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SEMIGROUPS CHARACTERIZED BY THEIR FUZZY IDEALS

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Abstract: It is shown that a semigroup is semisimple if and only if each fuzzy ideal is the intersection of fuzzy prime ideals. It is also shown that the set of proper fuzzy prime ideals of a semisimple semigroup S with a two–sided zero admits the structure of a topological space, and the lattice of open sets of this space is isomorphic to the fuzzy ideal lattice of S.

Keywords: Fuzzy semigroup; fuzzy ideal; fuzzy prime ideal; fuzzy irreducible ideal; fuzzy idempotent ideal; semisimple semigroup; regular semigroup; distributive lattice.

1. Introduction

Fuzzy semigroups were introduced by Kuroki [2, 3, 4] as a generalization of classical semigroups, using the concept of fuzzy set introduced by Zadeh in his pioneering paper [7] of 1965. Recently, Kuroki [5] has characterized several classes of semigroups by the properties of their fuzzy ideals. In particular, he has shown that a semigroup S is semisimple if and only if each fuzzy ideal of S is idempotent. The purpose of this short note is to report the additional characterization that a semigroup S is semisimple if and only if each proper fuzzy ideal of S is the intersection of fuzzy prime ideals (definitions follow). It is also shown that the set of proper fuzzy prime ideals of a semisimple semigroup with a two–sided zero admits the structure of a topological space whose open sets are in one–to–one correspondence with the set of fuzzy ideals of S.

2. Preliminaries

Let S be a semigroup. A function λ from S to the unit interval $[0,1]$ of real numbers is called a fuzzy subset of S. For any fuzzy subsets λ and μ of S, $\lambda \subseteq \mu$ means that for all x in S, $\lambda(x) \leq \mu(x)$. The symbols $\lambda \cap \mu$ and $\lambda \cup \mu$ will mean the following fuzzy subsets of S:

$$(\lambda \cap \mu)(x) = \inf\{\lambda(x), \mu(x)\}$$

$$(\lambda \cup \mu)(x) = \sup\{\lambda(x), \mu(x)\}.$$

More generally, if $\{\lambda_i : i \in I\}$ is a family of fuzzy subsets of S, then $\bigcap_i \lambda_i$ and $\bigcup_i \lambda_i$ are defined as follows:

$$\left(\bigcap_i \lambda_i\right)(x) = \inf(\lambda_i(x)) \qquad \text{and}$$

$$\left(\bigcup_i \lambda_i\right)(x) = \sup(\lambda_i(x))$$

and will be called the *intersection* and *union* of the family $\{\lambda_i : i \in I\}$ of fuzzy subsets of S. A fuzzy subset λ of S is called a *fuzzy right (left) ideal* of S if $\lambda(xy) \geq \lambda(x)$ $(\lambda(xy) \geq \lambda(y))$ for all x, y in S. A fuzzy subset of S which is both a fuzzy right and a fuzzy left ideal is called a *fuzzy ideal.* A fuzzy ideal λ is called *proper* if $\lambda \neq \rho$, where ρ is the fuzzy ideal defined by $\rho(x) = 1$ for all x in S. For fuzzy ideals λ and μ of S, we define the *product*, $\lambda\mu$, of λ and μ as follows:

$$\lambda\mu(x) = \begin{cases} \sup[\inf\{\lambda(y), \mu(z)\}] & \text{if } x \text{ is expressible as } x = yz \\ 0 & \text{otherwise} \end{cases}$$

A fuzzy ideal λ of S is called *idempotent* if $\lambda\lambda = \lambda^2 = \lambda$. A fuzzy ideal ξ of S is called *fuzzy prime* if for fuzzy ideals λ and μ, $\lambda\mu \subseteq \xi$ implies $\lambda \subseteq \xi$ or $\mu \subseteq \xi$; ξ is called *fuzzy irreducible* if $\lambda \cap \mu = \xi$ implies $\lambda = \xi$ or $\mu = \xi$. A semigroup S is called *regular* if

$x \in xSx$ for each $x \in S$ [1, 5, 6]; S is called *semisimple* if $I = I^2$ for each ideal I of S ([1, p. 76], [5]). Regular semigroups form a proper subclass of semisimple semigroups.

3. Characterizations of Semisimple Semigroups

Proposition 3.1. *The following conditions on a semigroup S are equivalent:*

1. *S is semisimple.*
2. *Each fuzzy ideal of S is idempotent.*
3. *For each pair of fuzzy ideals λ, μ of S, $\lambda\mu = \lambda \cap \mu$.*
4. *The set of all fuzzy ideals of S (ordered by inclusion) is a distributive lattice under the union and product of fuzzy ideals.*

Proof. (1) $\iff$ (2) $\iff$ (3): This is due to Kuroki [5, Thm. 7.2, p. 229].

(3) $\Rightarrow$ (4): The set of fuzzy ideals of S, ordered by inclusion, is clearly a distributive lattice under the union and intersection of fuzzy ideals. Since, by the hypothesis, intersection of any two fuzzy ideals coincide with their product, the desired implication follows.

(4) $\Rightarrow$ (1): Let λ be any fuzzy ideal of S. Then $\lambda = g.l.b.$ of $\{\lambda, \lambda\} = \lambda \cdot \lambda = \lambda^2$; so λ is idempotent. This implies (2), and hence (1).

Lemma 3.2. *Let S be a semisimple semigroup. If λ is a fuzzy ideal of S with $\lambda(x) = \alpha$, where x is an element of S and $\alpha \in [0, 1]$, then there exists a fuzzy prime ideal ξ of S such that $\lambda \subseteq \xi$ and $\xi(x) = \alpha$.*

Proof. Let $X = \{\mu : \mu$ is a fuzzy ideal of S, $\mu(x) = \alpha$ and $\lambda \subseteq \mu\}$. Then $X \neq \phi$, since $\lambda \in X$. By Zorn's lemma, there exists a fuzzy ideal ξ of S which is maximal with respect to the property that $\lambda \subseteq \xi$ and $\xi(x) = \alpha$. We show that ξ is a fuzzy irreducible ideal of S. Suppose $\xi = \delta_1 \cap \delta_2$, where δ_1 and δ_2 are fuzzy ideals of S. Hence $\xi \subseteq \delta_1$ and $\xi \subseteq \delta_2$. We claim that $\xi = \delta_1$ or $\xi = \delta_2$. Suppose $\xi \neq \delta_1$ and $\xi \neq \delta_2$. Since ξ is maximal with respect to the property that $\xi(x) = \alpha$, it follows that $\delta_1(x) \neq \alpha$ and $\delta_2(x) \neq \alpha$. Hence $\alpha = \xi(x) = (\delta_1 \cap \delta_2)(x) = \inf(\delta_1(x), \delta_2(x)) \neq \alpha$, which is absurd. Hence either $\xi = \delta_1$ or $\xi = \delta_2$. We now show that ξ is a fuzzy prime ideal. Let λ and μ be fuzzy ideals of S satisfying $\lambda\mu \subseteq \xi$. Hence $\lambda\mu \cup \xi = \xi$. Using the distributivity of the lattice of fuzzy ideals of S which, in fact, follows from Theorem 3.1(4) as S is semisimple, we have $\xi = \lambda\mu \cup \xi = (\lambda \cup \xi) \cdot (\mu \cup \xi)$. Hence by Thm. 3.1(3), we have $\xi = (\lambda \cup \xi) \cdot (\mu \cup \xi) = (\lambda \cup \xi) \cap (\mu \cup \xi)$. As we have already proved that ξ is irreducible, so it follows that $\lambda \cup \xi = \xi$ or $\mu \cup \xi = \xi$. This implies that $\lambda \subseteq \xi$ or $\mu \subseteq \xi$. Hence ξ is a fuzzy prime ideal.

We now prove the following characterization theorem for semisimple semigroups.

Theorem 3.3. *The following conditions on a semigroup S are equivalent:*

1. *S is semisimple.*

2. *Each proper fuzzy ideal of S is the intersection of fuzzy prime ideals.*

Proof. (1) $\Rightarrow$ (2): Let λ be a proper fuzzy ideal of S and let $\{\lambda_\alpha : \alpha \in \Omega\}$ be the family of all fuzzy prime ideals of S which contain λ. Obviously, $\lambda \subseteq \bigcap_{\alpha\in\Omega} \lambda_\alpha$. We now prove that $\bigcap_{\alpha\in\Omega} \lambda_\alpha \subseteq \lambda$. Let x be any element of S. By Lemma 3.2, there exists a fuzzy prime ideal λ_β (say) such that $\lambda \subseteq \lambda_\beta$ and $\lambda(x) = \lambda_\beta(x)$. Thus $\lambda_\beta \in \{\lambda_\alpha : \alpha \in \Omega\}$. Hence $\bigcap_{\alpha\in\Omega} \lambda_\alpha \subseteq \lambda_\beta$, so $\bigcap_{\alpha\in\Omega} \lambda_\alpha(x) \leq \lambda_\beta(x) = \lambda(x)$. This implies that $\bigcap_{\alpha\in\Omega} \lambda_\alpha \subseteq \lambda$. Hence

$\lambda = \bigcap_{\alpha \in \Omega} \lambda_\alpha$.

(2) $\Rightarrow$ (1): Let λ be any proper fuzzy ideal of S. Hence by the hypothesis, we can write $\lambda^2 = \bigcap_{\alpha \in \Omega} \lambda_\alpha$, where $\{\lambda_\alpha : \alpha \in \Omega\}$ is the family of all fuzzy prime ideals of S which contain λ^2. Hence $\lambda^2 \subseteq \lambda_\alpha$ for all $\alpha \in \Omega$, and since λ_α is a fuzzy prime ideal, it follows that $\lambda \subseteq \lambda_\alpha$ for all $\alpha \in \Omega$. Hence $\lambda \subseteq \bigcap_{\alpha \in \Omega} \lambda_\alpha = \lambda^2$. Now $\lambda^2 \subseteq \lambda$ is always true. Hence $\lambda^2 = \lambda$, and so by Proposition 3.1(2), S is semisimple.

Finally, as an application of the above characterization, we prove the following result. First we describe some notations. Let $\mathcal{L}_S$ denote the lattice of fuzzy ideals of S, and $\mathcal{FP}_S$ the set of all proper fuzzy prime ideals of S. Moreover, for any fuzzy ideal λ of S, $\Theta_\lambda = \{\mu \in \mathcal{FP}_S : \lambda \nsubseteq \mu\}$, and $\tau(\mathcal{FP}_S) = \{\Theta_\lambda : \lambda \in \mathcal{L}_S\}$.

Theorem 3.4. *If S is a semisimple semigroup with a two–sided zero 0, then the set $\tau(\mathcal{FP}_S)$ forms a topology (in the classical sense) on the set $\mathcal{FP}_S$ and the mapping $\lambda \mapsto \Theta_\lambda$ is an isomorphism between the lattice $\mathcal{L}_S$ of fuzzy ideals of S and the lattice of open subsets of $\mathcal{FP}_S$.*

Proof. First wee show that the family $\tau(\mathcal{FP}_S)$ constitutes a topology on the set $\mathcal{FP}_S$. Note that $\Theta_\Phi = \{\mu \in \mathcal{FP}_S : \Phi \nsubseteq \mu\} = \phi$, where ϕ is the usual empty set and Φ denotes the fuzzy zero ideal of S, defined by $\Phi(x) = 0$ for all $x \in S$. This follows since Φ is contained in every fuzzy (prime) ideal of S. Thus Θ_Φ represents the empty subset of $\tau(\mathcal{FP}_S)$. On the other hand, $\Theta\rho = \{\mu \in \mathcal{FP}_S : \rho \nsubseteq \mu\} = \mathcal{FP}_S$. This is true, since $\mathcal{FP}_S$ is the set of proper fuzzy prime ideals of S. Hence $\mathcal{FP}_S = \Theta\rho$ is an element of $\tau(\mathcal{FP}_S)$. Let $\Theta_{\lambda_1}, \Theta_{\lambda_2} \in \tau(\mathcal{FP}_S)$ with λ_1, λ_2 in $\mathcal{L}_S$. Then $\Theta_{\lambda_1} \cap \Theta_{\lambda_2} = \{\mu \in \mathcal{FP}_S : \lambda_1 \nsubseteq \mu$ and $\lambda_2 \nsubseteq \mu\}$. Since μ is a fuzzy prime ideal of S it follows that

$$\Theta_{\lambda_1} \cap \Theta_{\lambda_2} = \{\mu \in \mathcal{FP}_S : \lambda_1 \cap \lambda_2 \nsubseteq \mu\} = \Theta_{\lambda_1 \cap \lambda_2}.$$

Hence $\Theta_{\lambda_1} \cap \Theta_{\lambda_2} \in \tau(\mathcal{FP}_S)$. Now consider an arbitrary family $\{\lambda_i : i \in I\}$ of fuzzy ideals of S. We have

$$\begin{aligned} \cup\Theta_{\lambda_i} &= \cup\{\mu \in \mathcal{FP}_S : \lambda_i \not\subseteq \mu\} \\ &= \{\mu \in \mathcal{FP}_S : \bigcup_i \lambda_i \not\subseteq \mu\} \\ &= \Theta_{\bigcup_i \lambda_i}. \end{aligned}$$

Hence $\bigcup_i \Theta_{\lambda_i} \in \tau(\mathcal{FP}_s)$. Thus $\tau(\mathcal{FP}_S)$ forms a topology on the set $\mathcal{FP}_S$. Let $\phi : \mathcal{L}_S \to \tau(\mathcal{FP}_S)$ be the mapping defined by $\lambda \mapsto \Theta_\lambda$. It is easily checked that ϕ is a lattice homomorphism. We show that ϕ is bijective. In fact, we need to prove that $\lambda_1 = \lambda_2$ if and only if $\Theta_{\lambda_1} = \Theta_{\lambda_2}$ for λ_1, λ_2 in $\mathcal{L}_S$. Suppose that $\Theta_{\lambda_1} = \Theta_{\lambda_2}$. If $\lambda_1 \neq \lambda_2$ then there exists $x \in S$ such that $\lambda_1(x) \neq \lambda_2(x)$. Hence either $\lambda_1(x) > \lambda_2(x)$ or $\lambda_2(x) > \lambda_1(x)$. Suppose that $\lambda_1(x) > \lambda_2(x)$. By Lemma 3.2, there exists a fuzzy prime ideal μ of S such that $\lambda_2 \subseteq \mu$ and $\lambda_2(x) = \mu(x)$. Hence $\lambda_1 \not\leq \mu$ because $\lambda_1(x) > \lambda_2(x) = \mu(x)$. Hence $\mu \in \Theta_{\lambda_1}$. Our assumption is that $\Theta_{\lambda_1} = \Theta_{\lambda_2}$. Hence $\mu \in \Theta_{\lambda_2}$, that is $\lambda_2 \not\subseteq \mu$. This is a contradiction. Thus $\Theta_{\lambda_1} = \Theta_{\lambda_2}$ implies that $\lambda_1 = \lambda_2$. Conversely, if $\lambda_1 = \lambda_2$ then, by definition, it is obvious that $\Theta_{\lambda_1} = \Theta_{\lambda_2}$. Hence $\lambda_1 = \lambda_2$ if and only if $\Theta_{\lambda_1} = \Theta_{\lambda_2}$ for λ_1, λ_2 in $\mathcal{L}_S$.

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