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D - Spaces

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D- SPACES

by

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ABSTRACT: In 1968, N. Levine introduced the concept of a D-Space by calling a topological space (X,T) a D-space whenever every nonempty open set is dense in X. In this paper we study a characterization of D-spaces in terms of semi-open sets as well as mappings on D-spaces.

INTRODUCTION: In 1963, N. Levine introduced the concept of semi-open set by defining a subset A of a topological space X to be semi-open if there exists an open set U in X such that A contains U and A is contained in the closure of U in X. He proved that a set A in a topological space X is semi-open if and only if A is contained in the closure of the interior of A in X. SO(X) will denote the class of all semi-open sets in a topological space X. We note that every open set in a topological space X is a semi-open set but clearly a semi-open set may not be an open set in X . N. Levine also proved that the union of a collection of semi-open sets in a topological space is always semi-open. It is clear that a nowhere dense set in a space X is always not semi-open in X and the complement of a nowhere dense set in X is always semi-open in X. In particular for any semi-open set S in a space X, the difference of the closure of S and S is not semi-open in X. We observe that the intersection of two semi-open sets in a space X may not be a semi-open set in X. A point x in X is called a semi-limit point of a subset A of X if for each semi-open set U of X containing the point x, the intersection of A and U contains a point other than x. The set of all semi-limit points of a set A is called the semi-derived set of A and is denoted by A^{sd}. The semi-closure of a set A, denoted by scl(A), is the union of A and its semi-derived set A^{sd}. A function $f: (X,T) \longrightarrow (Y,T)$ is called semi-continuous (or irresolute) if the inverse image of each open (or semi-open) set in Y is semi-open in X.

A. CHARACTERIZATION OF D-SPACES

In this section, we give some characterizations of D-spaces.

THEOREM.1. In a topological space (X,T), the following are equivalent:

(i) (X,T) is a D-space.
(ii) Every pair of nonempty open sets has a nonempty intersection.
(iii) Every open set in X is connected.

PROOF: Easy.

THEOREM.2. A topological space X is a D-space if and only if for each nonempty semi-open set S in X, scl(S) = X.

PROOF: NECESSITY. Let S be a nonempty semi-open set in X. Let $x \in X$. Let $T \in SO(X)$ such that $x \in T$. Then there exist nonempty open sets U and V in X such that $U \subseteq T \subseteq \overline{T}$ and $V \subseteq S \subseteq \overline{V}$. By hypothesis, $\overline{U}=X=\overline{V}$. If $S \cap T=\phi$, then $U \cap V=\phi$. Then by theorem .1, X is not a D-space. But this is a contradiction. Hence we have $S \cap T \neq \phi$. Thus $x \in scl(S)$, so $X \subseteq scl\, S$. Hence, scl(S) = X.

SUFFICIENCY: Suppose that for each nonempty semi-open set S in X, we have scl(S) = X. Let U be a nonempty open set in X. Then U is a nonempty semi-open set in X. Hence by hypothesis, scl(U) = X. But scl(U) being the intersection of all semi-closed sets containing U, and each closed set being a semi-closed set, it follows immediately that scl(U) $\subseteq$ cl (U). Thus cl(U) = X. Therefore, X is a D-space.

THEOREM.3. A topological space X is a D-space if and only if every pair of nonempty semi-open sets has a non-empty intersection.

PROOF: NECESSITY. Suppose X is a D-space. Let S and T be nonempty semi-open sets in X. Then there exist U and V nonempty open sets in X such that
$U \subseteq S \subseteq cl(U)$, $V \subseteq T \subseteq cl(V)$. By Theorem.1, $U \cap V \neq \Phi$. Hence, $S \cap T \neq \Phi$.

SUFFICIENCY. Let S be a nonempty semi-open set in X. Let $x \in X$. Let $T \in SO(X)$ such that $x \in T$. By hypothesis, $S \cap T \neq \emptyset$. Hence, $x \in scl(S)$. Thus X = scl(S). So by Theorem.2, X is a D-space.

We note that a D-space is never a semi-T2 space though it can be a semi-T_1 space.

Now in the following, we define S-connected spaces and examine their relation with D-spaces.

DEFINITION.4. A space X is said to be S-connected if X cannot be written as the disjoint union of two nonempty semi-open sets.

We observe that an S-connected space is connected but a connected space may not be an S-connected space. For example, R with the usual topology.

THEOREM.5. Every D-space is S-connected and hence connected.

PROOF: Suppose X is a D-space. If X is not S-connected, then there exist two nonempty semi-open sets S and T such that $X = S \cup T$ and $S \cap T = \emptyset$. Hence scl(S) $\neq$ X. But then by Theorem.2, X is not a D-space which is a contradiction. Thus X is S-connected.

We note that a connected space may not be a D-space. For example, R with the usual topology.

THEOREM.6. A topological space X is a D-space if and only if every nonempty semi-open set in X is S-connected.

PROOF: NECESSITY. Let Y be a nonempty semi-open set in X. If Y is not S-connected, then there exist two nonempty semi-open sets T_1 and T_2 in Y such that $Y = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$. Then T_1 and T_2 are two disjoint nonempty semi-open sets in X. So by Theorem.3, X is not a D-space. But this is a contradiction. Hence Y is S-connected.

SUFFICIENCY. Suppose that X is not a D-space. Then by Theorem.1, there exist nonempty open sets U and V in X such that $U \cap V = \emptyset$. Let $Y = U \cup V$. Then Y is semi-open in X. Also U and V are open in Y and hence semi-open in X. Thus Y is not S-connected. This contradiction shows that X is a D-space.

THEOREM.7. A space with a dense D-subspace is itself a D-space.

PROOF: Theorem 5(i) of (LEVINE [1968]).

THEOREM.8. If A is a D-subspace of a space X, then scl(A) is also a D-subspace of X.

PROOF: Since A is dense in scl(A), the result follows from Theorem.7.

THEOREM.9. Suppose X is a D-space and C is a D-subset of X. Suppose further that the subspace X – C of X has an open set V. Then $C \cup V$ is a D-subset of X.

PROOF: Theorem 4.9 of (FATTEH AND SINGH [1983]).

Our next result is a generalization of this Theorem.

THEOREM.10. Suppose X is a D-space and C is a D-subset of X. Suppose further that the subspace X – C of X has a semi-open set S. Then $C \cup S$ is a D-subset of X.

PROOF: Since every nonempty semi-open set contains a nonempty open set, the result follows from Theorem 9.

In the following, we give a characterization of D-spaces in terms of regular semi-open sets. We need the following result.

THEOREM.11. A topological space X is a D-space if and only if X has no proper regularly open set.

PROOF: Theorem 3.2 of (FATTEH AND SINGH [1983]).

DEFINITION.12. A subset A of a topological space X is regular semi-open if there is a regular open set U such that $U \subseteq A \subseteq cl(U)$.

THEOREM.13. A topological space X is a D-space if and only if X has no proper regularly semi-open set.

PROOF: NECESSITY. If X has a proper regularly semi-open set A, then there exists a regularly open set U such that $U \subseteq A \subseteq cl(U)$. Clearly U is also proper. Hence by Theorem 11, X is not a D-space. This contradiction shows that X does not have any proper regularly semi-open set.

SUFFICIENCY. Suppose X is not a D-space. Then by Theorem 11, there exists a proper regularly open set U. Clearly U is a proper regularly semi-open set which is a contradiction. Thus X is a D-space.

Now we give a characterization of semi-open sets in a D-space. We need the following easy lemma.

LEMMA.14. Let (X,T) be a D-space. Let U and V be non-empty open sets in X. Then $cl(U \cap V) = (cl(U) \cap cl(V))$.

PROOF: Easy.

THEOREM.15. Let (X,T) be a D-space. Let $A, B \in SO(X)$. Then $A \cap B \in SO(X)$.

PROOF: Let $U, V \in T$ such that $U \subseteq A \subseteq cl(U)$, $V \subseteq B \subseteq cl(V)$.
Then $U \cap V \subseteq A \cap B \subseteq (cl(U)) \cap (cl(V))$. By applying Lemma. 14, we have:
$U \cap V \subseteq A \cap B \subseteq (cl(U \cap V))$. Thus $A \cap B \in SO(X)$.

THEOREM.16. If (X,T) is a topological space and T is finite, then (X,T) is a D-space if and only if the intersection of all non-void semi-open subsets of X is non-void.

PROOF: NECESSITY. If possible then suppose that $\cap(SO(X) - \{\emptyset\}) = \emptyset$. Then we assert that $\cap(T - \{\emptyset\}) = \emptyset$. For that let $x \in X$. Then there exists $S_x \in (SO(X) - \{\emptyset\})$ such that $x \notin S_x$. So there exists $U_x \in (T - \{\emptyset\})$ such that $U_x \subseteq S_x \subseteq cl(U_x)$. Hence $x \notin U_x$. Thus $\cap(T - \{\emptyset\}) = \emptyset$. Then it follows easily by Theorem.1 that X is not a D-space. By this contradiction, we conclude that $\cap(SO(X) - \{\emptyset\}) \neq \emptyset$.

SUFFICIENCY. Suppose that $\cap(SO(X) - \{\emptyset\}) \neq \emptyset$. Since $T \subseteq SO(X)$. Hence $\cap(T - \{\emptyset\}) \neq \emptyset$. Then by Theorem.1, it follows immediately that (X,T) is a D-space.

B. CHARACTERIZATION OF IMAGES AND PRE-IMAGES OF D-SPACES

In the present section, we will show that some mappings preserve the D-space structure of their images and pre-images.

THEOREM.17. The semi-continuous image of a D-space is a D-space.

PROOF: Let $f: X \rightarrow Y$ be a surjective semi-continuous function where X is a D-space. If Y is not a D-space, then by Theorem. 1, there exist nonempty open sets U and V in Y such that $U \cap V = \emptyset$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are nonemty disjoint semi-open sets. So by Theorem 3, X is not a D-space. But this is a contradiction. Hence Y is a D-space.

COROLLARY.18. The irresolute image of a D-space is a D-space.

THEOREM.19. Let (X,T) and (Y,T*) be topological spaces and suppose that $f: X \rightarrow Y$ is a function. Then prove the following.

(i) If f is one to one and semi-open and if (Y,T*) is a D-space, then (X,T) is a D-space.

(ii) If f is one to one and pre-semi open and if (Y,T*) is a D-space, then (X,T) is a D-space.

(iii) If f is onto and $SO(X) = \{f^{-1}(U) : U \in T^*\}$, then (X,T) is a D-space if and only if (Y,T*) is a D-space.

(iv) If f is onto and $SO(X) = \{f^{-1}(U) \mid U \in SO(Y)\}$, then (X,T) is a D-space if and only if (Y,T*) is a D-space.

PROOF: (i) If (X,T) is not a D-space, then by Theorem. 1, there exist disjoint nonempty open sets A and B in X. Then f(A) and f(B) are disjoint nonempty semi-open sets in Y and by Theorem. 3, Y is not a D-space. But this is a contradiction to our hypothesis, so (X,T) is a D-space.

(ii) The straight forward proof (similar to (i)) is omitted.

(iii) Necessity follows from Theorem. 17. To show the sufficiency, suppose that (X,T) is not a D-space. Then by Theorem. 3, there exist disjoint nonempty semi-open sets A and B in X. Then there exist non-empty open sets U and V in Y such that $A = f^{-1}(U)$ and $B = f^{-1}(V)$. Now $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) = A \cap B = \emptyset$. Since f is onto. So $U \cap V = \emptyset$. Thus U and V are disjoint non-empty open sets in Y. So by Theorem. 1, Y is not a D-space. But this is a contradiction. Hence (X,T) is a D-space.

(iv) The necessity follows from corollary. 18. To prove sufficiency, suppose (X,T) is not a D-space. Then by Theorem. 3, there exist disjoint nonempty semi-open sets A and B in X. Then $\emptyset = A \cap B = f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$ for some nonempty semi-open sets U and V in Y. Also U and V are disjoint as f is onto. Then by Theorem. 3, it follows that Y is not a D-space. This leads us to a contradiction. Hence it is proved that (X,T) is a D-space.

THEOREM.20. Every semi-continuous function from a D-space into a T_2 - space is a constant function.

PROOF: Let $f: X \longrightarrow Y$ be semi-continuous, where X is a D-space and Y is a T_2 - space. Let $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$. Since Y is T_2, so there exist disjoint open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$. Then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint nonempty semi-open sets in X. Hence by Theorem. 3, X is not a D-space. But this is a contradiction. Thus f is a constant function.

THEOREM.21. Prove that every irresolute function from a D - space into semi -T_2 space is constant.

PROOF: Let $f: X \longrightarrow Y$ be an irresolute function, where X is a D - space and Y is a semi-T_2 - space. Let $x,y \in X$ such that $f(x) \neq f(y)$. Since Y is semi -T_2, there exist disjoint semi-open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is an irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint nonempty semi-open sets in X. Hence by Theorem. 3, X is not a D-space. This gives a contradiction. Hence f is a constant function.

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