pi / 2$, and making a substitution $\sin^2 \phi = t$, we can write

$$I(\alpha) = \int_0^1 t^{-0.5} (1 - t)^{-0.5} \exp(-2\alpha^2 t)dt ,$$

and hence

$$I(\alpha) = \pi\, \Phi(0.5, 1; -2\alpha^2) , \qquad ..... (5)$$

where Φ is the confluent hypergeometric function [10].

As discussed by Lynch [9], for the purpose of determining the applicability of the VS theory to the experimental data of Noh *et al* [7], the parameter to be evaluated is the sum of variances in cosine and sine, *i. e.*

$$\Delta\cos^2\phi + \Delta\sin^2\phi = 1 - \langle|\cos\phi|\rangle^2 , \qquad ..... (6)$$

where $\Delta\cos^2\phi \equiv \langle\cos^2\phi\rangle - \langle\cos\phi\rangle^2$ is the variance in $\cos\phi$ and the symbol $\langle....\rangle$ represents the expectation value. In the present case, the expectation value will be found by integration over the phase distribution, *i. e.*

$$\langle f \rangle = \int_0^{\pi} f(\phi) p_E(\phi, |\alpha\rangle) d\phi . \qquad ..... (7)$$

Lynch [9] calculated the second term in equation (6) *viz* $\langle |\cos\phi| \rangle^2$, numerically. We present an analytical expression here for an exact computation.

From equations (4) and (7), and from the consideration of symmetry about the line $\phi = \pi/2$, we have

$$\langle |\cos\phi| \rangle = \{2 / I(\alpha)\} \int_0^{\pi/2} |\cos\phi| \exp\{-2\alpha^2 \sin^2\phi\} d\phi.$$

Substituting $2\alpha^2 \sin^2\phi = z^2$, we get

$$\langle |\cos\phi| \rangle = [2 / \{\sqrt{2}\, \alpha\, I(\alpha)\}] \int_0^{\sqrt{2}\,\alpha} \exp(-z^2) dz$$

$$= [1 / I(\alpha)\sqrt{\{\pi / 2\alpha^2\}}]\ \mathrm{Erf}\,(\sqrt{\{2\alpha^2\}}) ;$$

and using equation (5), this yields

$$\langle |\cos\phi| \rangle = \mathrm{Erf}\,(\sqrt{\{2\alpha^2\}}) / [\sqrt{\{2\pi\alpha^2\}}\ \Phi(0.5, 1; -2\alpha^2)] \qquad ..... (8)$$

The values of the confluent hypergeometric function and the error function are well documented in the published literature, and therefore, the above reference parameter can be easily evaluated.

For very small value of α, *i. e.* $\alpha \ll 1$, the asymptotic representation of the error function as well as the confluent hypergeometric function may be used to obtain

$$\Delta\cos^2\phi \ + \ \Delta\sin^2\phi = 1 - \langle|\cos\phi|\rangle^2$$

$$= 1 - (4/\pi^2) + 8\,\alpha^2/\pi^2 \ . \qquad \ldots.. \ (9)$$

In the limit when $\alpha = 0$, this gives $\Delta\cos^2\phi \ + \ \Delta\sin^2\phi = 0.594$.

The average photon number $\langle m_1 \rangle$ in the state $|\alpha\rangle$ is α^2. However, for a 50:50 beam splitter used in the incident beam of Noh *et al,* $\langle m_1 \rangle$ is $2\alpha^2$.

Fig 1 is a plot of $\Delta\cos^2\phi \ + \ \Delta\sin^2\phi = 1 - \langle|\cos^2\phi|\rangle$ as a function of $\langle m_1 \rangle = 2\alpha^2$ based on equation (8). The experimental data of Noh *et al* (taken from Fig 5 of their paper [7]) is also superimposed for comparison. This confirms the earlier conclusions of Lynch [9] that the VS theory gives excellent agreement with the experimental results of Noh *et al* for $\langle m_1 \rangle \geq 1$. However, below $\langle m_1 \rangle = 1$, there is disagreement. But, this is to be expected from the "blob" geometry assumed for the state in the phase space of the VS theory [6].

The principal utility of the present work is indeed in facilitating calculations for further analysis and comparisons with experimental results. In particular, equation (8) should prove useful when improvements are incorporated in the VS theory.

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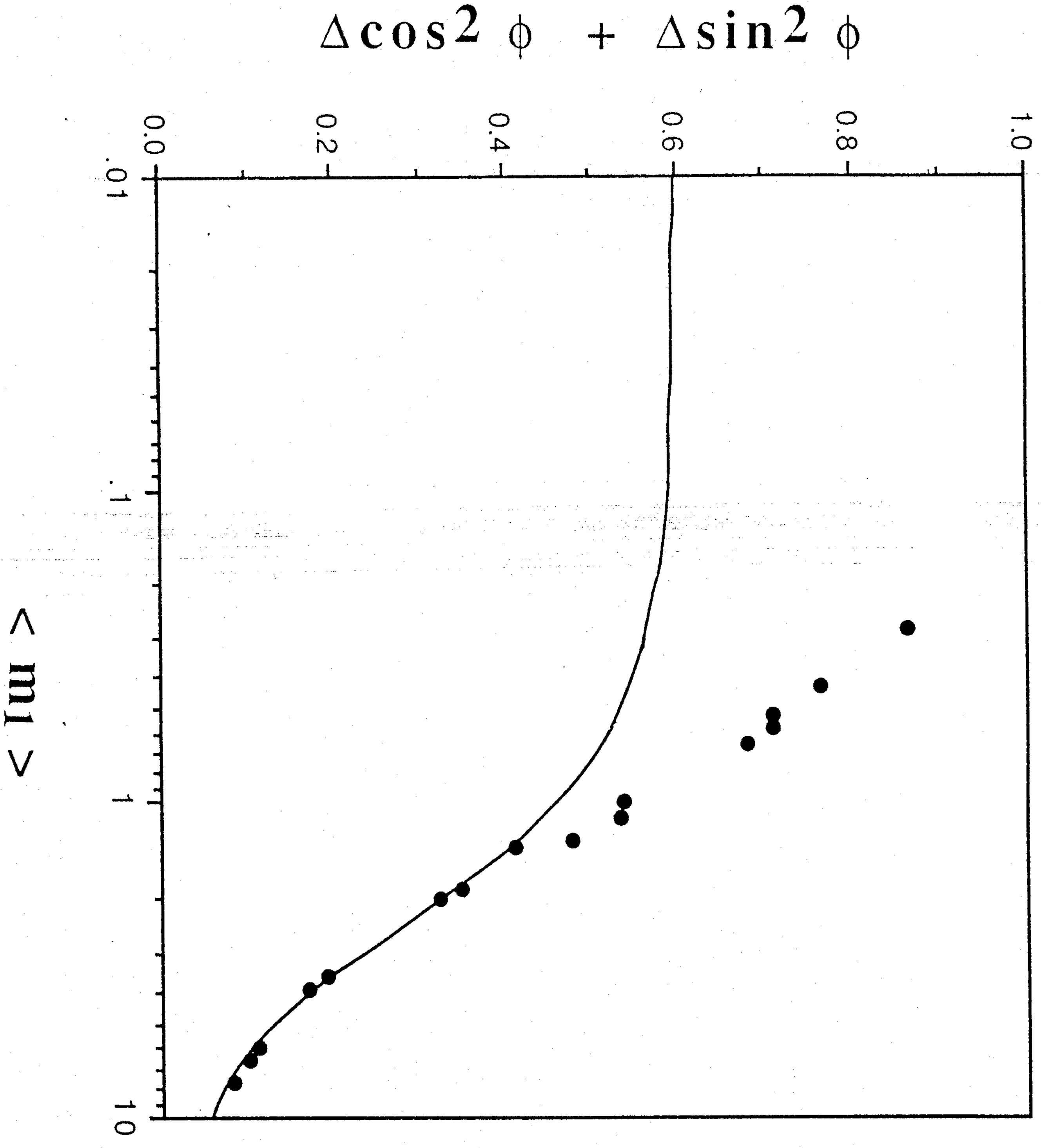
FIGURE CAPTIONS

Fig. 1: Vogel - Schleich (VS) theory predictions (—) versus the experimental measurements of Noh *et al* (full circles). Here, the mean of detected photon number of the incident beam, $\langle m_1 \rangle = 2\alpha^2$ as it enters the apparatus is plotted along the x - axis while the sum of variances in cosine and sine, $\Delta\cos^2\phi + \Delta\sin^2\phi = 1 - \langle|\cos^2\phi|\rangle$ is plotted along the y - axis.


Δcos2 φ + Δsin2 φ
0.0
0.2
0.4
0.6
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1.0
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