D-SPACES

by

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ABSTRACT: In 1968, N. Levine introduced the concept of a D-Space by calling a topological space \((X,T)\) a D-space whenever every nonempty open set is dense in \(X\). In this paper we study a characterization of D-spaces in terms of semi-open sets as well as mappings on D-spaces.

INTRODUCTION: In 1963, N. Levine introduced the concept of semi-open set by defining a subset \(A\) of a topological space \(X\) to be semi-open if there exists an open set \(U\) in \(X\) such that \(A\) contains \(U\) and \(A\) is contained in the closure of \(U\) in \(X\). He proved that a set \(A\) in a topological space \(X\) is semi-open if and only if \(A\) is contained in the closure of the interior of \(A\) in \(X\). \(SO(X)\) will denote the class of all semi-open sets in a topological space \(X\). We note that every open set in a topological space \(X\) is a semi-open set but clearly a semi-open set may not be an open set in \(X\). N. Levine also proved that the union of a collection of semi-open sets in a topological space is always semi-open. It is clear that a nowhere dense set in a space \(X\) is always not semi-open in \(X\) and the complement of a nowhere dense set in \(X\) is always semi-open in \(X\). In particular for any semi-open set \(S\) in a space \(X\), the difference of the closure of \(S\) and \(S\) is not semi-open in \(X\). We observe that the intersection of two semi-open sets in a space \(X\) may not be a semi-open set in \(X\). A point \(x\) in \(X\) is called a semi-limit point of a subset \(A\) of \(X\) if for each semi-open set \(U\) of \(X\) containing the point \(x\), the intersection of \(A\) and \(U\) contains a point other than \(x\). The set of all semi-limit points of a set \(A\) is called the semi-derived set of \(A\) and is denoted by \(A^{sd}\). The semi-closure of a set \(A\), denoted by \(scl(A)\), is the union of \(A\) and its semi-derived set \(A^{sd}\). A function \(f: (X,T) \rightarrow (Y,T)\) is called semi-continuous (or irresolute) if the inverse image of each open (or semi-open) set in \(Y\) is semi-open in \(X\).

A. CHARACTERIZATION OF D-SPACES

In this section, we give some characterizations of D-spaces.

THEOREM 1. In a topological space \((X,T)\), the following are equivalent:

(i) \((X,T)\) is a D-space.
(ii) Every pair of nonempty open sets has a nonempty intersection.
(iii) Every open set in \(X\) is connected.

PROOF: Easy.

THEOREM 2. A topological space \(X\) is a D-space if and only if for each nonempty semi-open set \(S\) in \(X\), \(scl(S) = X\).

PROOF: NECESSITY. Let \(S\) be a nonempty semi-open set in \(X\). Let \(x \in X\). Let \(T \in SO(X)\) such that \(x \in T\). Then there exist nonempty open sets \(U\) and \(V\) in \(X\) such that \(U \subseteq T \subseteq U\) and \(V \subseteq S \subseteq V\). By hypothesis, \(\overline{U} = X = \overline{V}\). If \(S \cap T = \emptyset\), then \(U \cap V = \emptyset\). Then by theorem 1, \(X\) is not a D-space. But this is a contradiction. Hence we have \(S \cap T \neq \emptyset\). Thus \(x \in scl(S)\), so \(X \subseteq scl S\). Hence, \(scl(S) = X\).
SUFFICIENCY: Suppose that for each nonempty semi-open set $S$ in $X$, we have $\text{scl}(S) = X$. Let $U$ be a nonempty open set in $X$. Then $U$ is a nonempty semi-open set in $X$. Hence by hypothesis, $\text{scl}(U) = X$. But $\text{scl}(U)$ being the intersection of all semi-closed sets containing $U$, and each closed set being a semi-closed set, it follows immediately that $\text{scl}(U) \subseteq \text{cl}(U)$. Thus $\text{cl}(U) = X$. Therefore, $X$ is a D-space.

THEOREM 3. A topological space $X$ is a D-space if and only if every pair of nonempty semi-open sets has a non-empty intersection.

PROOF: NECESSITY. Suppose $X$ is a D-space. Let $S$ and $T$ be nonempty semi-open sets in $X$. Then there exist $U$ and $V$ nonempty open sets in $X$ such that $U \subseteq S \subseteq \text{cl}(U), \ V \subseteq T \subseteq \text{cl}(V)$. By Theorem 1, $U \cap V \neq \emptyset$. Hence, $S \cap T \neq \emptyset$.

SUFFICIENCY. Let $S$ be a nonempty semi-open set in $X$. Let $x \in X$. Let $T \in \text{SO}(X)$ such that $x \in T$. By hypothesis, $S \cap T \neq \emptyset$. Hence, $x \in \text{scl}(S)$. Thus $X = \text{scl}(S)$. So by Theorem 2, $X$ is a D-space.

We note that a D-space is never a semi-T2 space though it can be a semi-T1 space.

Now in the following, we define S-connected spaces and examine their relation with D-spaces.

DEFINITION 4. A space $X$ is said to be S-connected if $X$ cannot be written as the disjoint union of two nonempty semi-open sets.

We observe that an S-connected space is connected but a connected space may not be an S-connected space. For example, $R$ with the usual topology.

THEOREM 5. Every D-space is S-connected and hence connected.

PROOF: Suppose $X$ is a D-space. If $X$ is not S-connected, then there exist two nonempty semi-open sets $S$ and $T$ such that $X = S \cup T$ and $S \cap T = \emptyset$. Hence $\text{scl}(S) \neq X$. But then by Theorem 2, $X$ is not a D-space which is a contradiction. Thus $X$ is S-connected.

We note that a connected space may not be a D-space. For example, $R$ with the usual topology.

THEOREM 6. A topological space $X$ is a D-space if and only if every nonempty semi-open set in $X$ is S-connected.

PROOF: NECESSITY. Let $Y$ be a nonempty semi-open set in $X$. If $Y$ is not S-connected, then there exist two nonempty semi-open sets $T_1$ and $T_2$ in $Y$ such that $Y = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$. Then $T_1$ and $T_2$ are two disjoint nonempty semi-open sets in $X$. So by Theorem 3, $X$ is not a D-space. But this is a contradiction. Hence $Y$ is S-connected.

SUFFICIENCY. Suppose that $X$ is not a D-space. Then by Theorem 1, there exist nonempty open sets $U$ and $V$ in $X$ such that $U \cap V = \emptyset$. Let $Y = U \cup V$. Then $Y$ is semi-open in $X$. Also $U$ and $V$ are open in $Y$ and hence semi-open in $X$. Thus $Y$ is not S-connected. This contradiction shows that $X$ is a D-space.

THEOREM 7. A space with a dense D-subspace is itself a D-space.

PROOF: Theorem 5(i) of (LEVINE [1968]).

THEOREM 8. If $A$ is a D-subspace of a space $X$, then $\text{scl}(A)$ is also a D-subspace of $X$. 
PROOF: Since A is dense in scl(A), the result follows from Theorem 7.

THEOREM 9. Suppose X is a D-space and C is a D-subset of X. Suppose further that the subspace X – C of X has an open set V. Then C ∪ V is a D-subset of X.

PROOF: Theorem 4.9 of (FATTEH AND SINGH [1983]).

Our next result is a generalization of this Theorem.

THEOREM 10. Suppose X is a D-space and C is a D-subset of X. Suppose further that the subspace X – C of X has a semi-open set S. Then C ∪ S is a D-subset of X.

PROOF: Since every nonempty semi-open set contains a nonempty open set, the result follows from Theorem 9.

In the following, we give a characterization of D-spaces in terms of regular semi-open sets. We need the following result.

THEOREM 11. A topological space X is a D-space if and only if X has no proper regularly open set.

PROOF: Theorem 3.2 of (FATTEH AND SINGH [1983]).

DEFINITION 12. A subset A of a topological space X is regular semi-open if there is a regular open set U such that U ⊆ A ⊆ cl(U).

THEOREM 13. A topological space X is a D-space if and only if X has no proper regularly semi-open set.

PROOF: NECESSITY. If X has a proper regularly semi-open set A, then there exists a regularly open set U such that U ⊆ A ⊆ cl(U). Clearly U is also proper. Hence by Theorem 11, X is not a D-space. This contradiction shows that X does not have any proper regularly semi-open set.

SUFFICIENCY. Suppose X is not a D-space. Then by Theorem 11, there exists a proper regularly open set U. Clearly U is a proper regularly semi-open set which is a contradiction. Thus X is a D-space.

Now we give a characterization of semi-open sets in a D-space. We need the following easy lemma.

LEMMA 14. Let (X, T) be a D-space. Let U and V be non-empty open sets in X. Then cl(U ∩ V) = (cl(U) ∩ cl(V)).

PROOF: Easy.

THEOREM 15. Let (X, T) be a D-space. Let A, B ∈ SO(X). Then A ∩ B ∈ SO(X).

PROOF: Let U, V ∈ T such that U ⊆ A ⊆ cl(U), V ⊆ B ⊆ cl(V).
Then U ∩ V ⊆ A ∩ B ⊆ (cl(U) ∩ cl(V)). By applying Lemma 14, we have:
U ∩ V ⊆ A ∩ B ⊆ (cl(U) ∩ cl(V)). Thus A ∩ B ∈ SO(X).

THEOREM 16. If (X, T) is a topological space and T is finite, then (X, T) is a D-space if and only if the intersection of all non-void semi-open subsets of X is non-void.
PROOF: NECESSITY. If possible then suppose that \( \bigcap (\text{SO}(X) - \{\emptyset\}) = \emptyset \). Then we assert that \( \bigcap (T - \{\emptyset\}) = \emptyset \). For that let \( x \in X \). Then there exists \( S_x \in (\text{SO}(X) - \{\emptyset\}) \) such that \( x \notin S_x \). So there exists \( U_x \in (T - \{\emptyset\}) \) such that \( U_x \subseteq S_x \subseteq \text{cl}(U_x) \). Hence \( x \notin U_x \). Thus \( \bigcap (T - \{\emptyset\}) = \emptyset \). Then it follows easily by Theorem 1 that \( X \) is not a D-space. By this contradiction, we conclude that \( \bigcap (\text{SO}(X) - \{\emptyset\}) \neq \emptyset \).

SUFFICIENCY. Suppose that \( \bigcap (\text{SO}(X) - \{\emptyset\}) \neq \emptyset \). Since \( T \subseteq \text{SO}(X) \). Hence \( \bigcap (T - \{\emptyset\}) \neq \emptyset \). Then by Theorem 1, it follows immediately that \( (X,T) \) is a D-space.

B. CHARACTERIZATION OF IMAGES AND PRE-IMAGES OF D-SPACES

In the present section, we will show that some mappings preserve the D-space structure of their images and pre-images.

THEOREM 17. The semi-continuous image of a D-space is a D-space.

PROOF: Let \( f : X \to Y \) be a surjective semi-continuous function where \( X \) is a D-space. If \( Y \) is not a D-space, then by Theorem 1, there exist nonempty open sets \( U \) and \( V \) in \( Y \) such that \( U \cap V = \emptyset \). Then \( f^{-1}(U) \) and \( f^{-1}(V) \) are nonempty disjoint semi-open sets. So by Theorem 3, \( X \) is not a D-space. But this is a contradiction. Hence \( Y \) is a D-space.

COROLLARY 18. The irresolute image of a D-space is a D-space.

THEOREM 19. Let \( (X,T) \) and \( (Y,T^*) \) be topological spaces and suppose that \( f : X \to Y \) is a function. Then prove the following.

(i) If \( f \) is one to one and semi-open and if \( (Y,T^*) \) is a D-space, then \( (X,T) \) is a D-space.

(ii) If \( f \) is one to one and pre-semi open and if \( (Y,T^*) \) is a D-space, then \( (X,T) \) is a D-space.

(iii) If \( f \) is onto and \( \text{SO}(X) = \{ f^{-1}(U) : U \in T^* \} \), then \( (X,T) \) is a D-space if and only if \( (Y,T^*) \) is a D-space.

(iv) If \( f \) is onto and \( \text{SO}(X) = \{ f^{-1}(U) : U \in \text{SO}(Y) \} \), then \( (X,T) \) is a D-space if and only if \( (Y,T^*) \) is a D-space.

PROOF: (i) If \( (X,T) \) is not a D-space, then by Theorem 1, there exist disjoint nonempty open sets \( A \) and \( B \) in \( X \). Then \( f(A) \) and \( f(B) \) are disjoint nonempty semi-open sets in \( Y \) and by Theorem 3, \( Y \) is not a D-space. But this is a contradiction to our hypothesis, so \( (X,T) \) is a D-space.

(ii) The straightforward proof (similar to (i)) is omitted.

(iii) Necessity follows from Theorem 17. To show the sufficiency, suppose that \( (X,T) \) is not a D-space. Then by Theorem 3, there exist disjoint nonempty semi-open sets \( A \) and \( B \) in \( X \). Then there exist non-empty open sets \( U \) and \( V \) in \( Y \) such that \( A = f^{-1}(U) \) and \( B = f^{-1}(V) \). Now \( f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) = A \cap B = \emptyset \). Since \( f \) is onto. So \( U \cap V = \emptyset \). Thus \( U \) and \( V \) are disjoint non-empty open sets in \( Y \). So by Theorem 1, \( Y \) is not a D-space. But this is a contradiction. Hence \( (X,T) \) is a D-space.
(iv) The necessity follows from corollary 18. To prove sufficiency, suppose \((X,T)\) is not a \(D\)-space. Then by Theorem 3, there exist disjoint nonempty semi-open sets \(A\) and \(B\) in \(X\). Then \(\emptyset = A \cap B = f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)\) for some nonempty semi-open sets \(U\) and \(V\) in \(Y\). Also \(U\) and \(V\) are disjoint as \(f\) is onto. Then by Theorem 3, it follows that \(Y\) is not a \(D\)-space. This leads us to a contradiction. Hence it is proved that \((X,T)\) is a \(D\)-space.

**THEOREM 20.** Every semi-continuous function from a \(D\)-space into a \(T_2\) - space is a constant function.

**PROOF:** Let \(f: X \rightarrow Y\) be semi-continuous, where \(X\) is a \(D\)-space and \(Y\) is a \(T_2\) - space. Let \(x_1, x_2 \in X\) such that \(f(x_1) \neq f(x_2)\). Since \(Y\) is \(T_2\), so there exist disjoint open sets \(V_1\) and \(V_2\) in \(Y\) such that \(f(x_1) \in V_1\) and \(f(x_2) \in V_2\). Then \(f^{-1}(V_1)\) and \(f^{-1}(V_2)\) are disjoint nonempty semi-open sets in \(X\). Hence by Theorem 3, \(X\) is not a \(D\)-space. But this is a contradiction. Thus \(f\) is a constant function.

**THEOREM 21.** Prove that every irresolute function from a \(D\) - space into semi -\(T_2\) space is constant.

**PROOF:** Let \(f: X \rightarrow Y\) be an irresolute function, where \(X\) is a \(D\) - space and \(Y\) is a semi-\(T_2\) - space. Let \(x, y \in X\) such that \(f(x) \neq f(y)\). Since \(Y\) is semi-\(T_2\), there exist disjoint semi-open sets \(U\) and \(V\) in \(Y\) such that \(f(x) \in U\) and \(f(y) \in V\). Since \(f\) is an irresolute, \(f^{-1}(U)\) and \(f^{-1}(V)\) are disjoint nonempty semi-open sets in \(X\). Hence by Theorem 3, \(X\) is not a \(D\)-space. This gives a contradiction. Hence \(f\) is a constant function.

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