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1. Introduction

In this work we refine a deep theorem of Jensen and Zimmermann–Huisgen [5] characterizing commutative artinian rings with infinitely many isomorphism classes of indecomposable modules via the algebraic compactness of certain reduced direct products. This refinement is performed along two main directions: direct sums and "thinned" direct products of [5] are replaced by the more general filter sums, and the cardinality of the local ring factors of the artinian ring is brought into play. Although several of our arguments, and particularly the construction in section 4, are adaptations, in the context of filters, of the techniques devised in [5], we introduce the notion of summable families of submodules, in order to extend the well–known and classical Chase's Lemma on direct products ([1] and [2]) to our present framework. We also had to establish a number of results on filters and filter sums, which enabled us to derive several other characterizations of the rings cited above. This article is organized as follows. In section 2, the basic concepts of purity, algebraic compactness and filters are given together with some needed results. Section 3 is devoted to the generalization of Chase's Lemma; some consequences of this are also included. The main result characterizing, in terms of filters, the artinian rings described above, is established in the fourth section along with a corollary that extends [5, Theorem 1]. Throughout this work, all rings are associative with 1 and all modules are left unital. A theory of ordinals is assumed where each cardinal is an initial ordinal, and each ordinal is the set of all preceding ordinals. Given a cardinal α, α^+ denotes the infinite successor cardinal of α and, for any set S, $|S|$ is the cardinality of S. Finite cardinals are denoted by $\aleph_{-1}$.

2. Notations and Preliminary Results

We start with the usual definitions of purity and compactness.

Definitions. Let M be an R–module and let α be an infinite cardinal.

1. A system of linear equations over M is α–solvable in M if every subsystem of it consisting of less than α equations is solvable in M. Hence, a system is finitely solvable precisely when it is $\aleph_0$–solvable.

2. M is α–compact if every $\aleph_0$–solvable system of equations over M is α^+–solvable.

3. M is algebraically compact if it is β–compact for all cardinals β.

4. A submodule N of M is α–pure in M if every system of equations over N which is α–solvable in M is also α–solvable in N. In particular, $\aleph_0$–pure means pure in the usual sense (Cohn [3]).

Remarks.

1. It can be shown (see for example [4]), that M is algebraically compact if and only if it is $(|R| + \aleph_0)$–compact.

2. Algebraic compactness coincides with pure–injectivity for modules [8]. Furthermore, every module category contains enough pure–injectives, in the sense that every module can be embedded in a pure–injective module as a pure submodule.

In fact, by [4], every module M has a pure–injective envelope $P(M)$, that is, a module satisfying: (a) M is pure in $P(M)$, (b) $P(M)$ is pure–injective and (c) the only submodule S of $P(M)$ such that $S \cap M = 0$ and $(M+S)/S$ pure in $P(M)/S$ is the zero module.

We shall also need the following definitions and results on filters.

Definitions. Let I be a non–empty set and let α be an infinite cardinal.

1. A subset φ of $\mathcal{P}(I)$ is an α–complete filter, or simply an α–filter on I, if (a) $I \in \varphi$, (b) φ is closed under taking intersections of less than α members and (c) φ is closed under taking supersets. An $\aleph_0$–filter is called a filter.

2. A filter φ on I is proper if $\phi \notin \varphi$. Note that $\mathcal{P}(I)$ is the only non–proper filter on I.

3. A filter φ on I is said to be non–principal if $\bigcap_{X \in \varphi} X \notin \varphi$, and is principal (generated by $\bigcap_{X \in \varphi} X$) otherwise.

4. A filter φ on I is free if $\bigcap_{X \in \varphi} X = \phi$.

Notation. Let φ be a filter on a set I, let α be an infinite cardinal or $\aleph_{-1}$ and let $\{M_i\}_{i \in I}$ be a family of R–modules.

1. The filter arising from φ by adding α intersections is denoted by φ_α. It is clear that φ_α is an α^+–filter on I, and that, if φ is α–complete, then $\varphi_\alpha = \varphi_{cf(\alpha)}$.

2. The set $I \setminus \bigcap_{X \in \varphi} X$ is denoted by I_φ. It is easy to see that φ is free if and only if $I_\varphi \in \varphi$.

3. The set $\{X \subseteq I : |I \backslash X| < \alpha\}$ is a filter on I denoted by $I(\alpha)$, and which is α–complete if α is regular. Note that when $I = \alpha$, this filter is simply the generalized Fréchet filter on α.

4. For each $m \in \prod_{i\in I} M_i$, define $z(m) = \{i \in I; m(i) = 0\}$ and denote the R–module $\left\{m \in \prod_{i\in I} M_i : z(m) \in \varphi\right\}$ by $\sum_\varphi M_i$. It is easy to see that the filter sum $\sum_\varphi M_i$ is an α–pure submodule of $\prod_{i\in I} M_i$ whenever φ is an α–filter on I. Moreover $\sum_{I(\aleph_0)} M_i = \bigoplus_{i\in I} M_i$ and $\sum_{\mathcal{P}(I)} M_i = \prod_{i\in I} M_i$, (note that $\mathcal{P}(I)$ is not a proper filter).

5. Let ψ be another filter on I with $\varphi \subseteq \psi$. It is clear that $\sum_\varphi M_i$ is a pure submodule of $\sum_\psi M_i$. The filter quotient $\sum_\psi M_i / \sum_\varphi M_i$ generalizes the concept of reduced product $\prod_{i\in I} M_i / \sum_\varphi M_i$.

The following results are required in section 4.

Proposition 1. *Let φ be a filter on a set I with $|I| = \alpha$ and let $\{M_i\}_{i\in I}$ be a family of R–modules. Then*

(i) φ is non–principal if and only if $\varphi \subset \varphi_\alpha$. In particular φ_α is principal (generated by $I \backslash I_\varphi$).

(ii) For each $m \in \sum_\varphi M$, $I \backslash I_\varphi$ is contained in $z(m)$ and $\sum_{\varphi_\alpha} M_i = \prod_{i\in I} M'_i$, where

$$M'_i = \begin{cases} M_i & \textit{if } i \in I_\varphi \\ 0 & \textit{otherwise} \end{cases}$$

Proof.

(i) If φ is principal, then clearly $\varphi = \varphi_\alpha$, and if φ is non–principal, then $\bigcap_{X\in\varphi} X \notin \varphi$. So $I_\varphi = \{x_j\}_{j<\alpha'}$, for some $x_j \in I$ with $\alpha' \leq \alpha$, i.e. $\bigcap_{X\in\varphi} X = \bigcap_{j<\alpha'} I \backslash \{x_j\}$. Now, for each $j < \alpha'$, $x_j \notin \bigcap_{X\in\varphi} X$, so that $X \subseteq I\backslash\{x_j\}$ for some $X \in \varphi$. This implies that $I\backslash\{x_j\} \in \varphi$ and therefore $\bigcap_{X\in\varphi} X \in \varphi_\alpha$, so that $\varphi \subset \varphi_\alpha$. Since $(\varphi_\alpha)_\alpha = \varphi_\alpha$, it follows that φ_α is principal.

(ii) Next, let $m \in \sum_\varphi M_i$. Then $z(m) \in \varphi$ and so, $I\backslash I_\varphi = \bigcap_{X\in\varphi} X \subseteq z(m)$. Also, for each $m' \in \sum_{\varphi_\alpha} M_i$, $z(m') \supseteq \bigcap_{X\in\varphi} X$ which means that $\sum_{\varphi_\alpha} M_i \subseteq \prod_{i\in I} M_i'$. Conversely, each $m'' \in \prod_{i\in I} M_i'$ satisfies $z(m'') \supseteq \bigcap_{X\in\varphi} X$. Since $\bigcap_{X\in\varphi} X = I\backslash I_\varphi \in \varphi_\alpha$, it follows that $z(m'') \in \varphi_\alpha$. This proves that $\prod_{i\in I} M_i' \subseteq \sum_{\varphi_\alpha} M_i$.

Proposition 2. [7]. *Let $0 \to A \to B \to C \to 0$ be a pure-exact sequence of R-modules and let α be an infinite cardinal. If both A and C are α-compact, then so too is B.*

Proposition 3. *Let φ be an α-filter on a set I, where $|I| = \alpha$, and let $\{M_i\}_{i\in I}$ be a family of α-compact R-modules. Then $\prod_{i\in I} M_i/\sum_\varphi M_i$ is α-compact.*

Proof. Consider the pure-exact sequence

$$0 \to \textstyle\sum_{\varphi_\alpha} M_i/\sum_\varphi M_i \to \prod_{i\in I} M_i/\sum_\varphi M_i \to \prod_{i\in I} M_i/\sum_{\varphi_\alpha} M_i \to 0.$$

The first term is α-compact by [7, Corollary 3], and the third is α-compact because φ_α is principal and $\prod_{i\in I} M_i/\sum_{\varphi_\alpha} M_i \cong \prod_{i\in I\backslash I_\varphi} M_i$. By Proposition 2, $\prod_{i\in I} M_i/\sum_\varphi M_i$ is α-compact.

Remark. The same conclusion in Proposition 3 is reached if the α-compactness of each M_i is replaced by the weaker conditions: $\{i \in I : M_i$ is β-compact for all $\beta < \alpha\} \in \varphi$ and $I\backslash I_\varphi \subseteq \{i : M_i$ is α-compact $\}$. A particular case of the next result was used in [5]. Its proof is straightforward.

Proposition 4. *Let $f : B \to C$ be an R-homomorphism of modules and let A be a* *submodule of B such that $A \cap \ker f = 0$ and $f(A)$ is α–pure in C for some cardinal α. Then A is isomorphic to an α–pure submodule of $B/\ker f$.*

Proposition 5. *Let φ be a filter on a set I and let α be an infinite cardinal such that $|I_\varphi| = |I|$ and $\varphi_\alpha \subseteq I_\varphi(|I|)$. Then φ is non–principal and $\alpha < |I|$.*

Proof. We have $|I \backslash (I \backslash I_\varphi)| = |I_\varphi| \not< |I|$, hence $I \backslash I_\varphi \notin I_\varphi(|I|)$, and so $I \backslash I_\varphi \notin \varphi$. This means φ is non–principal. Next, by Proposition 1, φ_α is principal, so $\bigcap_{X \in \varphi} X = I \backslash I_\varphi \in \varphi_\alpha$. On the other hand, $I \backslash I_\varphi \notin \varphi_\alpha$, by first part. Hence $\varphi_\alpha \subset \varphi_{|I|}$ and therefore $\alpha < |I|$.

3. Chase's Lemma Generalized

The main result in this section (Proposition 6) generalizes Chase's Lemma [2] in three directions. First we introduce the concept of summable families of submodules, we also consider the more general filter sums instead of restricting our study to direct sums or "thinned" direct products as in [5], and last, this will be done for arbitrary cardinals.

Definition. A family $\{A_i\}_{i \in I}$ of submodules of an R–module A is said to be *summable* in A if the sum function $s : \bigoplus_{i \in I} A_i \to A$, $s((a_i)_{i \in I}) = \sum_{i \in I} a_i$, can be extended to an R–homomorphism $\sigma : \prod_{i \in I} A_i \to A$.

Remarks.

1. It is clear that every finite family of submodules is summable.

2. If A is algebraically compact, then every family of submodules of A is summable. In the same vein, if $\bigoplus_{i \in I} A_i$ is a direct summand of $\prod_{i \in I} A_i$, then $\{A_i\}_{i \in I}$ is summable in any module containing the A_i as submodules. In particular this holds if $\bigoplus_{i \in I} A_i$

is algebraically compact. Note also that $\{A_i\}_{i\in I}$ is summable in $\prod_{i\in I} A_i$ if and only if $\bigoplus A_i$ is a direct summand of $\prod_{i\in I} A_i$.

3. Suppose that every family of κ copies of A, where κ is a cardinal, is summable (for example, if κ is finite), and that we have κ disjoint families of summable families $\{\{A_i\}_{i\in I_\tau}\}_{\tau<\kappa}$ of submodules of A. Then $\{A_i\}_{i\in \bigcup_{\tau<\kappa} I_\tau}$ is summable in A. It follows in particular, that if all families of size less than a singular cardinal λ of submodules of A are summable, so too are families of size λ.

The proof of the following result is adapted from the one given by Zimmermann–Huisgen in [5]. The new concepts we add here are those of summable families of modules and filter sums. In the following a family $\{F_i\}_{i\in I}$ of p–functors is said to be κ–filtered for some infinite cardinal κ if for each subset J of I with $|J| < \kappa$, there exists $i_0 \in I$ such that $F_{i_0} \subseteq \bigcap_{i\in J} F_i$. For a detailed discussion on p–functors, see [10] (note that p–functors commute with filter sums). If $\{B_h\}_{h\in H}$ is a family of R–modules and $H' \subseteq H$, the map $\pi_{H'}$ denotes the canonical projection $\prod_{h\in H} B_h \to \prod_{h\in H'} B_h$. If H' is a singleton $\{h\}$, $\pi_{H'}$ is written π_h.

Proposition 6. *Let $\{A_i\}_{i\in I}$ be a summable family of submodules of an R–module A, such that $|I| = \kappa$ is a an infinite cardinal, and let $\{B_h\}_{h\in H}$ be a family of R–modules. Suppose also that φ is a filter on H with $\varphi \subseteq H(\kappa)$ and that $f : A \to \sum_\varphi B_h$ is an R–homomorphism. Then, for any κ–filtered family $\{F_i\}_{i\in I}$ of p–functors, there exist $i_0 \in I$ and $H_0 \in \varphi$ such that $\pi_{H_0} f(F_{i_0} A_{i_0}) \subseteq \bigcap_{i\in I} \left(F_i \prod_{h\in H_0} B_h \right)$.*

Proof. Without loss of generality, we may assume that I is the well–ordered set κ. Assume that the conclusion of the theorem does not hold. We use transfinite induction to establish the existence of sequences $\{\alpha_i\}_{i<\kappa}$ in κ, $\{h_i\}_{i<\kappa}$ in H and elements $x_i \in F_{\alpha_i} A_{\alpha_i}$ such that the following four conditions hold:

(i) $F_{\alpha_i} < F_{\alpha_j}$ if $i < j < \kappa$

(ii) $h_i \neq h_j$ if $i \neq j$

(iii) $\pi_{h_i} f(x_i) \notin F_{\alpha_{i+1}} B_{h_i}$

(iv) $\pi_{h_i} f\sigma((x_j)_{j<i}) = 0$, where $\sigma : \prod_{i\in I} A_i \to A$ is the extension of the sum $s : \bigoplus_{i\in I} A_i \to A$.

Choose any $\alpha_0 < \kappa$. By our assumption, there exist $x_0 \in F_{\alpha_0} A_{\alpha_0}$, $\alpha'_0 < \kappa$ such that $f(x_0) \notin F_{\alpha'_0} \sum_\varphi B_h$. Let $\alpha_1 < \kappa$ be such that $F_{\alpha_1} \subseteq F_{\alpha_0} \cap F_{\alpha'_0}$ (this is possible because $\{F_i\}_{i<\kappa}$ is filtered). Then there exists $h_0 \in H$ with $\pi_{h_0} f(x_0) \notin F_{\alpha_1} B_{h_0}$. Suppose next that for some ordinal $\tau < \kappa$ we have obtained ordinals $\alpha_i, \alpha_{i+1} < \kappa$, elements $x_i \in F_{\alpha_i} A_{\alpha_i}$ and elements $h_i \in H$ for all $i < \tau$ satisfying conditions (i) – (iv) above. Assume first that τ is a limit ordinal. Since $|\{\alpha_i : i < \tau\}| < \kappa$, there exists $\alpha_\tau < \kappa$ such that $F_{\alpha_\tau} \supseteq \bigcap_{i<\tau} F_{\alpha_i}$. If we put $a = \sigma((x_i)_{i<\tau})$, then $a \in A$ and $z(f(a)) \in \varphi$. So, for some $h_\tau \in z(f(a))$, $\pi_{h_\tau} f(F_{\alpha_\tau} A_{\alpha_\tau}) \nsubseteq \bigcap_{i\in I} F_i B_{h_\tau}$. Thus, there exist $x_\tau \in F_{\alpha_\tau} A_{\alpha_\tau}$ and an ordinal $\alpha'_\tau < \kappa$ such that $\pi_{h_\tau} f(x_\tau) \notin F_{\alpha'_\tau} B_{h_\tau}$. Let $\alpha_{\tau+1} < \kappa$ be an ordinal such that $F_{\alpha_{\tau+1}} \subseteq F_{\alpha_\tau} \cap F_{\alpha'_\tau}$. Then for all $i \leq \tau$, we have $F\alpha_i < F\alpha_j$ when $i < j$, $\pi_{h_i} f(x_i) \notin F_{\alpha_{i+1}} B_{h_i}$ and $\pi_{h_i} f\sigma((x_j)_{j<i}) = 0$. Also, for each $i < \tau$, $h_i \notin z(f(a))$. For, write $a = \sigma((x_j)_{j<i}) + f(x_i) + f\sigma((x_j)_{i<j<\tau})$ so that $\pi_{h_i} f(a) = \pi_{h_i} f\sigma((x_j)_{j<i}) + \pi_{h_i} f(x_i) + \pi_{h_i} f\sigma((x_j)_{i<j<\tau})$. But the first term in the above sum is 0 by condition (iv), the second term is not in $F_{\alpha_{i+1}} B_{h_i}$, whereas the last term is in $F_{\alpha_{i+1}} B_{h_i}$, because $(x_j)_{i<j<\tau} \in F_{\alpha_{i+1}} \prod_{i<j<\tau} A_{\alpha_j}$ so that $\sigma((x_j)_{i<j<\tau}) \in F_{\alpha_{i+1}} A$. Consequently, $\pi_{h_i} f(a) \neq 0$, i.e. $h_i \neq h_\tau$ for all $i < \tau$. The case when τ is a successor ordinal holds verbatim. Finally, let $x = \sigma((x_i)_{i<\kappa})$. Then a similar argument as above yields that $\pi_{h_i} f(x) \neq 0$ for all $i < \kappa$, so that $z(f(x)) \subseteq H \backslash \{h_i\}_{i<\kappa}$. Hence $H \backslash \{h_i\}_{i<\kappa} \in \varphi$ contradicting our assumption on φ.

The first consequence of the theorem is the following generalization of [5, Lemma

5].

Corollary 1. *Let κ be an infinite cardinal, $\{A_i\}_{i\in I}$ and $\{B_h\}_{h\in H}$ be two families of R–modules with $|I| \geq \kappa$, φ be a filter on H such that $\varphi \subseteq H(\kappa)$ and let $f : \prod_{i\in I} A_i \to \sum_\varphi B_h$ be a an R–homomorphism. Then, for any κ–filtered family $\{F_\tau\}_{\tau<\kappa}$ of p–functors, there exist an ordinal $\tau_0 < \kappa$, a subset I_0 of I with $|I_0| = |I|$ and a member H_0 of φ such that*

$$\pi_{H_0} f\left(F_{\tau_0} \prod_{i\in I_0} A_i\right) \subseteq \bigcap_{\tau<k}\left(F_\tau \prod_{h\in H_0} B_h\right).$$

Proof. Clearly $|I| = \kappa|I|$, so there exists a family $\{I_\sigma\}_{\sigma<\kappa}$ of disjoint subsets of I such that $I = \bigcup_{\sigma<\kappa} I_\sigma$ and each $|I_\sigma| = |I|$. Consequently, $\prod_{i\in I} A_i \cong \prod_{\sigma<\kappa} M_\sigma$, where $M_\sigma = \prod_{i\in I_\sigma} A_i$. Since the I_σ are disjoint, the family $\{M_\sigma\}_{\sigma<\kappa}$ is summable in $\prod_{i\in I} A_i$, and therefore, by Proposition 6, there exist an ordinal $\sigma_0 < \kappa$ and $H_0 \in \varphi$ such that $\pi_{H_0} f(F_{\sigma_0} M_{\sigma_0}) \subseteq \bigcap_{\tau<\kappa} F_\tau \prod_{h\in H_0} B_h$.

The next result extends [2, Theorem 1.2] to the case of filter sums.

Corollary 2. *Let $\{A_i\}_{i\in N}$, $\{B_h\}_{h\in H}$ be families of R–modules, φ be a filter on H whose members are cofinite and $f : \prod_{i\in N} A_i \to \sum_\varphi B_h$ be an R–homomorphism. Then for any descending chain $\{a_i\}_{i\in N}$ of finitely generated right ideals of R, there exist $i_0 \in N$, a finite subset I_0 of N and $H_0 \in \varphi$ such that*

$$f\left(a_{i_0} \prod_{i\in N\setminus I_0} A_i\right) \subseteq \bigoplus_{h\in H\setminus H_0} B_h + \bigcap_{i\in N}\left(a_i \textstyle\sum_\varphi B_h\right).$$

Proof. Put $M_n = \prod_{i\geq n} A_i$, so that $\{M_n\}_{n\in N}$ is a summable family in $\prod_{i\in I} A_i$. Then, with $\kappa = \aleph_0$ in Proposition 6, there exist $i_0 \in N$ and $H_0 \in \varphi$ such that

$$f(a_{i_0} M_{i_0}) \subseteq \bigoplus_{h\in H\setminus H_0} B_h + \bigcap_{i\in N}\left(a_i \textstyle\sum_\varphi B_h\right).$$

Corollary 3. *Let A be an algebraically compact R–module and let $f : A \to \sum_{\varphi} B_h$ be an R–homomorphism, where $\{B_h\}_{h\in H}$ are R–modules and φ a filter on H with $\varphi \subseteq H(\aleph_0)$. Then for any descending chain $\{F_n\}_{n\in N}$ of p–functors, there exist $n_0 \in N$ and $H_0 \in \varphi$ such that* $\pi_{H_0} f(F_{n_0} A) \subseteq \bigcap_{n\in N} \left(F_n \prod_{h\in H_0} B_h \right)$.

Proof. This follows readily from Proposition 6, as the family $\{A_n\}_{n\in N}$, where each $A_n = A$, is summable in A.

4. ℵ–Compactness over Commutative Artinian Rings

In this section, we give a number of characterizations of commutative artinian rings of infinite representation type, involving the sizes of their local summands. We extend several results in [5] in the context of filter sums and, as a by–product, we construct over commutative local artinan rings, for each regular cardinal κ, modules which are λ–compact for all $\lambda < \kappa$ but not κ–compact. Some of the steps described in the proofs appearing in [5] are included here for the sake of completeness.

Let φ be a filter on a set H and ψ a proper filter on a set I, and let $\{M_h\}_{h\in H}$ be a family of R-modules. Put $M = \sum_{\varphi} M_h$, $N_h = M_h^I / \sum_{\psi} M_h$ and let $f : M^I / \sum_{\psi} M \to \prod_{h\in H} N_h$ be the R–homomorphism given by

$$f(((m_{hi})_{h\in H})_{i\in I} + \textstyle\sum_{\psi} M) = ((m_{hi})_{i\in I} + \textstyle\sum_{\psi} M_h)_{h\in H}$$

where $m_{hi} \in M_h$, and let $g : M \to M^I / \sum_{\psi} M$ be the (diagonal) homomorphism given by $g(m) = (\ldots, m, m, m, \ldots) + \sum_{\psi} M$. f is a well–defined map since $\{i \in I : (m_{hi})_{h\in H} = 0\} = \bigcap_{h\in H} \{i \in I : m_{hi} = 0\}$, so that if $((m_{hi})_{h\in H})_{i\in I} \in \sum_{\psi} M$, then $(m_{hi})_{i\in I} \in \sum_{\psi} M_h$ for each $h \in H$. Also, as ψ is a proper filter, g is easily shown to be a pure monomorphism. In fact, if ψ is α–complete for some cardinal α, then g is even α–pure. Next, let $((m_{hi})_{h\in H})_{i\in I} \in (\sum_{\varphi} M_h)^I$. Then, for each $i \in I$, $z((m_{hi})_{h\in H}) \in \varphi$, so that the set $S = \bigcap_{i \in I} z((m_{hi})_{h \in H}) \in \varphi_{|I|}$. It is clear that $(m_{hi})_{i \in I} = 0$ for all $h \in S$, and therefore Im $f \subseteq \sum_{\varphi_{|I|}} N_h$. We now claim that M is isomorphic to a pure submodule of $(M^I/\sum_\psi M)/ker f$. To prove this, we first show that fg is a monomorphism. Let $m = (m_h)_{h \in H} \in \sum_\varphi M_h$ with $fg(m) = 0$; then, for each $h \in H$, $(\ldots, m_h, m_h, \ldots) \in \sum_\psi M_h$, i.e. $\{i \in I : m_h = 0\} \in \psi$. As ψ is proper, this implies that each $m_h = 0$, so that $m = 0$, as required. Next, as g is a pure monomorphism, M is isomorphic to the pure submodule $g(M)$ of $M^I/\sum_\psi M$; also, $g(M) \cap ker f = 0$ by the previous claim, and $fg(M)$ is pure in Im f (since $\sum_\varphi M_h$ is pure in $\sum_\varphi N_h$, which in turn, is pure in $\prod_{h \in H} N_h$). Consequently, by Proposition 4, M is isomorphic to a pure submodule of $(M^I/\sum_\psi M)/ker f$. We summarize this in:

Proposition 7. *With the above notation:*

(i) Im $f \subseteq \sum_{\phi_{|I|}} N_h$

(ii) M *is isomorphic to a pure submodule of* $M^I/\sum_\psi M$ *and of* $(M^I/\sum_\psi M)/ker f$*, respectively.*

Remark. If we suppose that φ is α–complete on H and ψ is β–complete on I for some infinite cardinals α, β, we can sharpen (ii) above to

(ii)$'$ M is isomorphic to a $\min(\alpha, \beta)$–pure submodule of $(M^I/\sum_\psi M)/ker f$.

Notation.

Let R be a commutative local ring which is not a principal ideal ring (i.e. of infinite representation type). Suppose further that the radical J of R has a minimal generating set $\{u, v\}$ and that $|R/J| \geq \kappa$ for some infinite cardinal κ. Let H be a subset of $R \backslash J$ whose images in R/J are distinct, and for each $h \in H$, let $r_h = u - hv$ and $M_h = R/(r_h)$.

Proposition 8. *With the above notation, for any filter* φ *on* H*, the pure–injective* *envelope of* $\sum_\varphi M_h$ *is* $\prod_{h\in H_\varphi} M_h$. *In particular, if* φ *is non-principal, then* $\sum_\varphi M_h$ *is not algebraically compact.*

Proof. Set $P = \prod_{h\in H_\varphi} M_h$. Since R is algebraically compact (commutative artinian rings are always algebraically compact), it follows that each M_h is algebraically compact, as a finitely presented module, and so P is pure–injective. Clearly $\sum_\varphi M_h$ is pure in P and therefore P contains, necessarily as a direct summand, a pure–injective envelope Q of $\sum_\varphi M_h$, with $P = Q \oplus S$, say, for some submodule S of P. We first show that $JP \subseteq Q$. We have $JP = \prod_{h\in H_\varphi} (JM_h)$, since J is finitely generated. Let $m = (m_h)_{h\in H} \in \prod_{h\in H_\varphi} JM_h$. For any $h \in H_\varphi$, there exist $a, b \in R$, $m_h = au+bv+(r_h) = ar_h+c_hv+(r_h) = c_hv+(r_h)$, where $c_h = b+ah$. Hence every element m of JP is of the form $(c_hv+(r_h))_{h\in H_\varphi}$, for some $c_h \in R$. Denote by q_h $(h \in H_\varphi)$, the canonical composition $P \to M_h \to P$, so that $m = \sum_{h\in H} q_h(m)$. Next consider the system of equations with unknowns x, y_h $(h \in H_\varphi)$

$$x + r_hy_h = q_h(m) \qquad (h \in H_\varphi) \tag{1}$$

For each $h \in H_\varphi$, $H\backslash\{h\} \in \varphi$ (otherwise h would be in each element of φ, i.e. $h \in H\backslash H_\varphi$). Hence, as $H\backslash\{h\} \subseteq z(q_k(m))$, it follows that $q_k(m) \in \sum_\varphi M_h$. Also, if $(1)'$ is the subsystem obtained from (1) by restricting h to a finite subset $\{h_1, \ldots, h_n\}$ of H_φ, if $\mu(c_h + (r_h))_{h\in H_\varphi} \in \prod_{h\in H_\varphi} M_h$, and if w_{ij} $(1 \le i, \ \ j \le n, \ \ i \ne j)$ are elements of R with w_{ij} $(h_i - h_j) = 1$ (recall that $h_i - h_j \notin J$ and so $h_i - h_j$ is a unit, when $i \ne j$), then $m = \mu v$ and $x = \sum_{j=1}^{n} q_{h_j}(m)$, $\ y_{h_i} = \sum_{\substack{j=1\\ j\ne i}}^{n} w_{ij}q_{h_j}(\mu)$ $\quad (1 \le i \le n)$, is easily seen to be a solution of $(1)'$ in $\sum_\varphi M_h$. This means that (1) is finitely solvable, and as Q is pure–injective, it has a solution x_0, $(y_h)_{h\in H_\varphi}$ in Q. It is clear that for each $h \in H_\varphi$, $\ x_0(h) = q_h(m)$, so that $x_0 = \sum_{h\in H_\varphi} x_0(h) = \sum_{h\in H_\varphi} q_h(m) = m$. This shows that $JP \subseteq Q$. Now $JS \subseteq JP \cap S \subseteq Q \cap S = 0$, so $JS = 0$. We now end the proof by showing that $S = 0$, so that $P = Q$. Let $s = (s_h)_{h\in H_\varphi} \in S$, where $s_h \in M_h$. There exist $t_h \in R$ such that $s_h = t_h + (r_h)$. If $t_h \notin J$ for some $h \in H_\varphi$, then there exists $d \in R$

such that $ds_h = 1 + (r_h)$, and so, as $vds \in JS = 0$, it follows that $v \in (r_h)$ which is impossible (otherwise, there is $e \in R$ such that $v = e(u - hv)$, i.e. $eu - (1 + eh)v = 0$, contradicting that $\{u, v\}$ is a minimal generating set for J)). Hence, each $t_h \in J$, i.e. $S \subsetneq \prod_{h \in H_\varphi} (JM_h) = J\left(\prod_{h \in H_\varphi} M_h\right) = JP \subseteq Q$. This completes the proof that $S = 0$.

For the rest of this section, R always denotes a commutative artinian ring, J its Jacobson radical and if $R_1 \times R_2 \times \cdots \times R_n$ is the local ring decomposition of R, μ denotes $\min\{|R_i| : \ 1 \leq i \leq n\}$.

Proposition 9. *Suppose that $\mu \geq \kappa$ for some infinite regular cardinal κ. Then the following statements are equivalent:*

(i) R is not a principal ideal ring.

(ii) There exists a family $\{M_h\}_{h \in H}$ of κ cyclic modules over R such that for every proper filter ψ on a set I, $M^I / \sum_\psi M$ is not algebraically compact whenever $M = \sum_\varphi M_h$ and φ is any filter on H satisfying $|H_\varphi| = \kappa$ and $\varphi_{|I|} \subseteq H_\varphi(\kappa)$.

Proof. (ii) implies (i), since, over a commutative artinian ring which is a principal ideal ring, all modules are algebraically compact. Also, R is a principal ideal ring if and only if each of $R_1, \ldots, R_n$ is. We may therefore assume that R is local. Suppose that (i) holds. We first set to deduce (ii) in case $J^2 = 0$ and $\dim_{R/J} J = 2$. Using the Loewy series of R as in the proof of [5, Lemma 2] we obtain that $|R/J| \geq \kappa$, and so we can define a subset H of $R \backslash J$ with κ elements, elements $(r_h)_{h \in H}$ of R and R–modules $M_h = R/(r_h)$ $(h \in H)$ as in the notation preceding Proposition 8. Let φ be a filter on H and ψ a proper filter on a set I such that $|H_\varphi| = \kappa$ and $\varphi_{|I|} \subseteq H_\varphi(\kappa)$. Put $M = \sum_\varphi M_h$, $N_h = M_h^I / \sum_\psi M_h$ and assume that $M^I / \sum_\psi M$ is algebraically compact. By Proposition 7, M is isomorphic to a pure submodule of $M^I / \sum_\psi M$, which therefore contains a pure–injective envelope Q of M. Let

$$f : M^I/\sum_\psi M \to \prod_{h\in H} N_h$$

be as in Proposition 7. Since M is isomorphic to a pure submodule in $(M^I/\sum_\psi M)/\ker f$, so too it is in $(Q + kerf)/kerf$. Q is a pure–essential extension of M and $(Q + kerf)/kerf \cong Q/Q \cap kerf$; therefore, $Q \cap kerf = 0$, and the restriction $f|_Q$ is injective. Since $Q \cong \prod_{h\in H_\varphi} M_h$ (Proposition 8), it follows that there is a monomorphism $g : \prod_{h\in H_\varphi} M_h \to \sum_{\varphi_{|I|}} N_h$. Let $\{F_i\}_{i<\kappa}$ be p–functors: R–Mod→ Ab given by $F_i(A) = \bigcap_{j\leq i} \left(r_{h_j}A\right)$, where $\{h_j : j < \kappa\}$ is a well–ordering of H_φ. Since κ is regular, $\{F_i\}_{i<\kappa}$ is κ–filtered. By Corollary 1, there exist an ordinal $\tau < k$, a subset H_0 of H_φ with $|H_0| = \kappa$ and a member T of $\varphi_{|I|}$ such that $g\left(F_\tau \prod_{h\in H_0} M_h\right) \subseteq \prod_{h\in H\setminus T} N_h + \bigcap_{i<\kappa} \left(F_i \sum_{\varphi_{|I|}} N_h\right)$. For each $h \in H_\varphi$, there exists $i_h < \kappa$ such that $F_{i_h} M_h = 0$, and since each F_i commutes with filter sums, it follows that $\bigcap_{i<\kappa} \left(F_i \sum_{\varphi_{|I|}} N_h\right) = 0$. Also, for each $i < \kappa$ and $j < \kappa$,

$$F_i M_{h_j} = \begin{cases} JM_{h_j} & \text{if } j > 1 \\ 0 & \text{if } j \leq i. \end{cases}$$

It is clear that $T \cap \{h_j : j > \tau\} \in H_\varphi(\kappa)$, and therefore $T \cap \{h_j : j > \tau\} \not\subseteq H_\varphi \setminus H_0$ (otherwise $H_\varphi \setminus H_0 \in H_\varphi(\kappa)$ contradicting that $|H_0| = \kappa$). Hence the set $T \cap \{h_j : j > \tau\} \cap H_0$ is not empty and contains an element h_σ, say. Now M_{h_σ} is cyclic, and so $g(M_{h_\sigma}) = Rm$ for some element $m = (m_h)_{h\in H}$ in $\sum_{\varphi_{|I|}} N_h$. We have $\text{Ann}(m) = \text{Ann}(M_{h_\sigma}) = (r_{h_\sigma}) = \bigcap_{h\in H} \text{Ann}(m_h) = \text{Ann}(m_{h_\sigma})$ (recall that $(r_h) \cap (r_{h'}) = 0$ if $h \neq h'$). We claim that $u \notin \text{Ann}(m_{h_\sigma})$. For, if not, $u \in (r_{h_\sigma})$; which would yield, as $\{u, v\}$ is an R/J–basis for J, that $h_\sigma \in J$ and this is impossible. On the other hand, $Jm = g(JM_{h_\sigma}) = g(F_\tau M_{h_\sigma})$, since $h_\sigma \in \{h_j : j > \tau\}$. We thus infer that $Jm \subseteq \prod_{h\in H\setminus T} N_h$ and, as $h_\sigma \in T$, $Jm_{h_\sigma} = 0$. This means $um_{h_\sigma} = 0$, a contradiction. This proves (ii) in the special case when $J^2 = 0$ and $\dim_{R/J} J = 2$. For the general case, by [9], there exists an ideal I of R such that $J^2 \subseteq I \subseteq J$ and $\dim_{R/J} J/I = 2$. Let A be the quotient ring R/I, and J' its radical. Then A is a commutative artinian local ring with

$|A| \geq |R/I| \geq \kappa$, and $J'^2 = 0$, $\dim_{A/J'} J' = 2$. Clearly A is not a principal ideal ring, and hence, by the first part of this proof, there exists a family $\{B_h\}_{h \in H}$ of κ cyclic A–modules for which (ii) holds. Each B_h is easily seen to be a cyclic R–module, and since $B^I / \sum_\psi B$ where $B = \sum_\varphi B_h$ is not A–algebraically compact, it is not R–algebraically compact either. This completes the proof.

Corollary 4. *Suppose that $\mu \geq \kappa$ for some uncountable regular cardinal κ. The following statements are equivalent:*

(i) R is not a principal ideal ring.

(ii) There exists an R–module M which is λ–compact for all $\lambda < \kappa$ but is not algebraically compact.

(iii) There exists an R–module M such that $M^\lambda / \sum_\psi M$ is λ–compact for each cardinal $\lambda < \kappa$ and each λ–complete proper filter ψ on λ, but $M^\lambda / \sum_\psi M$ is not algebraically compact.

(iv) There exists a κ–generated R–module M such that $M^I / \sum_\psi M$ is not algebraically compact for any proper filter ψ on a set I such that $|I| < \kappa$.

Proof. It is clear that each of (ii), (iii) and (iv) implies (i). Assume now that (i) holds. We set to deduce each of (ii), (iii) and (iv). First let $\{M_h\}_{h \in H}$ and $|H| = \kappa$ as in Proposition 9, and let $\varphi^1 = H(\kappa)$. Then $|H_{\varphi^1}| = H$ and φ^1 is non–principal, so that by Proposition 8, $M = \sum_{\varphi^1} M_h$ is not algebraically compact. As $H(\kappa)$ is κ–complete on H, M is κ–pure in $\prod_{h \in H} M_h$, and M is therefore λ–compact for each $\lambda < \kappa$. This proves (ii). Moreover, for any cardinal $\lambda < \kappa$ and any λ–complete proper filter ψ on λ, $\varphi^1_\lambda = \varphi^1$. So, by Proposition 9, $M^\lambda / \sum_\psi M$ is not algebraically compact. However, by Proposition 3, $M^\lambda / \sum_\psi M$ is λ–compact and (iii) follows. Finally, let $\varphi^2 = H(\aleph_0)$ and let ψ be any proper filter on a set I with $|I| < \kappa$. It is easy to check that

$|H_{\varphi^2}| = |H| = \kappa$ and that $\varphi_\alpha^2 = H(|I|^+) \subseteq H(\kappa)$. By Proposition 9, $M^I / \sum_\psi M$ is not algebraically compact. Since $M = \bigoplus_{h \in H} M_h$ is κ–generated, (iv) follows.

Let R be a commutative local artinian ring with $|R| = \kappa$ for some infinite regular cardinal κ, and suppose that it is not a principal ideal ring. The R–module M in (ii) above is not κ–compact (see Remark 1 in §2). This provides a new approach to the construction of modules that are λ–compact for all $\lambda < \kappa$ but not κ–compact. Note that the construction of such modules (over principal ideal rings) was discussed in [4], but that it used a result that was shown in [6] to fail for weakly inaccessible cardinals.

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