King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 152

October 1993

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SEMI - OPEN SETS IN LOCALLY CONNECTED AND LOCALLY PATHWISE CONNECTED METRIC SPACES

by

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ABSTRACT: The intersection of two semi-open sets in a space X may not be a semi-open set in X. In this paper we characterize the intersection of semi-open sets in locally connected and locally pathwise connected metric spaces.

INTRODUCTION

In 1963, N. Levine introduced the concept of semi-open set by defining a subset A of a topological space X to be semi-open if there exists an open set U in X such that A contains U and A is contained in the closure of U in X. He proved that a set A in a topological space X is semi-open if and only if A is contained in the closure of the interior of A in X. SO(X) will denote the class of all semi-open sets in a topological space X. We note that every open set in a topological space X is a semi-open set but clearly a semi-open set may not be an open set in X. N. Levine also proved that the union of a collection of semi-open sets in a topological space is always semi-open. It is clear that a nowhere dense set in a space X is always not semi-open in X and the complement of a nowhere dense set in X is always semi-open in X. In particular for any semi-open set S in a space X, the difference of the closure of S and S is not semi-open in X.

MAIN RESULTS

In the following our first result gives an excellent characterization of the finite intersection of semi-open sets in a topological space.

THEOREM.1. The intersection of a finite number of semi-open sets in a space X is the union of an open set and a nowhere dense set in X.

PROOF: Suppose $A = S_1 \cap S_2 \cap \ldots \cap S_n$ where each S_i is semi-open in X. Then for each $i \in \{1, 2, \ldots, n\}$, there exists G_i open in X such that $G_i \subseteq S_i \subseteq \overline{G_i}$. Let $T_i = S_i - G_i$, for each $i \in \{1, 2, \ldots, n\}$. Then each $S_i = G_i \cup T_i$ and each T_i is nowhere dense in X

Now $A = \bigcap_{i=1}^{\infty} (G_i \cup T_i) = [\bigcap_{i=1}^{n} G_i] \cup [\bigcup_{i=1}^{2^n-1} (\bigcap_{k=1}^{n} E_{(i,k)})]$

where for all $i \in \{1, 2, \ldots, 2^n - 1\}$ $E_{(i,k)} = G_k$ or T_k, for some $k \in \{1, 2, \ldots, n\}$.

We notice that $\cap_{i=1}^{n} G_i$ is open, and all other sets in the union are nowhere dense being contained in nowhere dense set, whence, as is well known, their union is nowhere dense. Hence now the theorem follows by induction.

The following theorem deals with a locally connected metric space.

THEOREM.2. Let (X,ρ) be a locally connected metric space. Let A be a nonempty closed nowhere dense subset of X. Then A is the intersection of two semi-open sets in X.

PROOF: Define $f: X \to [0,\infty)$ by $f(x) = \rho(x,A)$, for all $x \in X$. Then f is continuous and $A = f^{-1}(\{0\})$.

For each $n \in N$, let $G_n = (\frac{1}{2n+1}, \frac{1}{2n})$, $H_n = (\frac{1}{2n}, \frac{1}{2n-1})$.

Let $G = \bigcup_{n=1}^{\infty} G_n$, $H = \bigcup_{n=1}^{\infty} H_n$. Then G and H are disjoint open sets in $[0,\infty)$ and $0 \in \overline{G}, 0 \in \overline{H}$. So $S = G \cup \{0\}$ and $T = H \cup \{0\}$ are semi-open sets in $[0,\infty)$ with $S \cap T = \{0\}$. Let $S^* = f^{-1}(S)$, $T^* = f^{-1}(T)$. Then $A = f^{-1}(\{0\}) = f^{-1}(S) \cap f^{-1}(T) = S^* \cap T^*$. Now we show that S^* is semi-open. Clearly $f^{-1}(G)$ is open and $f^{-1}(G) \subseteq f^{-1}(S) = S^* = f^{-1}(G) \cup f^{-1}(\{0\}) = f^{-1}(G) \cup A$. So it is enough to show that $A \subseteq \overline{f^{-1}(G)}$.

Let $x \in A$. Since X is locally connected, and let U be any open connected neighborhood of x. Then $f(x) = 0 \in f(U)$ and $f(U)$ is connected. Since A is nowhere dense and closed, we can fix $y \in U \cap (X - A)$ and hence $f(y) > 0$ and $f(y) \in f(U)$. While $f(U)$ being connected, is an open interval, so $[0, f(y)] \subseteq f(U)$. Let $n^* \in N$ such that $\frac{1}{2n^*} < f(y)$.

Then $[\frac{1}{2n^*+1}, \frac{1}{2n^*}] \subseteq f(U)$.

Let $$z^* = \frac{1}{2}(\frac{1}{2n^*} - \frac{1}{2n^*+1}) \in (\frac{1}{2n^*+1}, \frac{1}{2n^*}) = G_{n^*}.$$

Then $z^* \in f(U)$. So there exists $x^* \in U$ such that $f(x^*) = z^*$. But then $f(x^*) \in G_{n^*} \subseteq G$ implies that $x^* \in f^{-1}(G)$. Hence, $x^* \in U \cap f^{-1}(G) \neq \emptyset$. SO $x \in \overline{f^{-1}(G)}$. Thus $A \subseteq \overline{f^{-1}(G)}$ as required. Similarly, T^* is semi-open in X^*. Thus A is the intersection of two semi-open sets S^* and T^* in X.

COROLLARY.3. Let (X,ρ) be a locally arcwise connected metric space. Let A be a closed and nowhere dense subset of X. Then A is the intersection of two semi-open sets in X.

PROOF. It is enough to note that a locally arcwise connected metric space is locally connected and then the result follows from Theorem.2.

Our next result is concerned with a locally pathwise connected metric space.

THEOREM.4. Let (X,ρ) be a locally pathwise connected metric space. Let A be a closed and nowhere dense subset of X. Then there exist semi-open sets S^* and T^* in X such that $A = S^* \cap T^*$.

PROOF: Define $f: X \to [0,\infty)$ by $f(x) = \rho(x,A)$, for all $x \in X$. Then f is continuous and $A = f^{-1}(\{0\})$.

Now for each $n \in N$, let $G_n = (\frac{1}{2n+1}, \frac{1}{2n})$, $H_n = (\frac{1}{2n}, \frac{1}{2n-1})$.

Also put $G = \bigcup_{n=1}^{\infty} G_n$, and $H = \bigcup_{n=1}^{\infty} H_n$. Then G and H are disjoint open sets in $[0,\infty)$ and $0 \in \overline{G}, 0 \in \overline{H}$. Thus $S = G \cup \{0\}$ and $T = H \cup \{0\}$ are semi-open sets in $[0,\infty)$ with $S \cap T = \{0\}$. Let $S^* = f^{-1}(S)$, $T^* = f^{-1}(T)$.

Then $A = f^{-1}(\{0\}) = f^{-1}(S) \cap f^{-1}(T) = S^* \cap T^*$. Now we show that S^* is semi-open. Clearly $f^{-1}(G)$ is open as f is continuous. Also $f^{-1}(G) \subseteq f^{-1}(S) = S^* = f^{-1}(G) \cup f^{-1}(\{0\}) = f^{-1}(G) \cup A$. So it suffices to verify that $A \subseteq \overline{f^{-1}(G)}$. Let $x \in A$. By hypothesis, X is locally pathwise connected. So let U be any (basic) pathwise connected neighborhood of x. Then as A is closed nowhere dense, so we can fix $y \in U \cap (X - A)$.
Then $x \neq y$; $x, y \in U$ and U is pathewise connected. So there exists a continuous function $g : I = [0,1] \rightarrow X$ such that $g(I) \subseteq U$, $g(0) = x$, $g(1) = y$. Let $Y = g(I)$, $h = fog$. Then h is continuous and $h(I) = f(g(I)) = f(Y)$. Since h(I) is connected and so it is an interval. Also $h(0) = f(g(0)) = f(x) = 0 \in h(I)$, $h(1) = f(g(1)) = f(y) \in h(I)$, $f(y) > 0$ as $y \notin A$. Thus we have $[0,f(y)] = h(I)$. Hence there exists $n^* \in N$ such that

$$G_{n^*} = \left(\frac{1}{2n^* + 1}, \frac{1}{2n^*}\right) \subseteq [0,f(y)].$$

Fix $z^* \in G_{n^*}$. Then $z^* \in G$. Also $z^* \in h(I) = f(Y)$ implies that there exists $y^* \in Y$ such that $f(y^*) = z^*$. Hence $y^* \in U$ as $Y \subseteq U$. Also $y^* \in f^{-1}(G)$ as $z^* = f(y^*) \in G$. Thus $y^* \in U \cap f^{-1}(G)$ implies that $U \cap f^{-1}(G) \neq \phi$. Thus $x \in \overline{f^{-1}(G)}$. So it follows that $x \in \overline{f^{-1}(G)}$ as required. Similarly, one can prove that T^* is also semi-open. Also it is already noted that $A = S^* \cap T^*$.

ACKNOWLEDGEMENT: The author is greatly thankful to Dr. S. Willard for his personal interest and guidance in the preparation of this paper in the present form. The author is indebted to the University of Alberta and the King Fahd University of Petroleum and Minerals for providing all necessary research facilities during the completion of this paper.

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