Semigroups Characterized by their Fuzzy Ideals

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Abstract: It is shown that a semigroup is semisimple if and only if each fuzzy ideal is the intersection of fuzzy prime ideals. It is also shown that the set of proper fuzzy prime ideals of a semisimple semigroup $S$ with a two-sided zero admits the structure of a topological space, and the lattice of open sets of this space is isomorphic to the fuzzy ideal lattice of $S$.

Keywords: Fuzzy semigroup; fuzzy ideal; fuzzy prime ideal; fuzzy irreducible ideal; fuzzy idempotent ideal; semisimple semigroup; regular semigroup; distributive lattice.

1. Introduction

Fuzzy semigroups were introduced by Kuroki [2, 3, 4] as a generalization of classical semigroups, using the concept of fuzzy set introduced by Zadeh in his pioneering paper [7] of 1965. Recently, Kuroki [5] has characterized several classes of semigroups by the properties of their fuzzy ideals. In particular, he has shown that a semigroup $S$ is semisimple if and only if each fuzzy ideal of $S$ is idempotent. The purpose of this short note is to report the additional characterization that a semigroup $S$ is semisimple if and only if each proper fuzzy ideal of $S$ is the intersection of fuzzy prime ideals (definitions follow). It is also shown that the set of proper fuzzy prime ideals of a semisimple semigroup with a two-sided zero admits the structure of a topological space whose open sets are in one-to-one correspondence with the set of fuzzy ideals of $S$. 

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2. Preliminaries

Let $S$ be a semigroup. A function $\lambda$ from $S$ to the unit interval $[0, 1]$ of real numbers is called a fuzzy subset of $S$. For any fuzzy subsets $\lambda$ and $\mu$ of $S$, $\lambda \subseteq \mu$ means that for all $x$ in $S$, $\lambda(x) \leq \mu(x)$. The symbols $\lambda \cap \mu$ and $\lambda \cup \mu$ will mean the following fuzzy subsets of $S$:

$$(\lambda \cap \mu)(x) = \inf\{\lambda(x), \mu(x)\}$$

$$(\lambda \cup \mu)(x) = \sup\{\lambda(x), \mu(x)\}.$$ 

More generally, if $\{\lambda_i : i \in I\}$ is a family of fuzzy subsets of $S$, then $\bigcap_i \lambda_i$ and $\bigcup_i \lambda_i$ are defined as follows:

$$\left(\bigcap_i \lambda_i\right)(x) = \inf(\lambda_i(x))$$

and

$$\left(\bigcup_i \lambda_i\right)(x) = \sup(\lambda_i(x))$$

and will be called the intersection and union of the family $\{\lambda_i : i \in I\}$ of fuzzy subsets of $S$. A fuzzy subset $\lambda$ of $S$ is called a fuzzy right (left) ideal of $S$ if $\lambda(xy) \geq \lambda(x)$ ($\lambda(xy) \geq \lambda(y)$) for all $x, y$ in $S$. A fuzzy subset of $S$ which is both a fuzzy right and a fuzzy left ideal is called a fuzzy ideal. A fuzzy ideal $\lambda$ is called proper if $\lambda \neq \rho$, where $\rho$ is the fuzzy ideal defined by $\rho(x) = 1$ for all $x$ in $S$. For fuzzy ideals $\lambda$ and $\mu$ of $S$, we define the product, $\lambda \mu$, of $\lambda$ and $\mu$ as follows:

$$\lambda \mu(x) = \begin{cases} 
\sup\{\inf\{\lambda(y), \mu(z)\}\} & \text{if } x \text{ is expressible as } x = yz \\
0 & \text{otherwise}
\end{cases}$$

A fuzzy ideal $\lambda$ of $S$ is called idempotent if $\lambda\lambda = \lambda^2 = \lambda$. A fuzzy ideal $\xi$ of $S$ is called fuzzy prime if for fuzzy ideals $\lambda$ and $\mu$, $\lambda \mu \subseteq \xi$ implies $\lambda \subseteq \xi$ or $\mu \subseteq \xi$; $\xi$ is called fuzzy irreducible if $\lambda \cap \mu = \xi$ implies $\lambda = \xi$ or $\mu = \xi$. A semigroup $S$ is called regular if
\( x \in xSx \) for each \( x \in S \) \([1, 5, 6]\); \( S \) is called \textit{semisimple} if \( I = I^2 \) for each ideal \( I \) of \( S \) \([1, \text{p. 76}], [5]\). Regular semigroups form a proper subclass of semisimple semigroups.

3. Characterizations of Semisimple Semigroups

\textbf{Proposition 3.1.} The following conditions on a semigroup \( S \) are equivalent:

1. \( S \) is semisimple.

2. Each fuzzy ideal of \( S \) is idempotent.

3. For each pair of fuzzy ideals \( \lambda, \mu \) of \( S \), \( \lambda \mu = \lambda \cap \mu \).

4. The set of all fuzzy ideals of \( S \) (ordered by inclusion) is a distributive lattice under the union and product of fuzzy ideals.

\textbf{Proof.} (1) \( \iff \) (2) \( \iff \) (3): This is due to Kuroki \([5, \text{Thm. 7.2, p. 229}]\).

(3) \( \Rightarrow \) (4): The set of fuzzy ideals of \( S \), ordered by inclusion, is clearly a distributive lattice under the union and intersection of fuzzy ideals. Since, by the hypothesis, intersection of any two fuzzy ideals coincide with their product, the desired implication follows.

(4) \( \Rightarrow \) (1): Let \( \lambda \) be any fuzzy ideal of \( S \). Then \( \lambda = \text{g.l.b. of } \{\lambda, \lambda\} = \lambda \cdot \lambda = \lambda^2 \); so \( \lambda \) is idempotent. This implies (2), and hence (1).

\textbf{Lemma 3.2.} Let \( S \) be a semisimple semigroup. If \( \lambda \) is a fuzzy ideal of \( S \) with \( \lambda(x) = \alpha \), where \( x \) is an element of \( S \) and \( \alpha \in [0, 1] \), then there exists a fuzzy prime ideal \( \xi \) of \( S \) such that \( \lambda \subseteq \xi \) and \( \xi(x) = \alpha \).

\textbf{Proof.} Let \( X = \{\mu : \mu \) is a fuzzy ideal of \( S \), \( \mu(x) = \alpha \) and \( \lambda \subseteq \mu \}. \) Then \( X \neq \phi \),
since $\lambda \in X$. By Zorn's lemma, there exists a fuzzy ideal $\xi$ of $S$ which is maximal with respect to the property that $\lambda \subseteq \xi$ and $\xi(x) = \alpha$. We show that $\xi$ is a fuzzy irreducible ideal of $S$. Suppose $\xi = \delta_1 \cap \delta_2$, where $\delta_1$ and $\delta_2$ are fuzzy ideals of $S$. Hence $\xi \subseteq \delta_1$ and $\xi \subseteq \delta_2$. We claim that $\xi = \delta_1$ or $\xi = \delta_2$. Suppose $\xi \neq \delta_1$ and $\xi \neq \delta_2$. Since $\xi$ is maximal with respect to the property that $\xi(x) = \alpha$, it follows that $\delta_1(x) \neq \alpha$ and $\delta_2(x) \neq \alpha$. Hence $\alpha = \xi(x) = (\delta_1 \cap \delta_2)(x) = \inf(\delta_1(x), \delta_2(x)) \neq \alpha$, which is absurd. Hence either $\xi = \delta_1$ or $\xi = \delta_2$. We now show that $\xi$ is a fuzzy prime ideal. Let $\lambda$ and $\mu$ be fuzzy ideals of $S$ satisfying $\lambda \mu \subseteq \xi$. Hence $\lambda \mu \cup \xi = \xi$. Using the distributivity of the lattice of fuzzy ideals of $S$ which, in fact, follows from Theorem 3.1(4) as $S$ is semisimple, we have $\xi = \lambda \mu \cup \xi = (\lambda \cup \xi) \cdot (\mu \cup \xi)$. Hence by Thm. 3.1(3), we have $\xi = (\lambda \cup \xi) \cdot (\mu \cup \xi) = (\lambda \cup \xi) \cap (\mu \cup \xi)$. As we have already proved that $\xi$ is irreducible, so it follows that $\lambda \cup \xi = \xi$ or $\mu \cup \xi = \xi$. This implies that $\lambda \subseteq \xi$ or $\mu \subseteq \xi$. Hence $\xi$ is a fuzzy prime ideal.

We now prove the following characterization theorem for semisimple semigroups.

**Theorem 3.3.** The following conditions on a semigroup $S$ are equivalent:

1. $S$ is semisimple.

2. Each proper fuzzy ideal of $S$ is the intersection of fuzzy prime ideals.

**Proof.** (1) $\Rightarrow$ (2): Let $\lambda$ be a proper fuzzy ideal of $S$ and let $\{\lambda_\alpha : \alpha \in \Omega\}$ be the family of all fuzzy prime ideals of $S$ which contain $\lambda$. Obviously, $\lambda \subseteq \bigcap_{\alpha \in \Omega} \lambda_\alpha$. We now prove that $\bigcap_{\alpha \in \Omega} \lambda_\alpha \subseteq \lambda$. Let $x$ be any element of $S$. By Lemma 3.2, there exists a fuzzy prime ideal $\lambda_\beta$ (say) such that $\lambda \subseteq \lambda_\beta$ and $\lambda(x) = \lambda_\beta(x)$. Thus $\lambda_\beta \in \{\lambda_\alpha : \alpha \in \Omega\}$. Hence $\bigcap_{\alpha \in \Omega} \lambda_\alpha \subseteq \lambda_\beta$, so $\bigcap_{\alpha \in \Omega} \lambda_\alpha(x) \leq \lambda_\beta(x) = \lambda(x)$. This implies that $\bigcap_{\alpha \in \Omega} \lambda_\alpha \subseteq \lambda$. Hence
\[ \lambda = \bigcap_{\alpha \in \Omega} \lambda_\alpha. \]

(2) \Rightarrow (1): Let \( \lambda \) be any proper fuzzy ideal of \( S \). Hence by the hypothesis, we can write \( \lambda^2 = \bigcap_{\alpha \in \Omega} \lambda_\alpha \), where \( \{ \lambda_\alpha : \alpha \in \Omega \} \) is the family of all fuzzy prime ideals of \( S \) which contain \( \lambda^2 \). Hence \( \lambda^2 \subseteq \lambda_\alpha \) for all \( \alpha \in \Omega \), and since \( \lambda_\alpha \) is a fuzzy prime ideal, it follows that \( \lambda \subseteq \lambda_\alpha \) for all \( \alpha \in \Omega \). Hence \( \lambda \subseteq \bigcap_{\alpha \in \Omega} \lambda_\alpha = \lambda^2 \). Now \( \lambda^2 \subseteq \lambda \) is always true. Hence \( \lambda^2 = \lambda \), and so by Proposition 3.1(2), \( S \) is semisimple.

Finally, as an application of the above characterization, we prove the following result. First we describe some notations. Let \( \mathcal{L}_S \) denote the lattice of fuzzy ideals of \( S \), and \( \mathcal{FP}_S \) the set of all proper fuzzy prime ideals of \( S \). Moreover, for any fuzzy ideal \( \lambda \) of \( S \), \( \Theta_\lambda = \{ \mu \in \mathcal{FP}_S : \lambda \nsubseteq \mu \} \), and \( \tau(\mathcal{FP}_S) = \{ \Theta_\lambda : \lambda \in \mathcal{L}_S \} \).

**Theorem 3.4.** If \( S \) is a semisimple semigroup with a two-sided zero \( 0 \), then the set \( \tau(\mathcal{FP}_S) \) forms a topology (in the classical sense) on the set \( \mathcal{FP}_S \) and the mapping \( \lambda \mapsto \Theta_\lambda \) is an isomorphism between the lattice \( \mathcal{L}_S \) of fuzzy ideals of \( S \) and the lattice of open subsets of \( \mathcal{FP}_S \).

**Proof.** First we show that the family \( \tau(\mathcal{FP}_S) \) constitutes a topology on the set \( \mathcal{FP}_S \). Note that \( \Theta_\emptyset = \{ \mu \in \mathcal{FP}_S : \emptyset \nsubseteq \mu \} = \emptyset \), where \( \emptyset \) is the usual empty set and \( \emptyset \) denotes the fuzzy zero ideal of \( S \), defined by \( \emptyset(x) = 0 \) for all \( x \in S \). This follows since \( \emptyset \) is contained in every fuzzy (prime) ideal of \( S \). Thus \( \Theta_\emptyset \) represents the empty subset of \( \tau(\mathcal{FP}_S) \). On the other hand, \( \Theta_\rho = \{ \mu \in \mathcal{FP}_S : \rho \nsubseteq \mu \} = \mathcal{FP}_S \). This is true, since \( \mathcal{FP}_S \) is the set of proper fuzzy prime ideals of \( S \). Hence \( \mathcal{FP}_S = \Theta_\rho \) is an element of \( \tau(\mathcal{FP}_S) \). Let \( \Theta_{\lambda_1}, \Theta_{\lambda_2} \in \tau(\mathcal{FP}_S) \) with \( \lambda_1, \lambda_2 \) in \( \mathcal{L}_S \). Then \( \Theta_{\lambda_1} \cap \Theta_{\lambda_2} = \{ \mu \in \mathcal{FP}_S : \lambda_1 \nsubseteq \mu \) and \( \lambda_2 \nsubseteq \mu \} \). Since \( \mu \) is a fuzzy prime ideal of \( S \) it follows that

\[ \Theta_{\lambda_1} \cap \Theta_{\lambda_2} = \{ \mu \in \mathcal{FP}_S : \lambda_1 \cap \lambda_2 \nsubseteq \mu \} = \Theta_{\lambda_1 \cap \lambda_2}. \]
Hence \( \Theta_{\lambda_1} \cap \Theta_{\lambda_2} \in \tau(\mathcal{FP}_S) \). Now consider an arbitrary family \( \{\lambda_i : i \in I\} \) of fuzzy ideals of \( S \). We have

\[
\bigcup \Theta_{\lambda_i} = \bigcup \{\mu \in \mathcal{FP}_S : \lambda_i \not\subseteq \mu\} \\
= \{\mu \in \mathcal{FP}_S : \bigcup_{i} \lambda_i \not\subseteq \mu\} \\
= \Theta_{\bigcup_i \lambda_i}.
\]

Hence \( \bigcup \Theta_{\lambda_i} \in \tau(\mathcal{FP}_S) \). Thus \( \tau(\mathcal{FP}_S) \) forms a topology on the set \( \mathcal{FP}_S \). Let \( \phi : \mathcal{L}_S \rightarrow \tau(\mathcal{FP}_S) \) be the mapping defined by \( \lambda \mapsto \Theta_{\lambda} \). It is easily checked that \( \phi \) is a lattice homomorphism. We show that \( \phi \) is bijective. In fact, we need to prove that \( \lambda_1 = \lambda_2 \) if and only if \( \Theta_{\lambda_1} = \Theta_{\lambda_2} \) for \( \lambda_1, \lambda_2 \in \mathcal{L}_S \). Suppose that \( \Theta_{\lambda_1} = \Theta_{\lambda_2} \). If \( \lambda_1 \neq \lambda_2 \) then there exists \( x \in S \) such that \( \lambda_1(x) \neq \lambda_2(x) \). Hence either \( \lambda_1(x) > \lambda_2(x) \) or \( \lambda_2(x) > \lambda_1(x) \).

Suppose that \( \lambda_1(x) > \lambda_2(x) \). By Lemma 3.2, there exists a fuzzy prime ideal \( \mu \) of \( S \) such that \( \lambda_2 \subseteq \mu \) and \( \lambda_2(x) = \mu(x) \). Hence \( \lambda_1 \not\subseteq \mu \) because \( \lambda_1(x) > \lambda_2(x) = \mu(x) \). Hence \( \mu \in \Theta_{\lambda_1} \). Our assumption is that \( \Theta_{\lambda_1} = \Theta_{\lambda_2} \). Hence \( \mu \in \Theta_{\lambda_2} \), that is \( \lambda_2 \not\subseteq \mu \). This is a contradiction. Thus \( \Theta_{\lambda_1} = \Theta_{\lambda_2} \) implies that \( \lambda_1 = \lambda_2 \). Conversely, if \( \lambda_1 = \lambda_2 \) then, by definition, it is obvious that \( \Theta_{\lambda_1} = \Theta_{\lambda_2} \). Hence \( \lambda_1 = \lambda_2 \) if and only if \( \Theta_{\lambda_1} = \Theta_{\lambda_2} \) for \( \lambda_1, \lambda_2 \) in \( \mathcal{L}_S \).

Acknowledgement:

The first two authors would like to acknowledge the support provided by King Fahd University of Petroleum and Minerals.
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