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On Automorphisms of C^*–Algebras

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1. Introduction

In this paper we study some properties of automorphisms of C^*–algebras and von Neumann algebras relative to an operator equation. During the last ten years a lot of work has been done on the operator equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, where α and β are *–automorphisms of a von Neumann algebra M (say). To mention briefly, this operator equation arose for modular operators in the new approach of the Tomita–Takesaki theory [11]. Later on, this operator equation has been studied for arbitrary *–automorphisms as well as for one–parameter groups of *–automorphisms of von Neumann algebras. Among several decomposition results in this context, it is known (see e.g. [1,7]) that if M is a factor then either $\alpha = \beta$ or $\alpha = \beta^{-1}$. This operator equation in the commuting case (that is, when α and β commute) has been used in the study of Tomita–Takesaki theory [4, 5]. Batty [2] has studied this operator equation for C^*–algebras. Recently, Brešar [3] has studied this operator equation in a more general context of rings and has obtained some decomposition results showing a relationship between α and β. For more information about this operation equation we refer to [1, 7, 8, 9].

In this paper we consider this operator equation in a more general form $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$, where α and β are automorphisms for a von Neumann algebra or a C^*–algebra for an appropriate real or complex number c. The situation is more general because on the one hand we do not assume automorphisms to be necessarily *–automorphisms and on the other hand the automorphisms α^{-1} and β^{-1} are replaced by the

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mappings $c\alpha^{-1}$ and $c\beta^{-1}$ which are no longer automorphisms except when $c = 1$. We study some properties of the automorphisms in the framework of this operator equation. The main aim is to resolve this operator equation. We prove (Theorem 2.8) that if α and β are automorphisms of a von Neumann algebra M such that $\|\alpha - 1\| < 1$, $\|\beta - 1\| < 1$ and satisfy the equation $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$, where c is a real or complex number such that $|c| > \max\{1, \|\alpha\|,\}$, then $\alpha = \beta$. We are also able to extend this result for any C^*–algebra (Corollary 2.10). We also provide a similar result for *–automorphisms with a relatively simpler condition on the number c We also provide a characterization (Theorem 2.12) of the operator equation $uxu^{-1} + u^{-1} + u^{-1}xu^{-1} = vxv^{-1} + v^{-1}xv$ for any x in a von Neumann algebra M for commuting invertible normal operators u and v in M in terms of a central projection p in M. This offers a generalization of some decomposition results obtained in [1, 7, 9].

We shall follow Sakai [6] for the general theory of von Neumann algebras and C^*–algebras. A von Neumann algebra is assumed to be acting on a separable Hilbert space H and $B(H)$ denotes the von Neumann algebra of all bounded operators on H.

2. Pairs of Automorphisms

Let us recall that an automorphism α of a von Neumann algebra (or a C^*–algebra) M is said to be inner if there is an invertible element u in M such that $\alpha(x) = uxu^{-1}$ for all $x \in M$. We say that α is implemented by u. If u is unitary then α is a *–automorphism.

To begin with, we prove certain results about the commutativity of a pair of automorphisms.

Proposition 2.1. *Let M be a von Neumann algebra and α, β be inner automorphisms of M which are implemented by u and v respectively. Assume that $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$,* *where c is a real or complex number such that* $|c| > \max\{1, \|\alpha\|\}$. *Then* u, v *(and hence* α, β*) commute.*

Proof. Suppose that α and β are implemented by u and v respectively. Then for any $x \in M$,

$$\alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x)$$

implies $uxu^{-1} + cu^{-1}xu = vxv^{-1} + cv^{-1}xv$.

For $x = v$, we get

$$uvu^{-1} + cu^{-1}vu = (1+c)v.$$

That is, $\alpha(v) + c\alpha^{-1}(v) = (1+c)v$.

This implies $\alpha^2(v) + cv = (1+c)\alpha(v)$. Then $(\alpha - c)\ (\alpha - 1)v = 0$. Put $(\alpha - 1)v = y$. Then $\alpha(v) = v + y$ and $(\alpha - c)y = 0$ or $\alpha(y) = cy$. This implies $\alpha^2(v) = \alpha(v) + \alpha(y) = v + y + cy = v + (1+c)y$. Thus we obtain $\alpha^n(v) = v + (1 + c + c^2 + \cdots + c^{n-1})y$ for any natural number $n \geq 1$. Then

$$\|y\| \frac{|c^n - 1|}{|c-1|} = \|\alpha^n(v) - v\| \leq (\|\alpha\|^n + 1)\,\|v\|.$$

It follows that

$$\|y\| \leq \frac{(\|\alpha\|^n + 1)\,|c-1|\|v\|}{|c^n - 1|}.$$

Since $|c| > \max\{1,\ \|\alpha\|\}$, therefore, as $n \to \infty$, we get $\|y\| = 0$ and $y = 0$. So $\alpha(v) - v = 0$ and $uvu^{-1} = v$ or $uv = vu$. This implies that α and β commute.

Corollary 2.2. *Let M be a von Neumann algebra and α, β be automorphisms of M such that $\|\alpha - 1\| < 1$ and β is inner and implemented by v. Assume that $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ where c is a real or complex number such that $|c| > 2$. Then α and β commute.*

Proof. By Sakai [6, Theorem 4.1.19] α is inner. Also $\|\alpha - 1\| < 1$ implies $\|\alpha\| < 1 + 1 = 2 < c$ by assumption. The result now follows from the above proposition.

Corollary 2.3. *Let M be a von Neumann algebra and α, β be automorphisms of*

M such that $\|\alpha - 1\| < 1$, $\|B - 1\| < 1$, let c be a real or complex number such that $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ and either $|c| > \max\{1, \|\alpha\|\}$ or $|c| > 2$. Then α and β commute.

Proof. The assumptions $\|\alpha - 1\| < 1$ and $\|\beta - 1\| < 1$ imply that α and β are inner and the result follows from Proposition 2.1 and Corollary 2.2.

Proposition 2.4. *Let M be a von Neumann algebra and α be an inner automorphism of M which is implemented by $u \in M$. Then u is normal if and only if*

$$(\#) \qquad (\alpha((\alpha^{-1}(x))^*)^* = \alpha^{-1}((\alpha(x^*))^*), \quad (x \in M).$$

Proof. Since $\alpha(x) = uxu^{-1}$ for every $x \in M$, therefore it follows that

$$(\alpha((\alpha^{-1}(x)))^*)^* = u^{*-1}u^{-1}xuu^*$$

and

$$\alpha^{-1}(\alpha(x^*))^*) = u^{-1}u^{*-1}xu^*u.$$

Then (#) is satisfied if and only if $u^{*-1}u^{-1}xuu^* = u^{-1}u^{*-1}xu^*u$ if and only if $(u^*u)(uu^*)^{-1}x = x(u^*u)(uu^*)^{-1}$ if and only if $(u^*u)(uu^*)^{-1} = K$ lies in the center of M. If u is normal then $u^*u = uu^*$ and (#) follows.

Now assume that (#) holds. Then $u^*u = K(uu^*)$ for some operator K in the center of M. Moreover, K is nonnegative and invertible because uu^* and u^*u are nonnegative, commuting and invertible. To prove that $u^*u = uu^*$ it is enough to show that $K = 1$.

Assume further that $K \geq 1$ does not hold. Let (the compact Hausdorff space) X be the spectrum of the center of M. The complex valued function on X representing K is positive but not dominated by 1. A simple topological argument shows that there is an (orthogonal) projection Q in the center of M and a positive number d such that

$KQ-(1+d)Q \geq 0.$

Let $u = w(u^*u)^{1/2}$ be the polar decomposition of u where w is the appropriate unitary operator in M. Then $uu^* = w(u^*u)w^*$ and $uu^*Q = w(u^*uQ)w^*$. But u is unitary, so $\|u^*uQ\| = \|u^*uQ\|$. Since $u^*u = Kuu^*$, therefore

$$\|uu^*Q\| = \|u^*uQ\| = \|Kuu^*Q\| = \|KQ-(1+d)Q(uu^*Q)+(1+d)Q(uu^*Q)\|.$$

Since the operators $((1+d)Q)(uu^*Q)$ and $(1+d)Q(uu^*Q)$ are commuting and nonnegative, therefore,

$$\|uu^*Q\| \geq (1+d)\|Q(uu^*Q)\| = \|uu^*Q\| + d\|uu^*Q\|.$$

But $d > 0$, then $\|uu^*Q\| = 0$. Now, uu^* is invertible implies that $Q = 0$, a contradiction. This proves that $K \geq 1$. To prove that $K \leq 1$, we simply write $K^{-1}\ u^*u = uu^*$ and interchange the roles of u and u^* in the above argument and obtain that $K^{-1} \geq 1$ or $K \leq 1$. This proves that $K = 1$ and $u^*u = uu^*$ and hence u is normal. This completes the proof of the proposition.

Proposition 2.5. *Let u and v be invertible operators in $B(H)$, let $\epsilon > 0$ such that $0 \leq \epsilon < 1$ and let c be a real or complex number such that*

$$uxu^{-1} + cu^{-1}xu = vxv^{-1} + cv^{-1}xv$$

for all x in $B(H)$. Then there exists a positive real number δ such that if u_0 and v_0 are invertible operators in $B(H)$ with $\|u-u_0\| < \delta$, $\|v-v_0\| < \delta$, $\|u^{-1}-u_0^{-1}\| < \delta$, $\|v^{-1}-v_0^{-1}\| < \delta$, then

(i) $\|u_0xu_0^{-1} + cu_0^{-1}xu_0 - v_0xv_0^{-1} - cv_0^{-1}xv_0\| < \epsilon^2\|x\|$ *and*

(ii) $\|uvxv^{-1}u^{-1} - u_0v_0xv_0^{-1}u_0^{-1}\| < \epsilon\|x\|$ *for all $x \in B(H)$.*

Proof. For any $x \in B(H)$.

$$\|uxu^{-1}-u_0xu_0^{-1}\| \leq 1\|u-u_0\|\|x\|\|u^{-1}\|+\|u\|\|x\|\ \|u^{-1}-u_0^{-1}\|+\|u-u_0\|\|x\|\|u^{-1}-u_0^{-1}\|,$$

$$\|u^{-1}xu - u_0^{-1}xu_0\| \leq \|u^{-1} - u_0^{-1}\|\|x\|\|u\| + \|u^{-1}\|\|x\|\|u - u_0\| + \|u^{-1} - u_0^{-1}\|\|x\|\|u - u_0\|.$$

Put $\delta_1 = \epsilon^2(\|u\| + \|u^{-1}\| + 1)^{-1}(1 + |c|)^{-1}$. Then $\delta_1 < 1$ implies $(\|u\| + \|u^{-1}\| + \delta_1) < (\|u\| + \|u^{-1}\| + 1)$ and for $\|u - u_0\| < \delta_1$ and $\|u^{-1} - u_0^{-1}\| < \delta_1$, we have

$$\begin{aligned} \|uxu^{-1} - u_0xu_0^{-1}\| &\leq (\delta_1\|u^{-1}\| + \delta_1\|u\| + \delta_1^2)\|x\| \\ &\leq (\|u\| + \|u^{-1}\| + \delta_1)\|x\|(\|u\| + \|u^{-1}\| + 1)^{-1}(1 + |c|)^{-1}\epsilon^2 \\ &< \epsilon^2\|x\|(1 + |c|)^{-1}, \end{aligned}$$

and in the same way

$$\|u^{-1}xu - u_0^{-1}xu_0\| < \epsilon^2\|x\|(1 + |c|)^{-1}.$$

Finally,

$$\|u_0xu_0^{-1} + cu_0^{-1}xu_0 - v_0xv_0^{-1} - cv_0^{-1}xv_0\| < (1 + |c|)\epsilon^2\|x\|(1 + |c|)^{-1} = \epsilon^2\|x\|.$$

Now

$$\begin{aligned} \|uvxv^{-1}u^{-1} - u_0v_0xv_0^{-1}u_0^{-1}\| \leq\ & (\|uv - u_0v_0\|\|v^{-1}u^{-1}\| + \|uv\|\|v^{-1}u^{-1} - v_0^{-1}u_0^{-1}\| + \\ & \|uv - u_0v_0\|\|v^{-1}u^{-1} - v_0^{-1}u_0^{-1}\|)\|x\| \end{aligned}$$

and

$$\|uv - u_0v_0\| \leq \|u - u_0\|\ \|v\| + \|u\|\ \|v - v_0\| + \|u - u_0\|\ \|v - v_0\|$$

and similarly

$$\|v^{-1}u^{-1} - v_0^{-1}u_0^{-1}\| \leq \|v^{-1} - v_0^{-1}\|\ \|u^{-1}\| + \|v^{-1}\|\|u^{-1} - u_0^{-1}\| + \|v^{-1} - v_0^{-1}\|\ \|u^{-1} - u_0^{-1}\|.$$

Let

$$\delta_2 = \epsilon(\|u\| + \|v\| + \|u^{-1}\| + \|v^{-1}\| + 2)^{-1}(\|uv\| + \|v^{-1}u^{-1}\| + 1)^{-1}.$$

Then for $\|u - u_0\| < \delta_2$, $\|v - v_0\| < \delta_2$, we have

$$\begin{aligned} \|uv - u_0v_0\| &< \epsilon(\|u\| + \|v\| + 1)(\|u\| + \|v\| + \|u^{-1}\| + \|v^{-1}\| + 2)^{-1}(\|uv\| + \|v^{-1}u^{-1}\| + 1)^{-1} \\ &\leq \epsilon(\|uv\| + \|v^{-1}u^{-1}\| + 1)^{-1} \end{aligned}$$

and

$$\begin{aligned}
&\|v^{-1}u^{-1} - v_0^{-1}u_0^{-1}\| \\
&\quad < \epsilon(\|u^{-1}\| + \|v^{-1}\| + 1)(\|u\| + \|v\| + \|u^{-1}\| + \|v^{-1}\| + 2)^{-1}(\|uv\| + \|v^{-1}u^{-1}\| + 1)^{-1} \\
&\quad \leq \epsilon(\|uv\| + \|v^{-1}u^{-1}\| + 1)^{-1}.
\end{aligned}$$

So

$$\begin{aligned}
\|uvxv^{-1}u^{-1} - u_0v_0xv_0^{-1}u_0^{-1}\| &< \epsilon(\|uv\| + \|v^{-1}u^{-1}\| + 1)^{-1}(\|uv\| + \|v^{-1}u^{-1}\| + 1)\|x\| \\
&= \epsilon\|x\|.
\end{aligned}$$

Finally, let $\delta = \min(\delta_1, \delta_2)$ and the conclusion follows.

We need the following lemma in the proof of Proposition 2.7.

Lemma 2.6. *Let X be a compact Hausdorff space such that the closure of every open set in X is open. Let F be a complex-valued continuous function on X, and let $\epsilon > 0$. Then there exists a continuous function F_0 on X such that $|F - F_0| < \epsilon$ and the range $F_0(X)$ of F_0 is finite and $F_0(X) \subseteq F(X)$.*

Proof. We can cover X with open–closed sets U such that the diameter of the st $F(U)$ is less than ϵ for each such U. By compactness there are finitely many such sets $U_1, U_2, \ldots, U_n$ covering X. On the set U_1, choose a value in $F(U_1)$ for $F_0(U_1)$. On the set $U_2 - U_1$, choose a value in $F(U_2)$ for $F_0(U_2 - U_1)$. On the set $U_3 - (U_1 \cup U_2)$, choose a value in $F(U_3)$ for $F_0(U_3 - (U_1 \cup U_2))$. We continue this process for a finite number of times and the result follows.

We now prove one of our main results.

Proposition 2.7. *Let M be a von Neumann algebra and α, β be inner automorphisms of M which are implemented by normal operators u and v respectively. Suppose u, v*

commute and α, β satisfy the equation $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ where c is a real or complex number with $c^2 \neq 1$. Then $\alpha = \beta$.

Proof. Since u and v are normal and $uv = vu$. Then by the Fuglede theorem, $uv^* = v^*u$ and $u^*v = vu^*$, so any two of the operators u, v, u^*, v^* commute. Let N be the von Neumann subalgebra generated by u, v, u^*, v^*, and all the operators in the center of M. Then N is a commutative von Neumann algebra. Let Y be the spectrum of N. Then each operator t in N is represented in a natural way by a continuous complex valued function $\dot{t}$ on Y.

Pick any $\epsilon > 0$. By Proposition 2.5 and Lemma 2.6, there exists $\delta > 0$ and elements u_0 and v_0 in N such that $\|u - u_0\| < \delta, \ \|v - v_0\| < \delta$ so that

(1) $$\|u_0 x u_0^{-1} + c u_0^{-1} x u_0 - v_0 x v_0^{-1} - c v_0^{-1} x v_0\| < \epsilon^2 \|x\| \text{ for any } x \in M,$$

(2) $$\|uvxv^{-1}u^{-1} - u_0 v_0 x v_0^{-1} u_0^{-1}\| < \epsilon \|x\| \text{ for any } x \in M,$$

$\dot{u}_0(Y) \subset \dot{u}(Y), \ \dot{v}_0(Y) \subset \dot{v}(Y)$ and the set $\dot{u}_0(Y) \cup \dot{v}_0(Y)$ has only a finite number of points in it. Say $u_0 = \sum_{i-1}^{n} a_i P_i, \ \ v_0 = \sum_{i=1}^{n} b_i P_i, \ \ 1 = \sum_{i=1}^{n} P_i$ for appropriate projections $P_i \in N$.

Let Q denote the projection on the closure of $(MP_1)(H)$. Then Q is in the center of M and $Q \in N$. Reindex $P_2, P_3, \ldots, P_n$ so that Q is orthogonal to each projection $P_{j+1}, \ldots, P_n$ and Q is not orthogonal to any projection $P_1, \ldots, P_j$. Now $\dot{Q}$ and each $\dot{P}_i$ is the characteristic function of an open and closed subset of Y; moreover $\dot{Q}(Y) \subseteq \bigcup_{i=1}^{j} \dot{P}_i(Y), \dot{Q}(Y)$ is disjoint from $\bigcup_{i=j+1}^{n} \dot{P}_i(Y)$, but $\dot{Q}(Y)$ is not disjoint from any of the sets $\dot{P}_1(Y), \ldots, \dot{P}_j(Y)$.

There must be an operator $t \in M$ such that $P_j t P_1 \neq 0$. Let w be the partial isometry in the polar decomposition of $P_j t P_1$. Then $w \in M$,$(w^*w)(H) \subseteq P_1(H)$ and $(ww^*)(H) \subseteq P_j(H)$. Let z_1 be a unit vector in $(w^*w)(H)$ and note that the unit vector

$z_j = w(z_1)$ lies in $(ww^*)(H) \subseteq P_j(H)$. Now $u_0(z_1) = a_1z_1$, $u_0(z_j) = a_1z_1$, $v_0(z_1) = b_1z_1$, $v_0(z_j) = b_jz_j$, $w^*(z_j) = z_1$, $w(z_1) = z_j$. From the matrix equations

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1^{-1} & 0 \\ 0 & a_j^{-1} \end{pmatrix} + c \begin{pmatrix} a_1^{-1} & 0 \\ 0 & a_j^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_j \end{pmatrix} = \begin{pmatrix} 0 & a_1a_j^{-1} + ca_1^{-1}a_j \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_j \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1^{-1} & 0 \\ 0 & a_j^{-1} \end{pmatrix} + c \begin{pmatrix} a_1^{-1} & 0 \\ 0 & a_j^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_j \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_1^{-1}a_j + ca_j^{-1}a_1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} b_1 & 0 \\ 0 & b_j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1^{-1} & 0 \\ 0 & b_j^{-1} \end{pmatrix} + c \begin{pmatrix} b_1^{-1} & 0 \\ 0 & b_j^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_j \end{pmatrix} = \begin{pmatrix} 0 & b_1b_j^{-1} + cb_1^{-1}b_j \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} b_1 & 0 \\ 0 & b_j \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_1^{-1} & 0 \\ 0 & b_j^{-1} \end{pmatrix} + c \begin{pmatrix} b_1^{-1} & 0 \\ 0 & b_j^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_j \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b_1^{-1}b_j + cb_j^{-1}b_1 & 0 \end{pmatrix},$$

from the two–dimensional subspace of H spanned by vectors z_1 and z_j, and from (1), we conclude that

$$|a_1a_j^{-1} + ca_1^{-1}a_j - b_1b_j^{-1} - cb_1^{-1}b_j| < \epsilon^2,$$

$$|a_1^{-1}a_j + ca_j^{-1}a_1 - b_1^{-1}b_j - cb_j^{-1}b_1| < \epsilon^2,$$

or

$$(3) \qquad \begin{cases} \left|\frac{b_j}{b_1} - \frac{a_j}{a_1}\right| \left|\frac{a_1b_1}{a_jb_j} - c\right| < \epsilon^2, \\ \left|\frac{b_1}{b_j} - \frac{a_1}{a_j}\right| \left|\frac{a_jb_j}{a_1b_1} - c\right| < \epsilon^2, \end{cases}$$

We claim that either $\left|\frac{a_1b_1}{a_jb_j} - c\right| \geq \epsilon$ or $\left|\frac{a_jb_j}{a_1b_1} - c\right| \geq \epsilon$. Assume on the contrary. Now $|a_1| \leq \|u\|$, $\|a_j\| < \|u\|$, $|b_1| < \|v\|$, $|b_j| < \|v\|$, $|a_1^{-1}| < \|u^{-1}\|$, $|b_1^{-1}| < \|v^{-1}\|$, so we obtain $|a_1b_1 - ca_jb_j| < \epsilon|a_j|\,|b_j| \leq \epsilon\|u\|\,\|v\|$, $|a_jb_j - ca_1b_1| < \epsilon|a_1|\,|b_1| \leq \epsilon\|u\|\,\|v\|$. Multiply the second inequality by $|c|$ and obtain

$$|ca_jb_j - c^2a_1b_1| < \epsilon\|u\|\,\|v\|\,|c|.$$

Combine these to obtain $|a_1b_1 - c^2a_1b_1| < \epsilon\|u\|\,\|v\|(1+|c|)$. Dividing by a_1b_1 we obtain $|1 - c^2| < \epsilon\|u\|\,\|v\|\,\|u^{-1}\|\,\|v^{-1}\|\,(1 + |c|)$. Since $\epsilon > 0$ is arbitrary, therefore this last inequality is not possible. Therefore, we conclude that

$$(4)\qquad \begin{cases} \left|c-\frac{a_1b_1}{a_jb_j}\right|\ge\epsilon, \\ \text{or} \\ \left|c-\frac{a_jb_j}{a_1b_1}\right|\ge\epsilon, \end{cases}$$

By (3), either $\left|\frac{a_j}{a_1}-\frac{b_1}{b_j}\right|<\epsilon$ or $\left|\frac{a_j}{a_1}-\frac{b_j}{b_1}\right|<\epsilon$. Assume the latter. Then it follows that $|a_jb_1-a_1b_j|<\epsilon|a_1|\,|b_1|$ or $\left|\frac{b_1}{b_j}-\frac{a_1}{a_j}\right|<\epsilon|a_1|\,|b_1|\,|a_j^{-1}|\,|b_j^{-1}|<\epsilon\|u\|\,\|v\|\,\|u^{-1}\|\,\|v^{-1}\|$. Put $\epsilon'=\epsilon/\|u\|\,\|v\|\,\|u^{-1}\|\,v^{-1}\|$, then $\epsilon'<\epsilon$. Thus we obtain

$$(5)\qquad \begin{cases} \left|\frac{a_j}{a_1}-\frac{b_j}{b_1}\right|<\epsilon, \\ \text{or} \\ \left|\frac{a_1}{a_j}-\frac{b_1}{b_j}\right|<\epsilon. \end{cases}$$

In the same way, for any index $i=1,2,\ldots,j$, we have

$$\left|\frac{a_j}{a_1}-\frac{b_j}{b_1}\right|<\epsilon,\quad \left|\frac{a_1}{a_j}-\frac{b_1}{b_j}\right|<\epsilon.$$

So on the set $\bigcup_{i=1}^{j}\dot{P}_i(H)$ and on the set $\dot{Q}(Y)$ we have

$$(6)\qquad \left|\frac{\dot{u}_0}{a_1}-\frac{\dot{v}_0}{b_1}\right|<\epsilon,\quad \left|\frac{a_1}{\dot{u}_0}-\frac{b_1}{\dot{v}_0}\right|<\epsilon.$$

It follows that

$$(7)\qquad \|(a_1^{-1}u_0-b_1^{-1}v_0)Q\|<\epsilon,\quad \|(a_1u_0^{-1}-b_1v_0^{-1})Q\|<\epsilon.$$

But

$$|\dot{u}-\dot{u}_0|<\epsilon,\quad |a_1^{-1}\dot{u}-a_1^{-1}\dot{u}_0|<\epsilon|a_1|^{-1},$$

$$|\dot{v}-\dot{v}_0|<\epsilon,\quad |b_1^{-1}\dot{v}-b_1^{-1}\dot{v}_0|<\epsilon|b_1|^{-1},$$

$$|\dot{u}^{-1}-\dot{u}_0^{-1}|<\epsilon,\quad |a_1\dot{u}^{-1}-a_1\dot{u}_0^{-1}|<\epsilon|a_1|,$$

$$|\dot{v}^{-1}-\dot{v}_0^{-1}|<\epsilon,\quad |b_1\dot{v}^{-1}-b_1\dot{v}_0^{-1}|<\epsilon|b_1|.$$

Thus

$$(8)\qquad \begin{cases} \|a_1^{-1}u - a_1^{-1}u_0\| < \epsilon\|a_1\|^{-1}, & \|a_1u^{-1} - a_1u_0^{-1}\| < \epsilon|a_1|, \\ \|b_1^{-1}v - b_1^{-1}v_0\| < \epsilon\|b_1\|^{-1}, & \|b_1v^{-1} - b_1v_0^{-1}\| < \epsilon|b_1|. \end{cases}$$

It follows from (7) and (8) that

$$\|(a_1^{-1}u - b_1^{-1}v)Q\| \leq (|a_1|^{-1} + |b_1|^{-1} + 1), \quad \|(a_1u^{-1} - b_1v^{-1})Q\| \leq (|a_1| + |b_1| + 1),$$

and consequently

$$\begin{aligned} &\|((a_1^{-1}u)x(a_1^{-1}u)^{-1} - (b_1^{-1}v)x(b_1^{-1}v)^{-1}Q\| \\ &\leq \|((a_1^{-1}u) - (b_1^{-1}v))x(a_1^{-1}u)^{-1}Q\| + \|(b_1^{-1}v)x((a_1^{-1}u)^{-1} - (b_1^{-1}v)^{-1})Q\| \\ &\leq (|a_1|^{-1} + |b_1|^{-1} + 1)\|x\|\|a_1\|\|u^{-1}\| + (|a_1| + |b_1| + 1)\|x\|\|b_1^{-1}\|\|v\|. \end{aligned}$$

Now $a_1 \in u_0(Y) \subseteq u(Y)$ and $b_1 \in v_0(Y) \subseteq v(Y)$, so $|a_1| \leq \|u\|$, $|a_1|^{-1} \leq \|u^{-1}\|$, $|b_1| \leq \|v\|$, $|b_1|^{-1} \leq \|v^{-1}\|$.

It follows that

$$\|((a_1^{-1}u)x(a_1^{-1}u)^{-1} - (b_1^{-1}v)x(b_1^{-1}v)^{-1}Q\| \leq k\|x\|$$

where $k = (\|u\| + \|v\| + \|u^{-1}\| + \|v^{-1}\| + 1)(\|u\|\|u^{-1}\| + \|v\|\|v^{-1}\|)$. Note that k is independent of the choices of ϵ, x and the projections P_i.

To recapitulate our argument, we have found a central projection Q_1 such that $P_1 \leq Q_1$ and for any $x \in M$, $\|\alpha(xQ_1) - \beta(xQ_1)\| \leq k\|x\|$. Likewise for each index $i = 1, 2, \ldots, n$, there is a central projection Q_i such that $P_i \leq Q_i$ and for any $x \in M$, $\|\alpha(xQ_i) - \beta(xQ_i)\| \leq k\|x\|$.

Now define $S_1 = Q_1$, $S_2 = (Q_1 \vee Q_2) - Q_1$, $S_3 = (Q_1 \vee Q_2 \vee Q_3) - (Q_1 \vee Q_2), \ldots, S_n = (Q_1 \vee Q_2 \vee \cdots \vee Q_n) - (Q_1 \vee Q_2 \vee \cdots \vee Q_{n-1})$. Finally, each S_i is a projection in the center of M. Moreover, the S_i are mutually orthogonal and $\sum_{i=1}^{n} S_i = 1$, and

$$\|\alpha(xS_i) - \beta(xS_i)\| \leq k\|x\| \quad \text{for any } x \in M \text{ and } i = 1, 2, \ldots, n.$$

It follows that $\|\alpha(x) = \beta(x)\| \leq k\|x\|$ for any $\in M$. But ϵ is arbitrary, so we obtain that $\alpha(x) = \beta(x)$ for all $x \in M$. This concludes the proof of the proposition.

The preceding results combined together lead to the following.

Theorem 2.8. *Let M be a von Neumann algebra and α, β be automorphisms of M and c be a real or complex number such that*

1. *α and β satisfy relation (cf. Proposition 2.4).*

$$(\#) \qquad \begin{aligned} (\alpha((\alpha^{-1}(x))^*)^* &= \alpha^{-1}((\alpha(x^*))^*), \quad x \in M, \\ (\beta((\beta^{-1}(x))^*)^* &= \beta^{-1}((\beta(x^*))^*), \quad x \in M, \end{aligned}$$

2. $\|\alpha - 1\| < 1, \quad \|\beta - 1\| < 1,$

3. $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ *on M,*

4. $|c| > \max\{1, \ \|\alpha\| \}$.

Then $\alpha = \beta$.

Remark 2.9. We make the following observations that essentially explain the situations under which certain assumptions in the above theorem may be relaxed:

(a) Let $|c| > 1$ then we have the following observations.

   (i) $|c| > 4$ implies (4).

   (ii) if α and β are *–automorphisms then conditions (1) and (4) may be omitted.

   (iii) If α and β are *–automorphisms that leave the center of M pointwise fixed and if M is a type I von Neumann algebra, then we may discard (1), (2) and (4).

(iv) In case α and β are *–automorphisms and M has cyclic and separating vectors, then conditions (1), (2) and (4) follow and consequently assumption (3) alone ensures the equality $\alpha = \beta$.

(b) If $|c| < 1$, then we can write $c^{-1}\alpha + \alpha^{-1} = c^{-1}\beta + \beta^{-1}$ and impose the hypothesis on α^{-1} and β^{-1} instead of α and β.

(c) If we assume initially that α and β are inner automorphisms, we can omit condition (2).

(d) We get a weaker form of Theorem 2.8, if we replace (#) with a stronger but neater condition $(\alpha(x))^* = \alpha^{-1}(x^*)$ for $x \in M$. In this case, an argument similar to the proof of Proposition 2.4 shows that $u^{-1}u^*$ and $v^{-1}v^*$ are in the center of M and hence $uu^* = u^*u, \quad vv^* = v^*v$.

The above theorem can be extended to C^*-algebra and even more.

Corollary 2.10. *Let M be a unital normed *–algebra satisfying $\|x^*x\| = \|x\|^2; \quad x \in M$. Let α and β be *–automorphisms of M satisfying assumptions of Theorem 2.8. Then $\alpha = \beta$.*

Proof By (2), $\|\alpha(x) - x\| \le p\|x\|$ where $p = \|\alpha - 1\|$ and $x \in M$. Then $\|x\| - \|\alpha(x)\| \le p\|x\|$ and $\|\alpha(x)\| \ge (1-p)\|x\|$. Thus $\|\alpha^{-1}(y)\| \le (1-p)^{-1}\|y\|$ for $y \in M$ and $\|\alpha^{-1}\| \le (1-p)^{-1}$. Also, $\|\alpha\| - 1 \le \|\alpha - 1)\|$ and $\|\alpha\| \le \|\alpha - 1\| + 1$. So, α and α^{-1} are bounded automorphisms. Likewise β and β^{-1} are bounded.

It follows that α and β can be extended to automorphisms on the completion $\overline{M}$ of M. (Again, we denote these automorphism α and β.) Of course, $\overline{M}$ is a C^*–algebra. Then $\overline{M}$ can be regarded s a C^*–algebra of operators on a complex Hilbert space H. Without loss of generality, we delete from H all vectors v for which $\overline{M}v = (0)$. From (2) it follows that there are invertible operators $u, v \in \overline{M}$ such that $\alpha(x) = uxu^{-1}$

and $\beta(x) = vxv^{-1}$ for $x \in \overline{M}$. Then α and β extend in a natural manner to weakly bicontinuous automorphisms (we call them α and β again) of $B(H)$.

Let $\overline{\overline{M}}$ denote the weak closure of $\overline{M}$ in $B(H)$. From the weak continuity of $\alpha, \beta, \alpha^{-1}, \beta^{-1}$, we conclude that $\alpha(\overline{\overline{M}}) = M = \beta(\overline{\overline{M}})$ and the properties (1), (2), (3) and (4) are satisfied by α and β on $\overline{\overline{M}}$. But $\overline{\overline{M}}$ is a von Neumman algebra of operators on H (as is well–known. Note that the maximal projection in $\overline{\overline{M}}$ must be the identity on H because the null space of $\overline{\overline{M}}$ is (0)). By Theorem 2.8, $\alpha = \beta$ on $\overline{\overline{M}}$, and $\overline{M}$ and on M. This completes the proof.

Now we express the result for *–automorphisms.

Theorem 2.11. *Let M be a unital normed *–algebra satisfying $\|x * x\| = \|x\|^2, \quad x \in M$. Let α and β be *–automorphisms of M and let c be a real or complex number with $|c| \neq 1$. Suppose that $\|\alpha - 1\| < 1, \quad \|\beta - 1\| < 1$ and $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ on M. Then $\alpha = \beta$ on M.*

Proof. We prove the result for $|c| > 1$. If $|c| < 1$, we write $c^{-1}\alpha + \alpha^{-1} = c^{-1}\beta + \beta^{-1}$ and reverse the roles of α and β. Now $\|\alpha\| = \|\beta\| = 1$ and α and β satisfy condition (#) because α and β are *–automorphisms. So, α and β satisfy (1) and (4) of Theorem 2.8. Also (2) and (3) of Theorem 2.8 are satisfied. Corollary 2.10 implies $\alpha = \beta$.

We now prove a result that characterizes the operator considered in [7, 8, 10].

Theorem 2.12. *Let M be a von Neumann algebra and u and v be commuting invertible normal operators in M. Then the following are equivalent:*

(i) $uxu^{-1} + u^{-1}xu = vxv^{-1} + v^{-1}xv$ for all $x \in M$,

(ii) there is a central projection P in M such that Puv and $(1 - P)uv^{-1}$ are in the center of M

Proof. (i) $\Rightarrow$ (ii). Since u and v are normal and $uv = vu$. Then by the Fuglede theorem, $uv^* = v^*u$ and $u^*v = vu^*$, so any of the two operators u, v, u^* and v^* commute. Let N be the von Nuemann subalgebra generated by u, v, u^* and v^*. Then N is a commutative von Neumann algebra. Let Y denote the spectrum of N. Then each $t \in N$ is identified in the natural way with a continuous complex–valued function $\dot{t}$ on Y.

Select any ϵ with $0 < \epsilon < 1$. As in the proof of the Proposition 2.7, take operators u_0 and v_0 in N such that $\|u - u_0\| < \epsilon$, $\|v - v_0\| < \epsilon$ and

$$\begin{aligned}&\|u_0xu_0^{-1} + u_0^{-1}xu_0 - v_0xv_0^{-1} - v_0^{-1}xv_0\| \\ &\quad < \frac{1}{4}\epsilon^2(1+\|u\|)^{-7}(1+\|u^{-1}\|)^{-7}(1+\|v\|)^{7}(1+\|v^{-1}\|)^{-7}\end{aligned}$$

and $\dot{u}_0(Y) \subseteq \dot{u}(Y)$, $\dot{v}_0(Y) \subseteq \dot{v}(Y)$ and the set $\dot{u}_0(Y) \cup \dot{v}_0(Y)$ has only a finite number of points in it. Put $u_0 = \sum_{i=1}^{n} a_iP_i$ and $v_0 = \sum_{i=1}^{n} b_iP_i$ for appropriate mutually orthogonal projections $P_1, \ldots, P_n$ in N.

We construct a finite contracting sequence of projections in M dominated by P_1 and a finite sequence of partial isometries in M as follows: Let $W_1 = Q_1$ be any projection in M such that $Q_1 \leq P_1$. If $MQ_1(H)$ is not orthogonal to $P_i(H)$ for some index $i > 1$, reindex $P_2, P_3, \ldots, P_n$; $a_2, a_3, \ldots, a_n$; $b_2, b_3, \ldots, b_n$ so that $P_2(H)$ is one such space. Take a nonzero operator P_2tQ_1, $(t \in M)$ and let W_2 be the partial isometry of its polar decomposition such that $W_2^*W_2 = Q_2 \leq Q_1 \leq P_1$ and $W_2W_2^* \leq P_2$. If $MQ_2(H)$ is not orthogonal to $P_i(H)$ for some index $i > 2$, reindex $P_3, P_4, \ldots, P_n$; $a_3, \ldots, a_n$; $b_3, \ldots, b_n$ so that $P_3(H)$ is one such space. As before, there is a partial isometry $W_3 \in M$ such that $W_3^*W_3 = Q_3 \leq Q_2$ and $W_3W_3^* \leq P_3$. This process must cease with some W_q and Q_q because there are only n projections P_i. So $P_1 \geq Q_1 \geq \cdots \geq Q_q$, $W_k^*W_k = Q_k \leq Q_{k-1}$ and $W_kW_k^* \leq P_k$ for $k = 1, \ldots, q$. Also, $W_jW_k^*$ is a partial isometry in M and $(W_jW_j^*)^*(W_jW_k^*) \leq P_k$ and $(W_jW_k^*)(W_jW_j^*)^* \leq P_j$.

As in the proof of Proposition 2.7, (inequality (3)), we conclude that

$$\left|\frac{b_j}{b_i} - \frac{a_j}{a_i}\right| \left|\frac{a_ib_i}{a_jb_j} - 1\right| < \frac{1}{4}\epsilon^2(1+\|u\|)^{-7}(1+\|v\|)^{-7}\,(1+\|u^{-1}\|)^{-7}(1+\|v^{-1}\|)^{-7}$$

for $1 \leq i,\ j \leq q$. But $|b_j| \leq \|v\|$, $|b_i^{-1}| \leq \|v^{-1}\|$, so

$$\left|\frac{b_j}{b_i} - \frac{a_j}{a_i}\right| \left|\frac{a_i}{a_j} - \frac{b_j}{b_i}\right| < \frac{1}{2}\epsilon^2(1+\|u\|)^{-6}(1+\|v\|)^{-6}(1+\|u^{-1}\|)^{-6}(1+\|v^{-1}\|)^{-6}.$$

Either

$$\left|\frac{b_j}{b_i} - \frac{a_j}{a_i}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-}3(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3},$$

or

$$\left|\frac{a_i}{a_j} - \frac{b_j}{b_i}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-}3(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3}.$$

Lemma . *Let*

(A) $$\left|\frac{b_i}{b_1} - \frac{a_i}{a_1}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-}3(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3},$$

and

(B) $$\left|\frac{b_j}{b_1} - \frac{a_1}{a_j}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-3}(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3},$$

Then either

$$\left|\frac{b_j}{b_1} - \frac{a_j}{a_1}\right| < \epsilon \quad or \quad \left|\frac{b_i}{b_1} - \frac{a_1}{a_i}\right| < \epsilon.$$

Proof. **Case I.** $\left|\frac{b_i}{b_j} - \frac{a_j}{a_i}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-3}(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3}.$

Multiply the inequality

$$\left|\frac{b_j}{b_1} - \frac{a_1}{a_j}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-3}(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3}$$

by $|a_j|\,|b_1|/|(a_1|\,|b_j|$ to obtain

(C) $$\left|\frac{a_j}{a_1} - \frac{b_1}{b_j}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-2}(1+\|v\|)^{-2}(1+\|u^{-1}\|)^{-2}(1+\|v^{-1}\|)^{-2}.$$

Multiply the inequality

$$\left|\frac{b_i}{b_j} - \frac{a_j}{a_i}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-3}(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3}$$

by $|a_i|/|a_1|$ to obtain

$$\text{(D)} \qquad \left|\frac{b_i}{b_j}\frac{a_i}{a_1} - \frac{a_j}{a_1}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-3}(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3}.$$

It follows from (C) and (D) that

$$\text{(E)} \qquad \left|\frac{b_i}{b_j}\frac{a_i}{a_1} - \frac{b_1}{b_j}\right| < \epsilon(1+\|u\|)^{-3}(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3}.$$

Now multiply inequality (E) by $|b_j|\,|a_1|/|b_1|\,|a_i|$ to get

$$\left|\frac{b_i}{b_1} - \frac{a_1}{a_i}\right| < \epsilon.$$

This proves one part of the lemma.

Case II:

$$\text{(F)} \qquad \left|\frac{b_i}{b_j} - \frac{a_i}{a_j}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-3}(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3}.$$

Multiply inequality (F) by $|a_j|/|a_1|$ to get

$$\text{(G)} \qquad \left|\frac{b_i}{b_j}\frac{a_i}{a_1} - \frac{a_i}{a_1}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-2}(1+\|v\|)^{-2}(1+\|u^{-1}\|)^{-2}(1+\|v^{-1}\|)^{-2}.$$

It follows from (A) and (G) that

$$\text{(H)} \qquad \left|\frac{b_i}{b_j}\frac{a_j}{a_1} - \frac{b_i}{a_1}\right| < \epsilon(1+\|u\|)^{-2}(1+\|v\|)^{-2}(1+\|u^{-1}\|)^{-2}(1+\|v^{-1}\|)^{-2}.$$

Finally, multiply the inequality (H) by $|b_j|/|b_i|$ to obtain $\left|\frac{a_j}{a_1} - \frac{b_j}{b_1}\right| < \epsilon$. This completes the proof of the lemma.

To continue the proof of (i) $\Rightarrow$ (ii), we prove by induction on q that either $\left|\frac{a_i}{a_1} - \frac{b_i}{b_1}\right| < \epsilon$ or $\left|\frac{a_i}{a_1} - \frac{b_1}{b_i}\right| < \epsilon$ for all $i = 1, \ldots, q$. If $q = 1$ or $q = 2$, the conclusion is evident.

Suppose the conclusion holds for $(q-1)$.

Case 1. $\left|\frac{a_i}{a_1} - \frac{b_i}{b_1}\right| < \epsilon$ for $i = 1, \ldots, q-1$. Then either

$$\left|\frac{a_q}{a_1} - \frac{b_1}{b_q}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-3}(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3}$$

or

$$\left|\frac{a_q}{a_1} - \frac{b_q}{b_1}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-3}(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3}.$$

We can omit the second possibility because it would give the desired conclusion. Likewise, we assume there is some index $j < q$ such that $\left|\frac{a_j}{a_1} - \frac{b_1}{b_j}\right| \geq \epsilon$ and hence

$$\left|\frac{a_j}{a_1} - \frac{b_j}{b_1}\right| < \frac{1}{2}\epsilon(1+\|u\|)^{-3}(1+\|v\|)^{-3}(1+\|u^{-1}\|)^{-3}(1+\|v^{-1}\|)^{-3}.$$

Then by the above lemma, $\left|\frac{a_q}{a_1} - \frac{b_q}{b_1}\right| < \epsilon$ and this gives the desired conclusion.

Case 2. The conclusion $\left|\frac{a_i}{a_1} - \frac{b_1}{b_q}\right| < \epsilon$ for $i = 1, 2, \ldots, q-1$ follows by a similar argument as in case 1 with the roles of the terms $(a_i/a_1) - (b_i/b_1)$ and $(a_i/a_1) - (b_1/b_i)$ reversed.

Let Q denote the central projection with range $\overline{MQ_q(H)}$. Then $Q_q \leq Q \leq P_1 \vee \cdots \vee P_q$. It follows that either $\left\|\frac{\dot{u}_0}{a_1} - \frac{\dot{v}_0}{b_1}\right\| < \epsilon$ on $\dot{Q}(Y)$ or $\left\|\frac{\dot{u}_0}{a_1} - \frac{b_1}{\dot{v}_0}\right\| < \epsilon$ on $\dot{Q}(Y)$. Thus, we have

$$\text{(I)} \qquad \begin{cases} \text{either} & \|((u_0/a_1) - (v_0/b_1))Q\| < \epsilon \\ \text{or} & \|((u_0/a_1) - b_1 v_0^{-1})Q\| < \epsilon \end{cases}$$

Moreover, we have the equations

$$\begin{aligned}(u_0/a_1)(Qx)(u_0/a_1)^{-1} - (v_0/b_1)(Qx)(v_0/b_1)^{-1} &= ((u_0/a_1) - (v_0/b_1))(Qx)(u_0/a_1)^{-1} \\ &+ (u_0/a_1)x((v_0/b_1) - (u_0/a_1))Q(u_0/a_1)^{-1}(v_0/b_1)^{-1} - ((u_0/a_1) \\ &- (v_0/b_1))(Qx)((v_0/b_1) - (u_0/a_1))Q(u_0/a_1)^{-1}(v_0/b_1)^{-1}\end{aligned}$$

and

$$(u_0/a_1)(Qx)(u_0/a_1)^{-1} - (v_0/b_1)^{-1}(Qx)(v_0/b_1) = ((u_0/a_1) - (v_0/b_1)^{-1})(Qx)(u_0/a_1)^{-1}$$
$$+(u_0/a_1)x((v_0/b_1)^{-1} - (u_0/a_1))Q(u_0/a_1)^{-1}(v_0/b_1) - ((u_0/a_1)$$
$$-(v_0/b_1)^{-1})(Qx)((v_0/b_1)^{-1} - (u_0/a_1))Q(u_0/a_1)^{-1}(v_0/b_1).$$

But

$$\epsilon < 1,\ |a_1| \le \|u_0\| \le \|u\|,\ \ |a_1^{-1}| \le \|u_0^{-1}\| \le \|u^{-1}\|,\ |b_1| \le \|v_0\| \le \|v\|,\ \ |b_1^{-1}| \le \|v_0^{-1}\| \le \|v^{-1}\|.$$

It follows from the inequalities (I) that

$$\text{(J)}\quad \begin{cases} \text{either } \|u_0(Qx)u_0^{-1} - v_0(Qx)v_0^{-1}\| \le \|x\|\,\|u\|\,\|u^{-1}\|\,\|(1 + \|u\|\,\|u^{-1}\| \\ \qquad \|v\|\,\|v^{-1}\| + \|v\|\,\|v^{-1}\|)\epsilon \\ \text{or} \\ \qquad \|u_0(Qx)u_0 - v_0^{-1}(Qx)v_0\| \le \|x\|\,\|u\|\,\|u^{-1}\|(1 + \|u\|\,\|u^{-1}\| \\ \qquad \|v\|\,\|v^{-1}\| + \|v\|\,\|v^{-1}\|)\epsilon. \end{cases}$$

In an analogous way, we have

$$\text{(L)}\quad \begin{cases} \|u(Qx)u^{-1} - u_0(Qx)u_0^{-1}\| \le \|x\|(\|u\| + \|u\|\,\|u^{-1}\|^2 + \|u^{-1}\|^2)\epsilon, \\ \|v(Qx)v^{-1} - v_0(Qx)v_0^{-1}\| \le \|x\|(\|v\| + \|v\|\,\|v^{-1}\|^2 + \|v^{-1}\|^2)\epsilon, \\ \|v^{-1}(Qx)v - v_0^{-1}(Qx)v_0\| \le \|x\|(\|v^{-1}\| + \|v^{-1}\|\,\|v\|^2 + \|v\|^2)\epsilon. \end{cases}$$

We infer from the inequalities (J) and (L) that either

$$\|u(Qx)u^{-1} - v(Qx)v^{-1}\| \le \|x\|k\epsilon \quad \text{or} \quad \|u(Qx)u^{-1} - v^{-1}(Qx)v\| \le \|x\|k\epsilon$$

where

$$k = 3\,(\|u\|\,\|u^{-1}\| + \|u\|^2\|\,\|u^{-1}\|^2\,\|v\|\,\|v^{-1}\| + \|u\|\,\|u^{-1}\,\|v\|\,\|v^{-1}\|$$
$$+\|u\| + \|v\| + \|v^{-1}\| + \|u^{-1}\|^2 + \|v^{-1}\|^2 + \|v\|^2$$
$$+\|u\|\,\|u^{-1}\|^2 + \|v\|\,\|v^{-1}\|^2 + \|v^{-1}\|\,\|v\|^2).$$

We observe that k depends on u and v but k is independent of the choice of ϵ and the projections P_i.

Put $\alpha(x) = uxu^{-1}$ and $\beta(x) = vxv^{-1}$ for $x \in M$. Then either

$$\|(\alpha(x) - \beta(x))Q\| < \|x\|k \in \text{ for all } x \in M$$

or

$$\|(\alpha(x) - \beta^{-1}(x))Q\| < \|x\|k \in \text{ for all } x \in M.$$

For convenience, we say that a central projection S is even if $\|(\alpha(x) - \beta(x))S\| < \|x\|k\epsilon$ for all $x \in M$ and odd if $\|\alpha(x) - \beta^{-1}\beta(x))S\| < \|x\|k \in$ for all $x \in M$. We say that S is special if S is either even or odd. In particular, Q is special. We say that S is super–odd (respectively super–even) if $\alpha(x)S = \beta^{-1}(x)S$ (respectively if $\alpha(x)S = \beta(x)S$) for all $x \in M$. It is obvious that if S is an even (odd) projection and $\overline{S}$ is a central projection such that $\overline{S} \leq S$, then $\overline{S}$ is also even (odd).

Let $\{S_a\}$ be a maximal family of mutually orthogonal even central projections. From the weak continuity of α and β it follows that $\hat{S} = \bigvee_a S_a$ is also an even projection. In fact, $\hat{S}$ is the largest even projection. Likewise, there is a maximal odd central projection $\hat{T}$. In the same way, there is a maximal super–even (respectively super–odd) projection W_2 (respectively W_1) that dominates every super–even (respectively super–odd) projection.

Now $P_1 \leq \hat{S} \vee \hat{T}$; for otherwise we could repeat the entire process starting with $Q_1 = P_1 - (\hat{S} \vee \hat{T})$ and find a nonzero special projection orthogonal to $\hat{S} \vee \hat{T}$, contrary to the maximality of $\hat{S}$ and $\hat{T}$. But in the same way, $P_2, P_3, \ldots, P_n \leq \hat{S} \vee \hat{T}$. Hence $1 = \hat{S} \vee \hat{T}$.

Thus we have shown that for any integer $j > 0$ there are central projections $\hat{S}_j$ and $\hat{T}_j$ with $1 = \hat{S}_j \vee \hat{T}_j$ such that $\|(\alpha(x) - \beta(x))\hat{S}_j\| < \|x\|/j$ and $\|(\alpha(x) - \beta^{-1}(x))\hat{T}_j\| <$

$\|x\|/j$ and $\hat{S}_j$ and $\hat{T}_j$ are maximal with respect to this property. Clearly, $\hat{T}_1 \geq \hat{T}_2 \geq \cdots$ and $\hat{S}_1 \geq \hat{S}_2 \geq \cdots$. Moreover, $1-\hat{T}_1 < \hat{S}_j$ because $1 = \hat{T}_j \vee \hat{S}_j$. Also $1-\hat{T}_1 \leq 1-\hat{T}_2 \leq \cdots$, so each $1-\hat{T}_j$ and $\bigvee_{j=1}^{\infty}(1-\hat{T}_j)$ is a super even projection. Also, $\bigwedge_{j=1}^{\infty} \hat{T}_j$ is a super–odd projection. We conclude that there exists a nonzero super–odd or super–even projection.

Recall that W_1 is the maximal super–odd and W_2 is the maximal super–even projection. Indeed, $1 = W_1 \vee W_2$; for otherwise we could repeat the whole process on the space $(1-(W_1 \vee W_2))(H)$ and find a nonzero super–odd or super–even projection dominated by $1-(W_1 \vee W_2)$, contrary to the maximality of W_1 and W_2. Put $P = W_1$. Then $1 - P \leq W_2$ and P is super–odd and $1-P$ is super–even.

For any $x \in M$, $uPxu^{-1} = P\alpha(x) = P\beta^{-1}(x) = v^{-1}Pxv$ and $Puvx = xPuv$. It follows that Puv is in the center of M. Also $u(1-P)xu^{-1} = (1-P)\alpha(x) = (1-P)\beta(x) = v(1-P)xv^{-1}$ and $(1-P)uv^{-1}x = x(1-P)uv^{-1}$ for any $x \in M$. Thus $(1-P)uv^{-1}$ is in the center of M. This completes the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Let P be the projection satisfying condition (ii). Then for any $x \in M$,

$$u^{-1}((1-P)x)u = v^{-1}((1-P)x)v,$$
$$u(Px)u^{-1} = v^{-1}(Px)v, u^{-1}(Px)u = v(Px)v^{-1}, u((1-P)x)u^{-1} = v((1-P)x)v^{-1},$$

and

$$\begin{aligned} uxu^{-1} + u^{-1}xu &= Pu(Px)u^{-1} + Pu((1-P)x)u^{-1} + (1-P)u(Px)u^{-1} \\ &\quad + (1-P)u((1-P)x)u^{-1} + Pu^{-1}(Px)u + Pu^{-1}((1-P)x)u \\ &\quad + (1-P)u^{-1}(Px)u + (1-P)u^{-1}((1-P)x)u \\ &= Pv(Px)v^{-1} + Pv((1-P)x)v^{-1} + (1-P)v(Px)v^{-1} + (1-P)v((1-P)x)v^{-1} \\ &\quad + Pv^{-1}(Px)v + Pv^{-1}((1-P)x)v + (1-P)v^{-1}(Px)v + (1-P)v^{-1}((1-P)x)v \\ &= vxv^{-1} + v^{-1}xv. \end{aligned}$$

This proves (ii) $\Rightarrow$ (i). This completes the proof of the theorem.

If M is a factor then we conclude that v is a scalar multiple of either u or u^{-1}. Thus we have the following corollary that provides an alternate proof of a decomposition result of [7, Corollary 2.7] for inner automorphisms.

Corollary 2.13. *Let M be a von Neumann algebra and α, β be inner automorphisms of M which are implemented by normal commuting operators u and v respectively. Suppose α and β satisfy the operator equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, then $\alpha(x) = \beta(x)$ or $\alpha(x) = \beta^{-1}(x)$ for all $x \in M$.*

We now present a stronger form of Proposition 2.1 that may be of independent interest.

Proposition 2.14. *Let M be a unital normed *–algebra satisfying $\|xx^*\| = \|x\|^2$ for all $x \in M$. Let α, β be commuting inner automorphisms of M which are implemented by u and v respectively such that $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ for a compelx number c such that $c \neq -1$. Then u and v commute.*

Proof. Since α, β commute, therefore for any $x \in M$, we have $vuxu^{-1}v^{-1} = uvxv^{-1}u^{-1}$ and $xv^{-1}u^{-1}vu = v^{-1}u^{-1}vux$ and $v^{-1}u^{-1}vu = k$ lies in the center of M. Also

$$vu = kuv. \tag{1}$$

Substituting $x = v$ in $\alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x)$, we obtain

$$uvu^{-1} + cu^{-1}vu = (1 + c)v. \tag{2}$$

Multiply (2) on the right by u, and apply (1) to get

$$uv + cu^{-1}vu^2 = (1 + c)kuv, \tag{3}$$

or

$$\text{(4)} \qquad cu^{-1}vu^2 = (ck + k - 1)uv.$$

Multiply (4) on the left by u to obtain

$$\text{(5)} \qquad cvu^2 = (ck + k - 1)u^2v.$$

Multiply (1) on the left by ku to obtain

$$\text{(6)} \qquad kuvu = k^2u^2v.$$

Multiply (1) on the right by u to obtain

$$\text{(7)} \qquad vu^2 = kuvu.$$

From (6) and (7), we get

$$\text{(8)} \qquad cvu^2 = ck^2u^2v.$$

From (5) and (8), we obtain

$$\text{(9)} \qquad ck^2u^2v = (ck + k - 1)u^2v$$

or

$$\text{(10)} \qquad (ck^2 - ck - k + 1)u^2v = 0.$$

But u^2v is invertible, so we have

$$\text{(11)} \qquad (ck - 1)(k - 1) = ck^2 - ck - k + 1 = 0.$$

Now $kk^* = k^*k$ because k is in the center of M. From (11) we see that c^{-1} and 1 or the only possible spectral values of the normal element k, and hence c and 1 are the only possible spectral values of k^{-1}. It follows that either $k^{-1} = 1$ or $c = c^{-1}$; in the latter case $c \neq -1$, so $c = 1$ and $k^{-1} = 1$ and $k = 1$. Thus, it folloss form (1) that

$$vu = uv.$$

This completes the proof.

We conclude the paper with the following characterization of inner automorphisms.

Proposition 2.15. *Let M be a von Neumann algebra and α be an inner automorphism of M. Then $\alpha = 1$ if and only if $\alpha(\alpha(x)) = (\alpha(x^*))^*$ for all $x \in M$.*

Proof. Assume that $\alpha(\alpha(x)) = (\alpha(x^*))^*$ for all $x \in M$ and $\alpha(x) = uxu^{-1}$, $(u \in M)$. Then $u^2xu^{-2} = (ux^*u^{-1})^* = u^{-1^*}xu^* = u^{*^{-1}}xu^*$ and $u^*u^2x = xu^*u^2$ for all $x \in M$. It follows that $k = u^*u^2$ is in the center of M. Also $ku^{-2} = u^*$, so $u^*u = uu^*$. Moreover, $|k| = |u|^3$ is in the center of M. It follows that $|u| = |k|^{1/3}$ is in the center of M. Indeed, we can express $|k|$ as a nonnegative function on the spectrum of the commutative von Neumann algebra generated by u and u^*, and using an appropriate version of the Stone–Weierstrass theorem, we get $|k|^{1/3}$ as the uniform limit of certain polynomials in $|k|$. Then $\alpha(x) = (u|u|^{-1})x(u|u|^{-1})^{-1}$ for $x \in M$. But $u|u|^{-1}$ is a unitary operator, so $\alpha(x^*) = (\alpha(x))^*$ for $x \in M$. Finally, the given equation reduces to $\alpha(\alpha(x)) = ((\alpha(x)^*)^* = \alpha(x)$ and hence $\alpha(\alpha(x) - x) = 0$ implies $\alpha(x) = x$ for all $x \in M$. The converse is straightforward.

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