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Abstract

A method that combines the use of finite elements and a pedictor-corrector method is presented to solve an initial boundary value problem. The problem arises in the study of drainage of liquids in porous media. Brooks and Corey retention functions are used to descibe the capillary property of the drainage cycle. An algorithm and a numerical scheme of the solution are presented.

1 Introduction

In this paper we present a mathematical and numerical treatment of certain types of unsaturated flows of liquids in porous media. The model developed here arises in many practical problems encountred in the study of drainage, irrigation, movement of polllutants through soils, etc. [see [6],[9]]. Physically speaking consider an incompressible fluid moving in an underground pipe filled with an isotropic homogeneous porous medium. The height is represented by the variable x and $x = 0$ and $x = a$ are respectively the top and the bottom of the pipe. The vertical coordinate x is positive downward. The volumetric water content distribution θ is assumed to be

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initially known as a function of position x over the interval $0 \leq x \leq a$. The mathematical formulation of this phisical problem leads to a boundary value problem in one space dimension where we need to determine the volumetric water content distribution at any time $t > 0$.

Neglecting temperature effects within the unsaturated flow region, the equation of continuity for incompressible flow reads [see [2], [10]]

$$\frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1}$$

where θ is once again the volumetric water content, and q is the water flux. θ is related to the saturation S and the porosity ϕ by the relation $\theta = S\phi$.

Under condition of slow flow, the equation of motion is given by Darcy's law [see [5], [7]]

$$q = -k[\frac{1}{\theta}\frac{\partial}{\partial x}\theta p - \rho g] \tag{2}$$

where g is the acceleration due to gravity and k the permeability. The function k is assumed to be single-valued, sufficiently smooth function of θ. p is the fluid pressure which is related to the capillary pressure p_c by the equation [see [2] or [4]]

$$p_c(\theta) = p_a - p \tag{3}$$

where p_a is the atmospheric pressure, which is assumed to be constant and which we take to be zero.

Combining (1), (2), and (3) we obtain

$$\frac{\partial \theta}{\partial t} = -\frac{\partial}{\partial x}\left[k\frac{\partial H}{\partial x} + k\rho g\right] \tag{4}$$

where $H = H(\theta)$ is a function such that $\frac{\partial H}{\partial x} = \frac{1}{\theta}\frac{\partial \theta p_c(\theta)}{\partial x}$.

We now introduce the diffusion coefficient

$$D(\theta) = -k(\theta)\frac{\partial H}{\partial \theta}. \tag{5}$$

By substituting (5) into (4) we get

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x} - \rho g k(\theta)\right] \tag{6}$$

This is the governing equation for an unsaturated flow of an incompressible fluid in a homogeneous isotropic porous medium.

As written above the functions D and k are function of θ. Due to hysteresis effects, such relationships are emperical. However, Brooks and Corey [see [3]] have found that for $S < 1$

$$D(\theta) = \alpha\theta^{2+1/\lambda}, \qquad k(\theta) = -\beta\theta^{3+2/\lambda} \tag{7}$$

where α, λ, and β are positive constants which depend on the soil properties.

We now write the problem we want to solve as follows:

Problem (P1). Given T and $g(x)$, find $\theta(x,t)$ such that

$$\begin{aligned} \theta_t &= [D(\theta)\theta_x - K(\theta)]_x & (8)\\ \text{in } Q &= \{(x,t)\,|\,0 < x < a,\ 0 < t \le T\} & \\ \theta(x,0) &= g(x), \qquad 0 \le x \le a & (9)\\ \theta_x(0,t) &= 0, \qquad 0 < t \le T & (10)\\ \theta(a,t) &= 0, \qquad 0 < t \le T & (11) \end{aligned}$$

Here $()_t = \frac{\partial}{\partial t}$, $()_x = \frac{\partial}{\partial x}$, and $K(\theta) = \rho g k(\theta)$. It is also assumed that $g(x)$ is continuously differentiable.

The governing equation of problem (P1) is nonlinear because of the presence of the dependent variables $D(\theta)$ and $K(\theta)$. The aim of this paper is to present a simple numerical scheme that combines the use of finite elements and a predictor-corrector method, for finding the approximate solution.

2 Variational formulation

We now introduce a variational formulation of problem (P1). Denote by H^1 the Sobolev space.

For t fixed we define the space

$$\dot{H}^1 = \{w : w \in H^1(0,a) \text{ and } w(0) = 0\}.$$

We now choose an arbitary function w in $\dot{H}^1(0,a)$ and multiply equation (8) by this function and integrate over $(0,a)$ to get

$$\int_0^a \theta_t w dx = \int_0^a D(\theta)\theta_x w_x dx - \int_0^a K(\theta) w_x dx \tag{12}$$

Note that we used integration by part and condition (11) to obtain the right-hand side of (12). Using the $L_2(0,a)$ scalar product notation,

$$\int_0^a uvdx = (u,v)$$

the variational formulation of problem (P1) now reads:

Problem (P2): Find $\theta(x,t)$ such that

$$(\theta_t, w) = (D(\theta)\theta_x, w_x) - (K(\theta), w_x),\ \forall w \in \dot{H}^1 \tag{13}$$

with the initial condition:

$$\theta(x,0) = g(x).$$

3 Finite element discretization

In this section we are going to introduce a method of approximating problem (P1) based on the variational formulation (P2) [see [1],[9]].

Let us construct the following space of finite elements: given a subdivision of the interval $[0, a]$ in N equal subintervals of lenght $h = 1/N$, with $x_i = ih$ $(i = 0, 1, \ldots, N)$ the subdivision points; let us define

$$\dot{S}_h = \{v_h | v_h \in S_h \text{ and } v_h(0) = 0\}$$

where S_h is the space of continuous functions which are piecewise polynomial, defined in $[0, a]$ of degree less than an integer r. As a consequence $\dot{S}_h \subset \dot{H}^1$.

The discretized problem now reads as follows:
Problem (P3): Find $\theta_h(x, t)$ such that

$$(\theta_{h,t}, w) = (D(\theta_h)\theta_{h,x}, w_x) - (K(\theta_h), w_x), \ \forall w \in \dot{H}^1 \tag{14}$$

with the initial condition

$$\theta_h(x, 0) = P_h g(x), \quad 0 \leq x \leq a \tag{15}$$

where P_h is an appropriate projection into $\dot{S}_h$.

4 Numerical scheme

At first, we let $\dot{S}_h$ to be the space of continuous functions which are piecewise polynomial of third degree [see [12]]. Then for a fixed time interval $\Delta t =$

T/M, we approximate $\theta_{h,t}$ in (14) at $t = (n+1)\Delta t$ with the backward difference

$$\theta_{h,t} = \frac{\theta_h^{n+1} - \theta_h^n}{\Delta t} \tag{16}$$

where $\theta_h^n = \theta_h(x, n\Delta t)$. From (16), (14) takes the form

$$\frac{1}{\Delta t}\left[(\theta_h^{n+1}, w) - (\theta_h^n, w)\right] = (D(\theta_h^{n+1})\theta_{h,x}^n, w_x) - (K(\theta_h^{n+1}), w_x)$$

i.e.,

$$(\theta_h^{n+1}, w) - \Delta t(D(\theta_h^{n+1})\theta_{h,x}^n, w_x) + \Delta t(K(\theta_h^{n+1}), w_x) = (\theta_h^n, w) \tag{17}$$

We now seek an approximation solution of the discretized problem in the form

$$\theta_h^n(x) = \sum_{j=0}^{N} \alpha_j^n \psi_j(x) \tag{18}$$

where $\psi_j(x)$ are the basis functions of $\dot{S}_h$ and $\alpha_0^n, \ldots, \alpha_N^n$ are coefficients to be determined.

By substituting (18) into (17) and taking $w = \psi_i(x)$, $(i = 0, 1, \ldots, N)$, we arrive at the following system of equations for the determination of the coefficients α_j:

$$\sum_{j=0}^{N} \int_0^a \left[\alpha_j^{n+1}\psi_i(x)\psi_j(x) - \Delta t D(\theta_h^{n+1})\theta_{h,x}^n \psi_i'(x)\right.$$
$$\left. + \Delta t K(\theta_h^{n+1})\psi_i'(x)\right] dx = \int_0^a \theta_h^n(x)\psi_i(x) \tag{19}$$

$n = 0, 1, \ldots, M$ and $i, j = 0, 1, \ldots, N$.

The presence of the terms $D(\theta_h^{n+1})$ and $K(\theta_h^{n+1})$ in equation (19) makes it nonlinear. To get around these nonlinear terms, we will use a predictor-corrector method which consists first on predecting the terms $D(\theta_h^{n+1})$ and

$K(\theta_h^{n+1})$ and then with these terms known we will solve the linear system (19) for the unknowns $\alpha_0^{n+1}, \alpha_1^{n+1}, \ldots, \alpha_N^{n+1}$. The new θ_h^{n+1} will be used again to correct the nonlinear terms $D(\theta_h^{n+1})$ and $K(\theta_h^{n+1})$. The process continues until the changes in θ_h^{n+1} is sufficiently small.

We now describe the predictor-corrector method as follows:
(7) and (6) together yield

$$\theta_{h,t} = \mu_1(\theta_h^{3+1/\lambda})_{xx} + \rho g(\theta_h^{3+2/\lambda})_x. \tag{20}$$

where $\mu_1 = \alpha\lambda/(3\lambda+1)$ and $()_{xx} = \frac{\partial^2}{\partial x^2}$.

(20) has a similar form to the governing equation of a adiabatic flow of gases in a porous medium [see [11]].
By replacing the lefthand side of (20) by (15) and evaluating the terms θ_{xx} and θ_x at the previous time step $t = n\Delta t$, we get

$$\theta_h^{n+1} = \Delta t\left\{\mu_1[(\theta_h^n)^{3+1/\lambda}]_{xx} + \rho g[(\theta_h^n)^{3+2/\lambda}]_x\right\} - \theta_h^n \tag{21}$$

using the chain rule, we rewrite (21) as

$$\theta_h^{n+1} = \Delta t\left[\alpha\theta_{h,xx}^n(\theta_h^n)^{2+1/\lambda} + \mu_2(\theta_h^n)^{1+1/\lambda}(\theta_{h,x}^n)^2 + \mu_3(\theta_h^n)^{2+2/\lambda}\theta_{h,x}^n\right] - \theta_h^n \tag{22}$$

where $\mu_2 = \alpha(2+1/\lambda)$ and $\mu_3 = \rho g(3+2/\lambda)$.
Using (7), the righthand side of (22), may be written as

$$\theta_h^{n+1} = \Delta t\left\{\alpha\left[\sum_{j=0}^{N}\alpha_j^n\psi_j''(x)\right]\left[\sum_{j=0}^{N}\alpha_j^n\psi_j(x)\right]^{2+1/\lambda} + \mu_2\left[\sum_{j=0}^{N}\alpha_j^n\psi_j(x)\right]^{1+1/\lambda}\cdot\right.$$
$$\left.\left[\sum_{j=0}^{N}\alpha_j^n\psi_j'(x)\right]^2 + \mu_3\left[\sum_{j=0}^{N}\alpha_j^n\psi_j(x)\right]^{1+1/\lambda}\left[\sum_{j=0}^{N}\alpha_j^n\psi_j'(x)\right]\right\} - \sum_{j=0}^{N}\alpha_j^n\psi_j(x)$$

for $n = 1, 2, \ldots, M$

and

$$\theta_h^0 = \Delta t \left\{ \alpha g''(x)[g(x)]^{2+1/\lambda} + \mu_2[g(x)]^{1+1/\lambda}[g'(x)]^2 + \mu_3[g(x)]^{2+2/\lambda}[g'(x)] \right\} - g(x), \quad \text{for } n = 0 \tag{23}$$

Having obtained θ_h^{n+1} form (22), we use (7) to get $D(\theta_h^{n+1})$ and $K(\theta_h^{n+1})$.

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