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On the Projectivity of Pure–Injective envelopes

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1. Introduction

In [2, Theorem 4.2], it is proved that for a commutative noetherian ring R the following statements are equivalent:

(a) Every flat R–module is pure–injective.

(b) Every flat R–module is projective, i.e., R is perfect.

(c) The pure injective envelope of every flat R–module is projective.

(d) The pure injective envelope of every free R–module is projective.

Since a commutative noetherian ring is perfect if and only if it is artinian (direct products of copies of the ring are projective), we can add the following equivalent condition

(e) R is artinian.

Our main objective in this note is to obtain similar characterizations for more general rings. The proofs given in [2] rely on localizations and completions, and it should be pointed out that the cited theorem contains four more equivalent conditions on Spec(R) and the local rings R_P ($P \in$ Spec(R)). Our main argument here is based on the structure of the quotient ring $R/J(R)$, where $J(R)$ is the Jacobson radical of R. This will provide an alternative proof of the above cited result and in fact, we shall see that both the commutativity and the noetherianness of the ring can be replaced by weaker conditions.

Throughout, all rings are associative with 1 and modules are left unital. For any ring R and any set I, M^I ($M^{(I)}$) denotes the direct product (sum) of $|I|$ copies of M for any R–module M while $P(M)$ stands for the pure–injective envelope of M. Recall that an R–module M is algebraically compact (*a.c.*) if and only if it is pure–injective, that is, if it has the injective property with respect to short pure–exact sequences. If $M^{(I)}$ is algebraically compact for all sets I (equivalently, if $M^{(N)}$ is *a.c.*), M is said to be $\sum$–algebraically compact ($\sum$–*a.c.*).

2. Main Results

Proposition 1. *Let R be any ring. Then the following conditions are equivalent:*

(i) Every free left R–module is a.c., i.e., ${}_RR$ is $\sum$–a.c.

(ii) Every flat left R–module is a.c.

If, in addition, R is right coherent, these conditions are equivalent to each of:

(iii) Every flat left R–module is projective, i.e. R is left perfect.

(iv) The pure–injective envelope of any flat left R–module is projective.

(v) ${}_RR$ is $\prod$–projective, i.e. every product of copies of ${}_RR$ is projective.

Proof. (i) $\Rightarrow$ (ii). Let M be a flat R–module. Then there exists a pure exact sequence $0 \to K \to R^{(I)} \to M \to 0$ for some set I. Since R is $\sum$–*a.c.*, $R^{(I)}$ is also $\sum$–*a.c.* Pure submodules (and hence pure quotients) of $\sum$–*a.c.* modules being direct summands [ ], it follows that M is a direct summand of the *a.c.* module $R^{(I)}$, and so it is *a.c.* also.

(ii) $\Rightarrow$ (i) is trivial.

(ii) $\Rightarrow$ (iii). If every flat R–module is *a.c.*, the free module $R^{(N)}$ is *a.c.*, i.e. R is $\sum$–*a.c.* By repeating the above argument we obtain that every flat R–module is a direct summand of a free module, i.e. it is projective.

(iii) $\Rightarrow$ (iv). It is well–known (see for example [3]) that if R is right coherent and M is a flat left R–module then M^{**} (where $M^* = \mathrm{Hom}_Z(M, Q/Z)$) is also flat. Hence M^{**} is projective. It is clear that $P(M)$ is a direct summand of M^{**}, and therefore $P(M)$ is projective too.

(iv) $\Rightarrow$ (v). The pure injective envelope $P(R)$ of R is projective. Since R is cyclic as a left R–module and since it is pure in $P(R)$, it must be a direct summand of some free R–module containing $P(R)$. This implies that R is a direct summand of $P(R)$ and so $R = P(R)$ is *a.c.* On the other hand, for any set I, R^I is *a.c.* and is flat, since R is right coherent. Consequently R^I is its own pure–injective flat envelope, and so it is projective.

(v) $\Rightarrow$ (i). This follows from the known fact that $\prod$–projective modules are always $\sum$–*a.c.* (see for example [8]).

Remark. If R is right artinian then ${}_RR$ is $\sum$–*a.c.* [7], but the converse is false. In fact, as was constructed in [8] there are rings R such that ${}_RR$ is $\sum$–*a.c.* but R is not artinian on either side. However, as observed in [9], if both R and $\mathrm{Hom}_Z(R, Q/Z)$ are $\sum$–a.c. as left R–modules, then R is left artinian.

Proposition 2. *Let R be a ring with central idempotents and such that the ring $R/J(R)$ is (Goldie) finite–dimensional. Then the following statements are equivalent.*

(i) ${}_RR$ is $\sum$–a.c.

(ii) The pure–injective envelope of every flat R–module is projective.

(iii) The pure–injective envelope of every free R–module is projective.

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 1, and (ii) $\Rightarrow$ (iii) is trivial. Now assume that (iii) hods. Observe first that ${}_RR$, being cyclic and pure in $P(R)$, is a direct summand of $P(R)$. Therefore ${}_RR$ is algebraically compact. By [8, Theorem 9], the ring $S = R/J(R)$ is von Neumann regular. Since S has finite Goldie dimension, it must be semisimple, which yields that R is semiperfect. Let $\{e_1, \ldots, e_n\}$ be (central) orthogonal idempotents such that $R = \bigoplus_{i=1}^{n} Re_i$, and let κ be any cardinal greater than $|R| + \aleph_0$. Since $P(R^{(\kappa)})$ is projective, there exist sets A_i $(1 \le i \le n)$ such that $P(R^{(\kappa)}) \cong \bigoplus_{i=1}^{n} R_{e_i}(A_i)$ (see e.g. [1]). Assume, by way of contradiction, that A_{i_0} is finite for some i_0, and let f be the monomorphism $Re_{i_0}^{(\kappa)} \to P\left(\bigoplus_{i=1}^{n} Re_i^{(\kappa)}\right) \to \bigoplus_{i=1}^{n} Re_i^{(A_i)}$. Using orthogonality, we obtain that $f\left(Re_{i_0}^{(\kappa)}\right) \subseteq Re_{i_0}^{(A_{i_0})}$, so that $\left|Re_{i_0}^{(\kappa)}\right| \le \left|Re_{i_0}^{(A_{i_0})}\right|$. This means that $\kappa \le |R| \cdot |A_{i_0}| \le |R| + \aleph_0$, a contradiction. Therefore, each A_i is infinite, and $R^{(N)} = \bigoplus_{i=1}^{n} Re_i^{(N)}$ is a direct summand of $P(R^{(\kappa)})$, which implies that $R^{(N)}$ is algebraically compact. This proves (i).

Corollary. *For a local ring the following statements are equivalent.*

(i) ${}_RR$ *is* $\sum$*-a.c.*

(ii) The pure-injective envelope of every free R module is projective.

Proof. (i) $\Rightarrow$ (ii) is trivial. To see that (ii) $\Rightarrow$ (i), recall that in a local ring R, 0 and 1 are the only idempotents and that the field $R/J(R)$ has obviously a finite dimension.

Remarks.

1. The corollary can also be proved using the fact that projectives are free over local rings (see [4]).

2. It is clear that for any ring, (ii) above is equivalent to:

(ii) The pure–injective envelope of every projective module is projective. (Note that a similar condition with "pure–injective" replaced by "injective" has been discussed in [5].)

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