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Raja M. Latif

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

ON CHARACTERIZATIONS OF MAPPINGS

by

RAJA MOHAMMAD LATIF
Department of Mathematical Sciences
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia

ABSTRACT

In the literature various kinds of mappings between topological spaces have been defined. Their properties and characterizations also have been studied. The purpose of the present paper is to continue to explore further properties and characterizations of semi-continuous, irresolute, semi-open, pre-semi-open, and pre-semi-closed functions.

INTRODUCTION

In 1963, N. Levine introduced the concept of semi-open set by defining a subset A of a topological space X to be semi-open if there exists an open set U in X such that A contains U and A is contained in the closure of U in X. He proved that a set A in a topological space X is semi-open if and only if A is contained in the closure of the interior of A in X. SO(X) will denote the class of all semi-open sets in a topological space X. We note that every open set in a topological space X is a semi-open set but clearly a semi-open set may not be an open set in X. N. Levine also proved that the union of a collection of semi-open sets in a topological space is always semi-open. However the intersection of even two semi-open sets may not be a semi-open set. It is clear that a nowhere dense set in a space X is always not semi-open in X and the complement of a nowhere dense set in X is always semi-open in X. In particular for any semi-open set S in a space X, the difference of the closure of S and S is not semi-open in X.

A. SEMI-CONTINUOUS FUNCTIONS

In (LEVINE [1963]), N. Levine introduced the notion of semi-continuous mapping. In (BISWAS [1970]), N. Biswas gave a certain characterization of semi-continuous mappings. The purpose of this section is to investigate further properties and characterizations of such semi-continuous functions.

1. DEFINITION: Let $f: X \longrightarrow Y$ be single-valued where X and Y are topological spaces. The $f: X \longrightarrow Y$ is called semi-continuous if and only if, for any V open in Y, $f^{-1}(V)$ is semi-open in X.

2. DEFINITION: Let X be a topological space. We say that a subset M_X of X is a semi-neighbourhood of a point x of X if and only if there exists a semi-open set A such that x is in A and A is a subset of M_X.

3. DEFINITION: Let X be a toplogical space. Let p be any point of X and A be any subset of X. We will say that p is a semi-limit point of the set A if and only if U and A contain at least one common point other than that of p,

for every $U \in SO(X)$ such that $p \in U$. The set of all semi-limit points of A is said to be the semi-derived set of A and is denoted by A^{sd}.

$A \cup A^{sd}$ is defined to be the semi-closure of A and is denoted by $sc\ell\, A$.

4. THEOREM. *Let $f : X \to Y$ be single-valued where X and Y are topological spaces. Then the following are equivalent.*

(a) *The function f is semi-continuous.*

(b) *For each point p in X and each open set O in Y with $f(p) \in O$, there is a semi-open set A in X such that $p \in A$ and $f(A) \subseteq O$.*

(c) *For each x in X and each neighbourhood U of $f(x)$, there is a semi-neighbourhood $\mathcal{V}$ of x such that $f(\mathcal{V}) \subseteq U$.*

(d) *For each subset B of Y, $sc\ell[f^{-1}(B)] \subseteq f^{-1}(c\ell\, B)$.*

PROOF: Theorem 4 of (BISWAS [1970]).

5. THEOREM. *Let $f : X \to Y$ be single-valued where X and Y are topological spaces. Then the function f is semi-continuous if and only if $f(A^{sd}) \subseteq f(A) \cup [f(A)]^d$, for every $A \subseteq X$.*

PROOF: NECESSITY. Let $A \subseteq X$ and $a_0 \in A^{sd}$. Assume that $f(a_0) \notin f(A)$. Let $\mathcal{V}$ be a neighbourhood of $f(a_0)$. Since f is semi-continuous, so by Theorem 4, there exists a semi-neighbourhood U of a_0 for which

$f(U) \subseteq \mathcal{V}$. From $a_o \in A^{sd}$, it implies that $U \cap A \neq \phi$. Fix $a \in U \cap A$. Then $a \in U$ and $a \in A$. Hence $f(a) \in \mathcal{V}$ and $f(a) \in f(A)$. Since $f(a_0) \notin f(A)$. So $f(a) \neq f(a_0)$. Thus every neighbourhood of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$. Hence we conclude that $f(a) \in [f(A)]^d$. This proves necessity.

SUFFICIENCY. Assume that f is not semi-continuous. Then by Theorem .2, there exists $a_0 \in X$ and a neighbourhood $\mathcal{V}$ of $f(a_0)$ such that every semi-neighbourhood U of a_0 contains at least one element $a \in U$ such that $f(a) \notin \mathcal{V}$. Put $A = \{a \in X : f(a) \notin \mathcal{V}\}$. Then $a_0 \notin A$ since $f(a_0) \in \mathcal{V}$, and therefore $f(a_0) \notin f(A)$; also $f(a_0) \notin [f(A)]^d$ since

$$f(A) \cap (\mathcal{V} - \{f(a_0)\}) = \phi.$$

Thus $f(A^{sd})$ is not contained in $f(A) \cup [f(A)]^d$. But this is a contradiction to our hypothesis. Hence, f is semi-continuous.

.6. THEOREM. *Let X and Y be topological spaces. Let $f : X \to Y$ be an injection. Then f is semi-continuous if and only if*

$$f(A^{sd}) \subseteq [f(A)]^d, \quad \text{for every} \quad A \subseteq X.$$

PROOF: NECESSITY. Let $A \subseteq X, a_0 \in A^{sd}$ and $\mathcal{V}$ be a neighbourhood of $f(a_0)$. Since f is semi-continuous, by Theorem .4, there exists a semi-neighbourhood U of a_0 for which $f(U) \subseteq \mathcal{V}$. But $a_0 \in A^{sd}$, hence there exists an element $a \in U \cap A$ such that $a \neq a_0$; then $f(a) \in f(A)$ and, since f is an injection, $f(a) \neq f(a_0)$. Thus every neighbourhood $\mathcal{V}$ of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$; consequently $f(a_0) \in [f(A)]^d$. We have therefore, $f(A^{sd}) \subseteq [f(A)]^d$.

SUFFICIENCY. Follows from Theorem 5.

7. THEOREM. *Let (X,T) be a topological space. Let $A \subseteq X$. Then*

$$Int\ A = X - c\ell(X - A), \quad \text{for all} \quad A \subseteq X.$$

PROOF: This is easy.

8. DEFINITION: Let X be a topological space and let $A \subseteq X$. A point $x \in X$ is said to be a semi-interior point of A if and only if there exists $U \in SO(X)$ such that $x \in U \subseteq A$. The set of all semi-interior points of A is said to be the semi-interior of A and is denoted be $s\,\mathrm{Int}\,A$.

9. THEOREM. *Let (X,T) be a topological space. Let $B \subseteq X$. Then*

$$s\ Int\ B = X - sc\ell(X - B).$$

PROOF: Clearly, $s\,\mathrm{Int}\,B \subseteq B$ Thus $X - B \subseteq X - s\,\mathrm{Int}\,B$, from which it follows that $sc\ell(X - B) \subseteq sc\ell(X - s\,\mathrm{Int}\,B)$, i.e., $sc\ell(X - B) \subseteq X - s\,\mathrm{Int}\,B$. Hence $s\,\mathrm{Int}\,B \subseteq X - sc\ell(X - B)$. On the other hand if $x \in X - sc\ell(X - B)$, then $x \notin s\mathcal{cl}(X - B)$. Hence there exists $U_x \in SO(X)$ such that $x \in U_x$ and $U_x \cap (X - B) = \phi$. Then $x \in U_x \in SO(X)$ and $U_x \subseteq B$ so that $x \in s \text{ Int } B$. This shows that $X - s\mathcal{cl}(X - B) \subseteq s \text{ Int } B$ Thus, $s \text{ Int } B = X - s\mathcal{cl}(X - B)$.

10. THEOREM. *Let* X *and* Y *be topological spaces. Then a function* $f : X \to Y$ *is semi-continuous if and only if*

$$f^{-1}(\text{Int } B) \subseteq s \text{ Int } f^{-1}(B), \quad \text{for each } B \subseteq Y.$$

PROOF: NECESSITY. Let $B \subseteq Y$. Then by Theorem 7,

$$\text{Int } B = Y - \mathcal{cl}(Y - B).$$

Thus

$$\begin{aligned} f^{-1}(\text{Int } B) &= f^{-1}(Y - \mathcal{cl}(Y - B)) \\ &= X - f^{-1}\mathcal{cl}(Y - B). \end{aligned}$$

Since f is semi-continuous, we have by Theorem 4,

$$s\mathcal{cl} f^{-1}(Y - B) \subseteq f^{-1}(\mathcal{cl}(Y - B)).$$

Hence, $f^{-1}(\text{Int } B) \subseteq X - s\mathcal{cl}\, f^{-1}(Y - B)$. Thus $f^{-1}(\text{Int } B) \subseteq X - s\mathcal{cl}(X - f^{-1}(B))$. Now by applying Theorem 9, we conclude that

$$f^{-1}(\text{Int } B) \subseteq s \text{ Int } f^{-1}(B).$$

SUFFICIENCY. Let B be an arbitrary open set in Y. Then $B = \text{Int } B$. Hence, by hypothesis,

$$f^{-1}(B) \subseteq s \text{ Int } f^{-1}(B).$$

But $s \text{ Int } f^{-1}(B) \subseteq f^{-1}(B)$. Therefore, $f^{-1}(B) = s \text{ Int } f^{-1}(B)$. Thus the set $f^{-1}(B)$ is semi-open. Consequently f is semi-continuous.

11. THEOREM. *Let X and Y be topological spaces. Let $f : X \to Y$ be a mapping and $g : X \to X \times Y$ be the graph mapping of f, given by $g(x) = (x, f(x))$ for every point $x \in X$. Then g is semi-continuous if and only if f is semi-continuous.*

PROOF: NECESSITY. Let $x \in X$ and $\mathcal{V}$ be any open set containing $f(x)$. Then $X \times \mathcal{V}$ is an open set in $X \times Y$ containing $g(x)$. Since g is semi-continuous, there exists a semi-open set U containing x such that $g(U) \subseteq X \times \mathcal{V}$. Since g is the graph mapping of f, we have $f(U) \subseteq \mathcal{V}$. By Theorem 4, it follows that f is semi-continuous.

SUFFICIENCY. Suppose f is semi-continuous. Let $W = U \times \mathcal{V}$ be a basic open set in $X \times Y$. Then $g^{-1}(W) = U \cap f^{-1}(\mathcal{V})$. Then U is open in X and $f^{-1}(\mathcal{V})$ is semi-open in X. Hence $g^{-1}(W)$ being the intersection of an open set and a semi-open set, is semi-open. Since the union of a collection of semi-open sets is semi-open. Thus g is semi-continuous.

B. IRRESOLUTE FUNCTIONS

In this section, the functions to be considered are those for which inverses of semi-open sets are semi-open. We investigate some new properties and characterizations of such functions.

12. DEFINITION: Let X and Y be topological spaces. A function $f: X \to Y$ is said to be irresolute if and only if, for any semi-open subset S of Y, $f^{-1}(S)$ is semi-open in X.

13. THEOREM. *A function $f: X \to Y$, where X and Y are topological spaces, is irresolute if and only if for each point p in X and each semi-open set B in Y with $f(p) \in B$, there is a semi-open set A in X such that $p \in A$, $f(A) \subseteq B$.*

PROOF: NECESSITY. Let $p \in X$ and $B \in SO(Y)$ such that $f(p) \in B$. Let $A = f^{-1}(B)$. Since f is irresolute, A is semi-open in X. Also $p \in f^{-1}(B) = A$ as $f(p) \in B$. We also have

$$f(A) = ff^{-1}(B) \subseteq B.$$

SUFFICIENCY. Let $B \in SO(Y)$, let $A = f^{-1}(B)$. We show that A is semi-open in X. For this let $x \in A$. It implies that $f(x) \in B$. Then by hypothesis, there exists $A_x \in SO(X)$ such that $x \in A_x$ and $f(A_x) \subseteq B$. Then $A_x \subseteq f^{-1}f(A_x) \subseteq f^{-1}(B) = A$. Thus $A = \cup_{x \in A} A_x$. It follows that A is semi-open in X. Hence f is irresolute.

14. THEOREM. *A function* $f : X \to Y$ *is irresolute if and only if for each* x *in* X, *the inverse of every semi-neighbourhood of* $f(x)$ *is a semi-neighbourhood of* x.

PROOF: NECESSITY. Let $x \in X$ and let B be a semi-neighbourhood of $f(x)$. Then there exists $U \in SO(X)$ such that $f(x) \in U \subseteq B$. This implies that $x \in f^{-1}(U) \subseteq f^{-1}(B)$. Since f is irresolute, so $f^{-1}(U) \in SO(X)$. Hence $f^{-1}(B)$ is a semi-neighbourhood of x.

SUFFICIENCY. Let $B \in SO(Y)$. Put $A = f^{-1}(B)$. Let $x \in A$. Then $f(x) \in B$. But then B, being semi-open, is a semi-neighbourhood of $f(x)$. So by hypothesis, $A = f^{-1}(B)$ is a semi-neighbourhood of x. Hence by definition, there exists $A_x \in SO(X)$ such that $x \in A_x \subseteq A$. Thus

$$A = \bigcup_{x \in A} A_x.$$

It follows that A is semi-open in X. Therefore, f is irresolute.

15. THEOREM. *A function* $f : X \to Y$ *is irresolute if and only if for each* x *in* X *and each semi-neighbourhood* U *of* $f(x)$, *there is a semi-neighbourhood* $\mathcal{V}$ *of* x *such that* $f(\mathcal{V}) \subseteq U$.

PROOF: NECESSITY. Let $x \in X$ and let U be a semi-neighbourhood of $f(x)$. Then there exists $O_{f(x)} \in SO(Y)$ such that $f(x) \in O_{f(x)} \subseteq U$. It follows that $x \in f^{-1}(O_{f(x)}) \subseteq f^{-1}(U)$. By hypothesis, $f^{-1}(O_{f(x)}) \in SO(X)$.

Let $\mathcal{V} = f^{-1}(U)$. Then it follows that $\mathcal{V}$ is a semi-neighbourhood of x and $f(\mathcal{V}) = ff^{-1}(U) \subseteq U$.

SUFFICIENCY. Let $B \in SO(Y)$. Put $O = f^{-1}(B)$. Let $x \in O$. Then $f(x) \in B$. Thus B is a semi-neighbourhood of $f(x)$. So by hypothesis, there exists a semi-neighbourhood $\mathcal{V}_x$ of x such that $f(\mathcal{V}_x) \subseteq B$. Thus it follows that $x \in \mathcal{V}_x \subseteq f^{-1}f(\mathcal{V}_x) \subseteq f^{-1}(B) = O$. Since $\mathcal{V}_x$ is a semi-neighbourhood of x, so there exists an $O_x \in SO(X)$ such that $x \in O_x \subseteq \mathcal{V}_x$. Hence $x \in O_x \subseteq O, O_x \in SO(X)$. Thus $O = \cup_{x \in O} O_x$. It follows that O is semi-open in X. Therefore, f is irresolute.

16. THEOREM. *A function* $f : X \to Y$ *is irresolute, if and only if, for all* $B \subseteq Y$, $s c\ell[f^{-1}(B)] \subseteq f^{-1}[s c\ell(B)]$.

PROOF: Theorem 1.6 of [CROSSLEY AND HILDEBRAND: 1972]

17. THEOREM. *Let* $f : X \to Y$ *be a function. Then* f *is irresolute if and only if* $f^{-1}(s\ Int\ B) \subseteq s\ Int\ f^{-1}(B)$, *for all* $B \subseteq Y$.

PROOF: NECESSITY. Let $B \subseteq Y$. Then by Theorem 9,

$$s \text{ Int } B = Y - s c\ell(Y - B).$$

Hence

$$f^{-1}(s\ \mathrm{Int}\ B) = f^{-1}(Y - s\ell(Y - B))$$

$$= f^{-1}(Y) - f^{-1}(s\ell(Y - B))$$

$$= X - f^{-1}(s\ell(Y - B)).$$

Since f is irresolute, so by Theorem 16,

$$s\ell\ f^{-1}(Y - B) \subseteq f^{-1}[s\ell(Y - B)].$$

Hence $f^{-1}(s\ \mathrm{Int}\ B) \subseteq X - s\ell[f^{-1}(Y - B)]$. Thus $f^{-1}(s\ \mathrm{Int}\ B) \subseteq X - s\ell[f^{-1}(Y) - f^{-1}(B)]$. Therefore $f^{-1}(s\ \mathrm{Int}\ B) \subseteq X - s\ell[X - f^{-1}(B)]$. By Theorem 9, it follows that

$$f^{-1}(s\ \mathrm{Int}\ B) \subseteq s\ \mathrm{Int}\ f^{-1}(B).$$

SUFFICIENCY. Let $B \in SO(Y)$. Then clearly $B = s\ \mathrm{Int}\ B$. So by hypothesis, $f^{-1}(B) \subseteq s\ \mathrm{Int}\ f^{-1}(B)$. But we know that $s\ \mathrm{Int}\ f^{-1}(B) \subseteq f^{-1}(B)$. Thus $f^{-1}(B) = s\ \mathrm{Int}\ f^{-1}(B)$. Hence $f^{-1}(B) \in SO(X)$. Therefore, f is irresolute.

18. THEOREM. *A function* $f : X \to Y$ *is irresolute if and only if*

$$f(A^{sd}) \subseteq f(A) \cup (f(A))^{sd}, \quad \text{for all} \quad A \subseteq X.$$

PROOF: NECESSITY. Let $f : X \to Y$ be an irresolute.

Let $A \subseteq X$, and $a_0 \in A^{sd}$. Assume that $f(r_0) \notin f(A)$ and let $\mathcal{V}$ denote a semi-neighbourhood of $f(a_0)$. Since f is irresolute, so by Theorem 15, there exists a semi-neighbourhood U of a_0 such that $f(U) \subseteq \mathcal{V}$. From $a_0 \in A^{sd}$, it follows that $U \cap A \neq \phi$; there exists, therefore, at least one element $a \in U \cap A$ such that $f(a) \in f(A)$ and $f(a) \in \mathcal{V}$. Since $f(a_0) \notin f(A)$, we have $f(a) \neq f(a_0)$. Thus every semi-neighbourhood of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$, consequently, $f(a_0) \in (f(A))^{sd}$. This proves necessity of the condition.

SUFFICIENCY. Assume that f is not irresolute. Then by Theorem 15, there exists $a_0 \in X$ and a semi-neighbourhood $\mathcal{V}$ of $f(a_0)$ such that every semi-neighbourhood U of a_0 contains at least one element $a \in U$ for which $f(a) \notin \mathcal{V}$. Put $A = \{a \in X : f(a) \notin \mathcal{V}\}$. Then $a_0 \notin A$ since $f(a_0) \in \mathcal{V}$, and therefore $f(a_0) \notin f(A)$; also $f(a_0) \notin [f(A)]^{sd}$ since $f(A) \cap (\mathcal{V} - \{f(a_0)\}) = \phi$. It follows that $f(a_0) \in f(A^{sd}) - [f(A) \cup (f(A))^{sd}] \neq \phi$, which is a contradiction to the given condition. The condition of the Theorem is therefore sufficient and the theorem is proved.

19. THEOREM. *Let $f : X \to X^*$ be a 1-1 function. Then f is an irresolute if and only if*

$$f(A^{sd}) \subseteq (f(A))^{sd}, \quad \textit{forall} \quad A \subseteq X.$$

PROOF: NECESSITY. Let f be irresolute. Let $A \subseteq X$, $a_0 \in A^{sd}$ and $\mathcal{V}$ be a semi-neighbourhood of $f(a_0)$. Since f is irresolute, so by Theorem

.15, there exists a semi-neighbourhood U of a_0 such that $f(U) \subseteq \mathcal{V}$. But $a_0 \in A^{sd}$; hence there exists an element $a \in U \cap A$ such that $a \neq a_0$; then $f(a) \in f(A)$ and, since f is 1-1, $f(a) \neq f(a_0)$. Thus every semi-neighbourhood $\mathcal{V}$ of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$; consequently $f(a_0) \in (f(A))^{sd}$. We have therefore $f(A^{sd}) \subseteq (f(A))^{sd}$.

SUFFICIENCY: Follows from Theorem 18.

20. THEOREM. *Let X and Y be topological spaces and $Z = X \times Y$ be the topological product. Let $A \in SO(X)$ and $B \in SO(Y)$. Then $A \times B \in SO(X \times Y)$.*

PROOF: Theorem 11 of (LEVINE [1963]).

21. THEOREM. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ be the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. If g is irresolute, then f is irresolute.*

PROOF: Let $x \in X$ and $\mathcal{V} \in SO(Y)$ such that $f(x) \in \mathcal{V}$. Then by Theorem 20, $X \times \mathcal{V}$ is a semi-open subset of $X \times Y$ containing $g(x)$ and hence by Theorem 13, there exists $U \in SO(X)$ such that $x \in U$ and $g(U) \subseteq X \times \mathcal{V}$. By the definition of g, we have $f(U) \subseteq \mathcal{V}$. Therefore, by Theorem 13, f is irresolute.

The converse of Theorem 21 is not true as the following example shows.

22. EXAMPLE. *Let* $X = \{1,2,3\}$ *and*

$$T = \{\phi, \{1\}, \{2\}, \{1,2\}, X\}.$$

Let $f : (X,T) \to (X,T)$ *be a function defined as follows:*

$$f(1) = 2, \quad f(2) = 1 \quad \text{and} \quad f(3) = 3.$$

Then f *is irresolute but*

$$g : (X,T) \to (X \times X, T \times T)$$

is not irresolute. Let $\mathcal{V} = \{(1,1),(1,3),(3,3)\}$. *Then clearly* $\mathcal{V}$ *is semi-open in* $X \times X$ *but* $g^{-1}(\mathcal{V}) = \{3\}$ *is not semi-open in* X. *This shows that* g *is not irresolute.*

23. DEFINITION: A topological space X is said to be semi-T_2 if for each two distinct points $x, y \in X$, there exist $U, \mathcal{V} \in SO(X)$ such that $x \in U$, $y \in \mathcal{V}$ and $U \cap \mathcal{V} = \phi$.

We note that every T_2-space is a semi-T_2-space but a semi-T_2-space need not be a T_2-space. For example, let $X = \{1,2,3\}$ and

$$T = \{\phi, \{1\}, \{2\}, \{1,2\}, X\}$$

be a topology on X. Then (X,T) is a semi-T_2-space but it is not a T_2-space.

.24. THEOREM. *If* $f : X \to Y$ *is an irresolute injection and* Y *is semi-*T_2*, then* X *is semi-*T_2*.*

PROOF: Let x_1 and x_2 be two distinct points of X. Since f is injective and Y is semi-T_2, there exist $\mathcal{V}_1, \mathcal{V}_2 \in SO(Y)$ such that $f(x_1) \in \mathcal{V}_1$, $f(x_2) \in \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \phi$. Then $x_1 \in f^{-1}(\mathcal{V}_1)$, $x_2 \in f^{-1}(\mathcal{V}_2)$ and $f^{-1}(\mathcal{V}_1) \cap f^{-1}(\mathcal{V}_2) = \phi$. Since f is irresolute, so $f^{-1}(\mathcal{V}_1), f^{-1}(\mathcal{V}_2) \in SO(X)$. Thus X is semi-T_2.

25. THEOREM. *If* $f : X \to Y$ *is irresolute and* Y *is semi-*T_2*, then the set* $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ *is semi-closed in* $X \times X$.

PROOF: Let $(x_1, x_2) \in (X \times X) - A$. Then $f(x_1) \neq f(x_2)$. Since Y is semi-T_2, there exist $\mathcal{V}_1, \mathcal{V}_2 \in SO(X)$ such that $f(x_1) \in \mathcal{V}_1$, $f(x_2) \in \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \phi$. Then $x_1 \in f^{-1}(\mathcal{V}_1)$, $x_2 \in f^{-1}(\mathcal{V}_2)$ and $f^{-1}(\mathcal{V}_1) \cap f^{-1}(\mathcal{V}_2) = \phi$. Since f is irresolute, $f^{-1}(\mathcal{V}_1)$ and $f^{-1}(\mathcal{V}_2)$ are semi-open in X. Put $U = f^{-1}(\mathcal{V}_1) \times f^{-1}(\mathcal{V}_2)$. Then $(x_1, x_2) \in U \in SO(X \times X)$ by Theorem 20. Also clearly $U \cap A = \phi$. Therefore, $(x_1, x_2) \notin scl\, A$ and hence A is semi-closed in $X \times X$.

26. THEOREM. *If* $f : X \to Y$ *is irresolute and* Y *is semi-*T_2*, then the graph* $G(f)$ *is semi-closed in* $X \times Y$.

PROOF: Let $(x, y) \notin G(f)$. Then $y \neq f(x)$. Since Y is semi-T_2, there exist $\mathcal{V}, W \in SO(Y)$ such that $y \in \mathcal{V}$, $f(x) \in W$ and $\mathcal{V} \cap W = \phi$.

Since f is irresolute, so by Theorem 13, there exists $U \in SO(X)$ such that $x \in U$ and $f(U) \subseteq W$. Then $f(U) \cap \mathcal{V} = \phi$. Therefore, $(x, y) \in U \times \mathcal{V} \in SO(X \times Y)$ and $(U \times \mathcal{V}) \cap G(f) = \phi$. This shows that $(x, y) \notin sc\ell(G(f))$ and thus $G(f)$ is semi-closed in $X \times Y$.

C. SEMI-OPEN FUNCTIONS

In (BISWAS [1969]), N. Biswas defined semi-open mappings as a generalization of open mappings and investigated several properties of such mappings. In (NOIRI [1973]), T. Noiri gave some more properties of semi-open mappings. The purpose of this section is to add another characterization of a semi-open mapping.

27. DEFINITION: Let X and Y be topological spaces. A function $f : X \to Y$ is called semi-open if and only if for each open set U in X, $f(U)$ is a semi-open set in Y.

28. THEOREM. *Let X and Y be two topological spaces. A necessary and sufficient condition for a mapping $f : X \to Y$ to be semi-open is that $f^{-1}(sc\ell B) \subseteq c\ell\, f^{-1}(B)$ for every subset B of Y.*

PROOF: Theorem 2 of (NOIRI [1973]).

29. THEOREM. *Let* X *and* Y *be topological spaces. A function* $f : X \to Y$ *is semi-open if and only if*

$$Int[f^{-1}(B)] \subseteq f^{-1}(s\ Int\ B), \quad \text{for all} \quad B \subseteq Y.$$

PROOF: NECESSITY. Let $B \subseteq Y$. Then $\text{Int}\, f^{-1}(B)$ is open in X. Since f is semi-open so $f[\text{Int}\, f^{-1}(B)]$ is semi-open in Y. Also we have

$$f[\text{Int}\, f^{-1}(B)] \subseteq ff^{-1}(B) \subseteq B.$$

Hence, $f[\text{Int}\, f^{-1}(B)] \subseteq s\ \text{Int}\ B$. Thus $\text{Int}\, f^{-1}(B) \subseteq f^{-1}(s\ \text{Int}\ B)$.

SUFFICIENCY. Let $B \subseteq Y$. Then $B^c = (Y - B) \subseteq Y$. By hypothesis, $\text{Int}\, f^{-1}(B^c) \subseteq f^{-1}[s\ \text{Int}(B^c)]$. By Theorem 7 and Theorem 9, we get

$$X - c\ell[X - f^{-1}(B^c)] \subseteq f^{-1}[Y - sc\ell(Y - B^c)].$$

Hence, $X - c\ell[f^{-1}(Y) - f^{-1}(B^c)] \subseteq X - f^{-1}(sc\ell\ B)$. Thus $X - c\ell[f^{-1}(B)] \subseteq X - f^{-1}(sc\ell\ B)$. Therefore $f^{-1}(sc\ell\ B) \subseteq c\ell[f^{-1}(B)]$. Then by Theorem 28, it follows that f is semi-open.

D. PRE-SEMI-OPEN FUNCTIONS

The purpose of this section is to discuss some new properties and characterizations of pre-semi-open functions.

30. DEFINITION: Let X and Y be topological spaces. Then a function $f : X \to Y$ is said to be pre-semi-open if and only if for each $A \in SO(X)$, $f(A) \in SO(Y)$.

31. THEOREM. *Let $f : X \to Y$ and $g : Y \to Z$ be any two pre-semi-open functions. Then the composition function $g \circ f : X \to Z$ is a pre-semi-open function.*

PROOF: Let $U \in SO(X)$. Then $f(U) \in SO(Y)$ since f is pre-semi-open. But then $g(f(U)) \in SO(Z)$ as g is pre-semi-open. Thus $(g \circ f)(U)$ is semi-open in Z. Hence, $g \circ f$ is pre-semi-open.

32. THEOREM. *$f : X \to Y$ is pre-semi-open if and only if for each $x \in X$ and for every $U \in SO(X)$ such that $x \in U$, there exists $\mathcal{V} \in SO(Y)$ such that $f(x) \in \mathcal{V}$ and $\mathcal{V} \subseteq f(U)$.*

PROOF: Routine.

33. THEOREM. *$f : X \to Y$ is pre-semi-open if and only if for each $x \in X$ and for every semi-neighbourhood U of x in X, there exists a semi-neighbourhood $\mathcal{V}$ of $f(x)$ in Y such that $\mathcal{V} \subseteq f(U)$.*

PROOF: NECESSITY. Let $x \in X$ and let U be a semi-neighbourhood of x. Then there exists $W \in SO(X)$ such that $x \in W \subseteq U$. Then

$f(x) \in f(W) \subseteq f(U)$. But $f(W) \in SO(Y)$ as f is pre-semi-open. Hence $\mathcal{V} = f(W)$ is a semi-neighbourhood of $f(x)$ and $\mathcal{V} \subseteq f(U)$.

SUFFICIENCY. Let $U \in SO(X)$. Let $x \in U$. Then U is a semi-neighbourhood of x. So by hypothesis, there exists a semi-neighbourhood $\mathcal{V}_{f(x)}$ of $f(x)$ such that $f(x) \in \mathcal{V}_{f(x)} \subseteq f(U)$. It follows at once that $f(U)$ is a semi-neighbourhood of $f(x)$. Thus $f(U)$ is a semi-neighbourhood of each of its points. Thus $f(U)$ is semi-open. Hence f is pre-semi-open.

34. THEOREM. *Let X and Y be topological spaces. Then a function $f : X \to Y$ is pre-semi-open if and only if $f(s\ \mathrm{Int}\ A) \subseteq s\ \mathrm{Int}\ f(A)$, forall $A \subseteq X$.*

PROOF: NECESSITY. Let $A \subseteq X$ Let $x \in s$ Int A. Then there exists $U_x \in SO(X)$ such that $x \in U_x \subseteq A$. So $f(x) \in f(U_x) \subseteq f(A)$ and by hypothesis, $f(U_x) \in SO(Y)$. Hence $f(x) \in s$ Int $f(A)$. Thus

$$f(s\ \mathrm{Int}\ A) \subseteq s\ \mathrm{Int}\ f(A).$$

SUFFICIENCY. Let $U \in SO(X)$. Then by hypothesis,

$$f(s\ \mathrm{Int}\ U) \subseteq s\ \mathrm{Int}\ f(U).$$

Since s Int $U = U$ as U is semi-open. Also s Int $f(U) \subseteq f(U)$. Hence $f(U) = s$ Int $f(U)$. Thus $f(U)$ is semi-open in Y. So f is pre-semi-open.

We remark that the equality does not hold in the above Theorem as the following example shows.

35. EXAMPLE: Let $X = Y = \{1,2\}$. Suppose X is antidiscrete and Y is discrete. Let $f = Id., A = \{1\}$. Then $\phi = f(s \text{ Int } A) \neq \text{Int } f(A) = \{1\}$.

36. THEOREM. *A function* $f : X \to Y$ *is pre-semi-open if and only if*

$$s \text{ Int } f^{-1}(B) \subseteq f^{-1}(s \text{ Int } B), \quad \text{for all} \quad B \subseteq Y.$$

PROOF: NECESSITY. Let $B \subseteq Y$. Since $s \text{ Int } f^{-1}(B)$ is semi-open in X and f is pre-semi-open, $f[s \text{ Int } f^{-1}(B)]$ is semi-open in Y. Also we have

$$f[s \text{ Int } f^{-1}(B)] \subseteq ff^{-1}(B) \subseteq B.$$

Hence, $f[s \text{ Int } f^{-1}(B)] \subseteq s \text{ Int } B$. Therefore $s \text{ Int } f^{-1}(B) \subseteq f^{-1}(s \text{ Int } B)$.

SUFFICIENCY. Let $A \subseteq X$. Then $f(A) \subseteq Y$.

Hence by hypothesis, we obtain

$$s \text{ Int } A \subseteq s \text{ Int } f^{-1}[f(A)] \subseteq f^{-1}[s \text{ Int } f(A)].$$

This implies that $f(s \text{ Int } A) \subseteq ff^{-1}[s \text{ Int } f(A)] \subseteq s \text{ Int } f(A)$. Thus $f(s \text{ Int } A) \subseteq s \text{ Int } f(A)$, for all $A \subseteq X$. Hence, by Theorem .34, f is pre-semi-open.

37. THEOREM. *Let* X *and* Y *be topological spaces. A necessary and sufficient condition for a mapping* $f : X \to Y$ *to be pre-semi-open is that* $f^{-1}(s\ell\, B) \subseteq s\ell\, f^{-1}(B)$ *for every subset* B *of* Y.

PROOF: NECESSITY. Let $B \subseteq Y$. Then $(Y-B) \subseteq Y$. By applying Theorem 36,

$$s\,\mathrm{Int}[f^{-1}(Y-B)] \subseteq f^{-1}[s\,\mathrm{Int}(Y-B)].$$

This implies that

$$s\,\mathrm{Int}[X-f^{-1}(B)] \subseteq f^{-1}[s\,\mathrm{Int}(Y-B)].$$

Now by applying Theorem 9, we obtain

$$X - scl\, f^{-1}(B) \subseteq f^{-1}(Y - scl\, B).$$

So $X - scl\, f^{-1}(B) \subseteq X - f^{-1}(scl\, B)$. Hence $f^{-1}(scl\, B) \subseteq scl\, f^{-1}(B)$.

SUFFICIENCY Let $B \subseteq Y$. Then $(Y-B) \subseteq Y$. By hypothesis, $f^{-1}[scl(Y-B)] \subseteq scl\, f^{-1}(Y-B)$. This implies $X - scl\, f^{-1}(Y-B) \subseteq X - f^{-1}[scl(Y-B)]$. Hence $X - scl[X - f^{-1}(B)] \subseteq f^{-1}[Y - scl(Y-B)]$. By applying Theorem 9, we have

$$s\,\mathrm{Int}\, f^{-1}(B) \subseteq f^{-1}(s\,\mathrm{Int}\, B).$$

Now from Theorem 36, it follows that f is pre-semi-open.

38. THEOREM. *Let X and Y be two topological spaces. If $f: X \to Y$ is an open and semi-continuous mapping, then the inverse image $f^{-1}(B)$ of each semi-open set B in Y is semi-open in X.*

PROOF: Theorem 3 of (NOIRI [1973]).

39. THEOREM. *Let* X, Y *and* Z *be three topological spaces, and let* $f : X \to Y$, $g : Y \to Z$ *be two mappings so that* $g \circ f : X \to Z$ *a pre-semi-open mapping. Then*

(1) *If* f *is open, semi-continuous and surjective, then* g *is pre-semi-open.*

(2) *If* g *is open, semi-continuous and injective, then* f *is pre-semi-open.*

PROOF:

(1) Let $\mathcal{V}$ be an arbitrary semi-open set in Y. Then $f^{-1}(\mathcal{V})$ is semi-open in X because f is irresolute by Theorem 38. Since $g \circ f$ is pre-semi-open and f is surjective, $g(\mathcal{V}) = (g \circ f)[f^{-1}(\mathcal{V})]$ is semi-open in Z. This shows that g is a pre-semi-open mapping.

(2) We note that $f(A) = g^{-1}[g(f(A))]$ for every subset A of X because g is injective. Let U be an arbitrary semi-open set in X. Then $(g \circ f)(U)$ is semi-open. Now $f(U)$ is semi-open in Y since g is irresolute by Theorem 38. This shows that f is a semi-open mapping.

40. THEOREM. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two mappings such that* $g \circ f : X \to Z$ *is irresolute. Then*

(1) *If* g *is a pre-semi-open injection, then* f *is irresolute.*

(2) *If* f *is a pre-semi-open surjection, then* g *is irresolute.*

PROOF:

(1) Let $U \in SO(Y)$. Then $g(U) \in SO(Z)$ since g is pre-semi-open. Also $g \circ f$ is irresolute. Therefore, we have $(g \circ f)^{-1}(g(U)) \in SO(X)$.

Since g is an injection, so we have:

$$(g \circ f)^{-1} g(U) = (f^{-1} \circ g^{-1}) g(U) = f^{-1}[g^{-1} g(U)] = f^{-1}(U).$$

Consequently $f^{-1}(U)$ is semi-open in X. This proves that f is irresolute.

(2) Let $\mathcal{V} \in SO(Z)$. Then $(g \circ f)^{-1}(\mathcal{V}) \in SO(X)$ since $g \circ f$ is irresolute. Also f is pre-semi-open, so $f(g \circ f)^{-1}(\mathcal{V})$ is semi-open in Y. Since f is surjective, we note that

$$f \circ (g \circ f)^{-1}(\mathcal{V}) = f \circ (f^{-1} \circ g^{-1})(\mathcal{V}) = (f \circ f^{-1}) \circ g^{-1}(\mathcal{V}) = g^{-1}(\mathcal{V}).$$

It follows that $g^{-1}(\mathcal{V}) \in SO(Y)$. Thus g is an irresolute function.

E. PRE-SEMI-CLOSED FUNCTIONS

In this section, we explore certain new properties and characterizations of pre-semi-closed functions.

41. Definition: A function $f : X \to Y$ is pre-semi-closed if and only if the image set $f(A)$ is semi-closed for each semi-closed subset A of X.

42. Theorem. *The composition of two pre-semi-closed mappings is a pre-semi-closed mapping.*

Proof: The straight forward proof is omitted.

43. THEOREM. *A function* f *is a pre-semi-closed mapping if and only if*

$$scl\, f(A) \subseteq f[scl(A)] \quad \text{for every subset } A \text{ of } X.$$

PROOF: NECESSITY. Suppose f is pre-semi-closed and A is an arbitrary subset of X. Then $f[scl(A)]$ is semi-closed in Y. Since $f(A) \subseteq f[scl(A)]$, we obtain $scl\, f(A) \subseteq f[scl(A)]$.

SUFFICIENCY. Suppose F is an arbitrary semi-closed set in X. By hypothesis, we obtain

$$f(F) \subseteq scl[f(F)] \subseteq f[scl(F)] = f(F).$$

Hence $f(F) = scl[f(F)]$. Thus $f(F)$ is semi-closed in Y. It follows that f is semi-closed.

44. THEOREM. *Prove that a function* $f : X \to Y$ *is a pre-semi-closed mapping if and only if* $Int\, cl[f(A)] \subseteq f(sclA)$ *for every subset* A *of* X.

PROOF: NECESSITY. Suppose f is a pre-semi-closed mapping and A is an arbitrary subset of X. Then $f(sclA)$ is semi-closed in Y. So by definition there exists a closed subset F of Y such that $\text{Int}\, F \subseteq f(scl\, A) \subseteq F$. This implies that $\text{Int}\, cl[f(scl\, A)] \subseteq \text{Int}\, F \subseteq f(scl\, A)$. But also $\text{Int}\, cl[f(A)] \subseteq \text{Int}\, cl[f(scl\, A)]$. Hence $\text{Int}\, cl[f(A)] \subseteq f(scl\, A)$.

SUFFICIENCY. Suppose F is an arbitrary semi-closed set in X. Then by hypothesis, we have

$$\text{Int}\ c\ell[f(F)] \subseteq f(sc\ell\ F) = f(F).$$

Take $E = c\ell\ f(F)$. Then E is closed in Y. Also it implies that $\text{Int}\ E \subseteq f(F) \subseteq E$. Hence $f(F)$ is semi-closed in Y. This implies that f is pre-semi-closed.

.45. THEOREM. *Let* $f : X \to Y$ *be a pre-semi-closed function, and* $B, C \subseteq Y$.

(1) *If* U *is a semi-open neighbourhood of* $f^{-1}(B)$, *then there exists a semi-open neighbourhood* $\mathcal{V}$ *of* B *such that* $f^{-1}(B) \subseteq f^{-1}(\mathcal{V}) \subseteq U$.

(2) *If* f *is also onto, then if* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint semi-neighbourhoods, so have* B *and* C.

PROOF:

(1) Let $\mathcal{V} = Y - f(X - U)$. Then $\mathcal{V}^c = Y - \mathcal{V} = f(U^c)$. Since f is pre-semi-closed, so $\mathcal{V}$ is semi-open. Since $f^{-1}(B) \subseteq U$, we have

$$\mathcal{V}^c = f(U^c) \subseteq f[f^{-1}(B^c)] \subseteq B^c.$$

Hence, $B \subseteq \mathcal{V}$, and thus $\mathcal{V}$ is a semi-open neighbourhood of B. Further $U^c \subseteq f^{-1}f(U^c) = f^{-1}(\mathcal{V}^c) = [f^{-1}(\mathcal{V})]^c$. This proves that $f^{-1}(\mathcal{V}) \subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint semi-neighbourhoods M and N, then by (1), we have semi-open neighbourhoods U and $\mathcal{V}$ of B and C respectively such that

$$f^{-1}(B) \subseteq f^{-1}(U) \subseteq s \operatorname{Int} M$$

and

$$f^{-1}(C) \subseteq f^{-1}(\mathcal{V}) \subseteq s \operatorname{Int} N.$$

Since M and N are disjoint, so are $s \operatorname{Int} M$ and $s \operatorname{Int} N$, and hence so $f^{-1}(U)$ and $f^{-1}(\mathcal{V})$ are disjoint as well. It follows that U and $\mathcal{V}$ are disjoint too as f is onto.

46. THEOREM. *A surjective mapping* $f : X \to Y$ *is pre-semi-closed if and only if for each subset* B *of* Y *and each semi-open set* U *in* X *containing* $f^{-1}(B)$, *there exists a semi-open set* $\mathcal{V}$ *in* Y *containing* B *such that* $f^{-1}(\mathcal{V}) \subseteq U$.

PROOF: NECESSITY. This follows from (1) of Theorem 45.

SUFFICIENCY. Suppose F is an arbitrary semi-closed set in X. Let y be an arbitrary point in $Y - f(F)$. Then $f^{-1}(y) \subseteq X - f^{-1}[f(F)] \subseteq (X - F)$ and $(X - F)$ is semi-open in X. Hence by hypothesis, there exists a semi-open set $\mathcal{V}_y$ containing y such that $f^{-1}(\mathcal{V}_y) \subseteq X - F$. This implies that $y \in \mathcal{V}_y \subseteq Y - f(F)$. Thus

$$Y - f(F) = \cup\{\mathcal{V}_y : y \in Y - f(F)\}.$$

Hence $Y - f(F)$, being a union of semi-open sets is semi-open. Thus its complement $f(F)$ is semi-closed. This shows that f is semi-closed.

47. THEOREM. *Let $f : X \to Y$ be a bijection. Then the following are equivalent.*

(a) f is pre-semi-closed.

(b) f is pre-semi-open.

(c) f^{-1} is an irresolute.

PROOF: (a) $\implies$ (b) : Let $U \in SO(X)$. Then $X - U$ is semi-closed in X. By (a), $f(X - U)$ is semi-closed in Y. But

$$f(X - U) = f(X) - f(U) = Y - f(U).$$

Thus $f(U)$ is semi-open in Y. This shows that f is pre-semi-open.

(b) $\implies$ (c) : Let $A \subseteq X$. Since f is pre-semi-open, by Theorem 37,

$$f^{-1}[scl\, f(A)] \subseteq scl\, f^{-1} f(A)$$

i.e., *scl* f(A) is a subset of f(*scl* A). Thus scl $(f^{-1})^{-1}$ (A) is contained in $(f^{-1})^{-1}$(scl A), for every subset A of X. Then by Theorem 16, it follows that f^{-1} is an irresolute.

(c) ⟹ (a) : Let A be an arbitrary semi-closed set in X. Then X – A is semi-open in X. Since f^{-1} is an irresolute, $(f^{-1})^{-1}$ (X – A) is semi-open in Y. But $(f^{-1})^{-1}$ (X – A) = f(X – A) = Y – f(A). Thus f(A) is semi-closed in Y. This shows that f is pre-semi-closed.

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Dr. Raja Mohammad Latif
Department of Mathematical Sciences
King Fahd University of Petroleum & Minerals
Dhahran 31261 Saudi Arabia