King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 156

November 1993

On Certain Classes of TC Semigroups

R.J. Warne

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

ON CERTAIN CLASSES OF TC SEMIGROUPS

R.J. WARNE
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia

Abstract

An algebra A satisfies TC (the term condition) if $p(a,\tilde{x}) = p(a,\tilde{y})$ iff $p(b,\tilde{x}) = p(b,\tilde{y})$ for any $a,b \in A$; $\tilde{x},\tilde{y} \in A^n$ and any $n+1$–ary term p. TC algebras have been extensively studied. We previously determined the structure of all TC semigroups. We utilize this theorem throughout the paper. Let S be a TC semigroup. Then S is $\mathcal{R}$–trivial if and only if $S_E = \{a \in S | ax$ is an idempotent for some $x \in S\} = \phi$ or $S_E = W \cup I$ where W and I are disjoint sets under the multiplication $ab = a\alpha(a,b \in W)$, $ai = a\alpha(i \in I)$, $ia = i$, and $ii_1 = i(i_i \in I)$ where α is a mapping of W into I. We then show S is a semisimple TC semigroup if and only if $S \cong H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a right zero semigroup. We give another proof of our structure theorem for quasi–regular TC semigroups $S(a \in S$ implies a^n is a regular element of S for some positive integer $n)$ based on our structure theorem for TC semigroups. We note that a TC semigroup S is quasi–regular iff $S_E = S$, i.e. S is E–inversive. We finally determine the members of any Hamiltonian variety γ of semigroups. We show $S \in \gamma$ implies $S \cong G \times I \times J$ where G is a periodic abelian group, I is a left zero semigroup, and J is a right zero semigroup, i.e. S is a periodic semisimple TC semigroup and, that $G \times I \times J$ is a Hamiltonian semigroup.

The term condition for algebras (TC) was introduced by McKenzie [6] and algebras obeying TC (also called abelian algebras) have been extensively studied. A semigroup satisfies TC if and only if $(c1)$ $xy = xz$ implies $uy = uz$ $(c2)$ $yx = zx$ implies $yu = zu$ $(c3)$ $y_1xy_2 = z_1xz_2$ implies $y_1uy_2 = z_1uz_2$. TC semigroups were first considered by Taylor [10] and McKenzie [7]. In [7], McKenzie characterized TC semigroups of finite exponent and posed the problem of characterizing all TC semigroups. In [11, Theorem 1.13], we determined the structure of all TC semigroups and in [11, Theorem 2.11] we gave a more detailed structure theorem for periodic TC semigroups. In [11, Corollary 2.6], we gave the structure of TC regular semigroups. In [12, Theorem 2.1], we specialized [11, Theorem 1.13] to obtain the structure of reversible TC semigroups S ($aS \cap bS \neq \phi$ and $Sa \cap Sb \neq \phi$ for all $a, b \in S$). In [12, Theorem 3.8], we determined the structure of quasi–regular TC semigroups. In [12, Theorem 4.2], we showed that a TC semigroup S has the congruence extension property (CEP) if and only if S is periodic (a semigroup S has CEP if for every subsemigroup T of S and congruence relation ρ on T, there is a congruence relation $\overline{\rho}$ on S such that $\overline{\rho} \cap (T \times T) = \rho$). In [13, Theorem 3.4], we determined the structure of a class of semigroups S that satisfies $(c1)'$ (if $x \in S$, there exists $u^2 = u \in S$ such that $(c1)$ is valid for $y, z \in S'$ (S with appended identity) and $(c2)'$ and additional conditions.

In section 1, we characterize $\mathcal{R}$–trivial TC semigroups S in terms of S_E (Theorem 1.21). In section 2, we determine the structure of semisimple TC semigroups (Theorem 2.6). We also give another proof (proof of Theorem 2.5) of our structure theorem for TC regular semigroups [11, Corollary 2.6]. In section 3, we give another proof (proof of Theorem 3.3) of our structure theorem for quasi–regular TC semigroups [12, Theorem 3.8]. Our present proof is based on [11, Theorem 1.13] while the original proof was independent of this theorem. As a corollary (Corollary 3.4) to Theorem 3.3, we show a semigroup S is TC and E–inversive if and only if S is an inflation of a direct product of an abelian group and a rectangular band.

We also show (Corollary 3.2) that a TC semigroup is E–inversive if and only if it is quasi–regular. In section 4, we show (Theorem 4.5) that any member of a Hamiltonian variety of semigroups is a direct product of a periodic abelian group and a rectangular band and (Proposition 4.6) that such a direct product is a Hamiltonian semigroup.

For universal algebra concepts not defined here such as algebra, term, or variety, we refer the reader to [1] or [8]. For algebraic semigroup concepts not defined here such as Green's relation ($\mathcal{R}$ and $\mathcal{J}$), right zero semigroup, left zero semigroup, rectangular band, idempotent, identity, regular element, regular semigroup, ideal, principal factor, periodic semigroup, simple semigroup, 0–simple semigroup, and congruence, we refer the reader to [2], [4], or [8].

A semigroup S is termed $\mathcal{R}$–trivial if each $\mathcal{R}$–class of S consists of a single element. If S is a semigroup, $E(S)$ will denote the set of idempotents of S and S_{Reg} will denote the set of regular elements of S. $\dot{\bigcup}$ will denote disjoint unions, ϕ denotes the empty set, and $\cong$ means "isomorphic to".

1. $\mathcal{R}$–Trivial TC Semigroups

In this section we characterize $\mathcal{R}$–trivial TC semigroups S in terms of the structure of $S_E = \{a \in S | ax \in E(S) \text{ for some } x \in S\}$ (Theorem 1.21). We first state our structure theorem [11, Theorem 1.13] for TC semigroups (Theorem 1.1). In this section, S will denote a TC semigroup unless otherwise specified. We describe $E(S)$ (Lemma 1.4) and S_{Reg} (Lemma 1.5) in terms of the parameter ϕ_{ij} of Theorem

1.1. We also give the structure of S_{Reg} (Lemma 1.7) and S_E (Lemma 1.11). We determine $\mathcal{R}$ (Lemma 1.12 and Lemma 1.16). We show S is an ideal extension of an $\mathcal{R}$–trivial TC semigroup without idempotent by $(S_E)^0$ (S_E with appended zero) (Lemma 1.13 and Lemma 1.14). We note (Remark 1.22) that S is $\mathcal{R}$–trivial iff $S_E = \phi$ or S_E is $\mathcal{R}$–trivial.

Construction of TC Semigroups

Let G be an abelian group, I be a left zero semigroup, and J be a right zero semigroup. Let V be a subsemigroup of $G \times I \times J$. Let $(X_v : v \in V)$ be a collection of pairwise disjoint sets and let $X = \cup(X_v : v \in V)$. For $v = (m, i, j)$ and $x \in X_v$, define $m(x) = m$, $i(x) = i$, and $j(x) = j$. Let $M_i = \{m : \{m, i, j) \in V$ for some $j\}$ and $N_j = \{m : (m, i, j) \in V$ for some $i\}$. For $(i, j) \in Pr_I\, V \times Pr_J\, V$, let ϕ_{ij} be a function from $M_i \times M_j \to X$ such that

1. $\phi_{ij}(m, n) \in X_{(mn,i,j)}$

2. $\phi_{ij}(mk, n) = \phi_{ij}(m, kn)$ for $k \in UM_i$

3. $\phi_{ij}(m, kn) = \phi_{ij}(p, kq)$ implies $\phi_{ij}(m, sn) = \phi_{ij}(p, sq)$ for $k, s \in UM_i$.

Let (X, V, ϕ) denote X under the multiplication

4. $xy = \phi_{i(x)j(y)}(m(x), m(y))$.

Theorem 1.1 or Remark 1.2 will be used in the proof of Lemmas 1.3 – 1.7, 1.9, 1.11 – 1.14 and 2.2, Theorem 2.5, Proposition 3.1 and Theorem 3.3.

Theorem 1.1. *S is a TC semigroup if and only if $S \cong (X, V, \phi)$ for some X, V,* *and* ϕ.

Remark 1.2 Let $C = (m(a) : a \in X)$. Thus, using (4) and (1) of Theorem 1.1, m is a homomorphism of X onto C. Hence, C is an abelian cancellative semigroup. Furthermore, $\varphi(a) = (m(a), i(a), j(a))$ defines a homomorphism of $X = S$ onto V. Let U denote the group of units of C.

We often use Theorem 1.1 and Remark 1.2 and their notation without explicit mention.

Lemma 1.3, Lemma 1.6, and Lemma 1.7 are [11, Lemma 2.1], [11, Lemma 2.4], and [11, Lemma 2.5]. Different proofs using Theorem 1.1 are given here.

Lemma 1.3 will be used in the proof of Lemmas 1.10 and 1.16 and Proposition 3.1.

Lemma 1.3. *If* $x, y \in S$ *and* $e \in E(S)$, $xey = xy$.

Proof: $(xe)y = \phi_{i(x)j(y)}(m(x), m(y)) = xy$.

Lemma 1.4 will be used in the proof of Lemma 1.9.

Lemma 1.4. $E(S) \neq \phi$ *if and only if* C *has an identity element 1 and, then,* $E(S) = (\phi_{ij}(1,1) : 1 \in M_i, 1 \in N_j)$.

Proof. Suppose C has an identity element 1. Then, $m(e) = 1$ for some $e \in S$. Let $t = \phi_{i(e)j(e)}(m(e), m(e))$. Then, $tt = \phi_{i(e)j(e)}(m(e), m(e)) = t$. So,$t \in E(S)$. Conversely, suppose $E(S) \neq \phi$. Let $a \in E(S)$. Thus, $aa = \phi_{i(a)j(a)}(m(a), m(a)) = a$. Hence, $(m(a))^2 = m(a)$ and, hence, $m(a)$ is an identity for C. Furthermore,

$a = \phi_{i(a)j(a)}(1,1)$ with $1 \in M_{i(a)}$ and $1 \in N_{j(a)}$. Let $t = \phi_{ij}(1,1)$ where $1 \in M_i$ and $1 \in N_j$. Then, $tt = \phi_{ij}(1,1)\phi_{ij}(1,1) = \phi_{ij}(1.1) = t$ since $\phi_{ij}(1,1) \in X_{(1,i,j)}$.

Lemma 1.5 will be used in the proof of Lemma 1.6 and Proposition 3.1.

Lemma 1.5. *$S_{Reg} = \{\phi_{i(a)j(a)}(1, m(a)) : m(a) \in U\}$.*

Proof. Suppose $a \in S_{\text{Reg}}$. Then $(ax)a = a$ for some $x \in S$. So, $a = \phi_{i(a)j(a)}(1, m(a))$. Since $m(ax) \in E(C)$, $m(a) \in U$. Conversely, if $m(a) \in U$, let $m(x) = (m(a))^{-1}$. So, $\phi_{i(a)j(a)}(1, m(a))\phi_{i(a)j(x)}(1, m(x)))\phi_{i(a)j(a)}(1, m(a)) = \phi_{i(a)j(x)}(m(a), m(x)) \cdot \phi_{i(a)j(a)}$ $(1, m(a)) = \phi_{i(a)j(a)}(1, m(a))$.

Lemma 1.6 will be used in the proof of Lemma 1.7 and Proposition 3.1.

Lemma 1.6. *$S_{Reg} = \phi$ or S_{Reg} is a regular subsemigroup of S.*

Proof. If $m(a), m(b) \in U$, let $x = \phi_{i(a)j(a)}(1, m(a))$ and $y = \phi_{i(b)j(b)}(1, m(b))$. Thus $xy = \phi_{i(a)j(b)}(m(a), m(b))$. Let $m(a) = m(a))^{-1}$. Then, $\phi_{i(a)j(a)}(m(a), m(b)) = \phi_{i(a)j(b)}(m(a)m(x)m(a), m(b)) = \phi_{i(ab)j(ab)}(1, m(ab))$. Since $m(ab) = m(a)m(b) \in U$, $xy \in S_{\text{Reg}}$ by Lemma 1.5. Since $a \in S_{\text{Reg}}$ implies $axa = a$ and $x = xax$ for some $x \in S$, S_{Reg} is a regular subsemigroup of S.

Lemma 1.7 or its proof will be used in the proof of Remark 1.8, Lemma 1.11, and Theorem 2.5

Lemma 1.7. *$S_{Reg} = \phi$ or $S_{Reg} \cong H \times A \times B$ where H is an abelian group, A is a left zero semigroup and B is a right zero semigroup.*

Proof. Let $H = (m(a) : a \in S_{\text{Reg}}), A = (i(a) : a \in S_{\text{Reg}}), B = (j(a) : a \in S_{\text{Reg}})$.

Since H is a regular subsemigroup of C by Remark 1.2 and Lemma 1.6, H is an abelian group. Also, $A \subseteq I$ and $B \subseteq J$. If $a \in S_{\text{Reg}}, \varphi(a) = (m(a), i(a), j(a))$ defines a homomorphism of S_{Reg} into $H \times A \times B$. We next show φ is "onto". Let $(x, y, z) \in H \times A \times B$. Thus, $x = m(a), y = i(b)$, and $z = j(c)$ for some $a, b, c \in S_{\text{Reg}}$. There exists $r, s \in S(= X)$ such that $brb = b$ and $csc = c$. Thus, $(x, y, z) = (m(brasc), i(brasc), j(brasc)) = \varphi(brasc)$ with $brasc \in S_{\text{Reg}}$ by Lemma 1.6. We next show φ is one–to–one. Suppose $\varphi(a) = \varphi(b)$ where $a, b \in S_{\text{Reg}}$. Suppose $axa = a$ and $byb = b$. So, $a = (ax)a = \phi_{i(a)j(a)}(1, m(a))$ and $b = (by)b = \phi_{i(b)j(b)}(1, m(b))$. So, $a = b$.

Remark 1.8. If $a, b \in S_{\text{Reg}}$ and $a, b \in X_v$, then $\varphi(a) = \varphi(b) = v$. Thus, $a = b$ by the proof of Lemma 1.7.

An element a in an arbitrary semigroup S is termed E–inversive if there exists $b \in S$ such that $ab \in E(S)$. We will denote the set of E–inversive elements of S by S_E. Following [2], a semigroup S is termed E–inversive if each element of S is E–inversive. Let $a \in S_E$. Then, there exists $y \in S$ such that $ay, ya \in E(S)$. If $ax \in E(S)$, just let $y = xax$.

Lemma 1.9 will be used in the proof of Lemmas 1.12, 1.13 and 2.3 and Proposition 3.1.

Lemma 1.9. *$a \in S_E$ if and only $m(a) \in U$.*

Proof. Let $a \in S_E$. Then, there exists $b \in S$ such that $ab \in E(S)$. So, $m(a)m(b) = m(b)m(a) = 1$, the identity of C. Hence, $m(a) \in U$. Conversely, let $m(a) \in U$. So, $m(a)m(b) = m(b)m(a) = 1 = m(e)$ for some $e \in S$. Thus, $ab = \phi_{i(a)j(b)}(m(a), m(b)) = \phi_{i(a)j(b)}(m(a), m(b)m(e)) = \phi_{i(a)j(b)}(m(a)m(b), m(e)) = \phi_{i(a)j(b)}(1,1)$. (Note, $ab \in X_{(m(e),i(a),j(b))}$. Thus, $1 = m(e) \in M_{i(a)}$ and $1 = m(e) \in N_{j(b)}$). Hence, $ab \in E(S)$ by Lemma 1.4.

Lemma 1.10 will be used in the proof of Lemmas 1.11 and 2.4.

Lemma 1.10. *$S_E = \phi$ or S_E is an E-inversive semigroup and $S_E^2 \subseteq S_{Reg}$.*

Proof. Suppose $S_E \neq \phi$. Let $a, b \in S_E$. Then, there exist $x, y \in S$ such that $xa, by \in E(S)$. Hence, using Lemma 1.3, $ab(yx)ab = a(by)(xa)b = ab$. So, $ab \in S_{\text{Reg}}$. If $a \in S_E$, there exists $y \in S$ such that $ay, ya \in E(S)$. Thus, $y \in S_E$.

Lemma 1.11 will be used in the proof of Lemmas 1.12 and 1.17, Remark 1.15, Theorem 3.3, and Remark 1.22.

Lemma 1.11. *$S_E = \phi$ or $S_E = W \dot{\bigcup} H \times A \times B$ where W, A, and B are sets and H is an abelian group under the following product: if $a, b \in W$ and $(g,i,j), (h,k,\ell) \in H \times A \times B$, then $ab = (a\phi b\phi, a\alpha, b\beta)$, $(g,i,j)a = (g(a\phi), i, a\beta)$, $a(g,i,j) = (a\phi g, a\alpha, j)$, and $(g,i,j)(h,k,\ell) = (gh, i, \ell)$ where ϕ, α, and β are functions of W into H, A, and B respectively and $S_{Reg} = H \times A \times B$.*

Proof. Assume $S_E \neq \phi$. Thus, $S_{\text{Reg}} \neq \phi$ and $S_{\text{Reg}} \subseteq S_E$. By the proof of Lemma 1.7, $\varphi(a) = (m(a), i(a), j(a))(a \in S_{\text{Reg}})$ defines an isomorphism of S_{Reg} onto $H \times A \times B$ (notation of Lemma 1.7 and its proof). Let $a \in S_E$. Thus, there exist x and y in S such that $ax, ya \in E(S)$. Hence, $\varphi(axaya) = (m(axaya), i(axaya), j(axaya)) = (m(a), i(a), j(a)) = \varphi(a)$. Since $axaya \in S_{\text{Reg}}$

by Lemma 1.10, φ defines a homomorphism of S_E onto $H \times A \times B$. We will repeatedly use the fact that $u, v \in S_E$ implies $uv \in S_{\text{Reg}}$ (Lemma 1.10). Let $W' = S_E - S_{\text{Reg}}$. So, $S_E = W' \dot{\bigcup} S_{\text{Reg}}$. There exists a one–to–one mapping θ of W' onto a set W with $W \cap \varphi(S_{\text{Reg}}) = \phi$. Let $\overline{S_E} = W \dot{\bigcup} \varphi(S_{\text{Reg}})$. Define $\delta(x) = \theta(x)$ if $x \in W'$ and $\delta(x) = \varphi(x)$ if $x \in S_{\text{Reg}}$. Then, δ is a one–to–one mapping of S_E onto $\overline{S_E}$. Define a product on $\overline{S_E}$ by the rule $x \circ y = \delta(\delta^{-1}(x)\delta^{-1}(y))$. Then, $(\overline{S_E}, \circ)$ is a groupoid isomorphic to S_E under δ. Define $a\phi = m\delta^{-1}(a), a\alpha = i\delta^{-1}(a)$, and $a\beta = j\delta^{-1}(a)$ for $a \in W$. Thus, ϕ, α, and β are functions of W into H, A, and B respectively. If $a, b \in W, a \circ b = \delta(\delta^{-1}(a)\delta^{-1}(b)) = (m(\delta^{-1}(a)\delta^{-1}(b)), i(\delta^{-1}(a)\delta^{-1}(b)), j(\delta^{-1}(a)\delta^{-1}(b))) = (a\phi b\phi, a\alpha, b\beta)$. If $(g, i, j) \in H \times A \times B$, then there exists $c \in S_{\text{Reg}}$ such that $m(c) = g, i(c) = i$, and $j(c) = j$. Thus, $a \circ (g, i, j) = \delta(\delta^{-1}a\delta^{-1}(g, i, j)) = \delta(\delta^{-1}(a)c) = (m(\delta^{-1}(a)c), i(\delta^{-1}(a)c), j(\delta^{-1}(a)c)) = (a\phi g, a\alpha, j)$. Similarly, $(g, i, j) \circ a = (g(a\phi), i, a\beta)$ and $(g, i, j) \circ (h, k, \ell) = (gh, i, \ell)$ where $(h, k, \ell) \in H \times A \times B$. We may find an isomorphism λ of S onto a semigroup $\overline{S}$ with subsemigroup $\overline{S_E}$ such that $\lambda | S_E = \delta$ with $\overline{S}_E = \overline{S_E}$. So, we identify $\overline{S}$ with S and $\overline{S_E}$ with S_E. Clearly, $S_{\text{Reg}} = H \times A \times B$.

Lemma 1.12 will be used in the proof of Remark 1.15 and Lemmas 1.16, 1.19, and 1.20.

Lemma 1.12. *$a\mathcal{R}b$ implies $a = b$ or $a = bs$ and $b = at$ where $s, t \in S_E$ and $st, ts \in E(S)$.*

Proof. Suppose $a \neq b$. Thus, there exists $s, t \in S$ such that $a = bs$ and $b = at$. Hence, $a = ats$. So, $m(a) = m(a)m(ts)$. Thus, $m(a)m(ts) = m(a)(m(ts))^2$. Hence,

$m(ts) \in E(C)$. So, $m(t)m(s) = m(s)m(t) = 1$. Thus, $s, t \in S_E$ by Lemma 1.9. Furthermore, $ats = a(ts)^2$. Hence, $(ts)^2 = (ts)^3$ by TC. Since $s, t \in S_E, ts = (g, i, j) \in H \times A \times B$ by Lemma 1.11 (notation of Lemma 1.11). So, $g^2 = g^3$ and, hence, $g = e$, the identity of H. Thus, $ts \in E(S)$. Similarly, $st \in E(S)$.

Lemma 1.13 will be used in the proof of Remark 1.15 and Theorem 1.21.

Lemma 1.13. *$S = S_E \dot{\bigcup} D$ where $S_E = m^{-1}(U)$ and $D = m^{-1}(C - U)$. D is an ideal of S.*

Proof. Using Lemma 1.9, $S_E = m^{-1}(U)$. It is easily checked that D is an ideal of S.

Lemma 1.14 is used in the proof of Theorem 1.21.

Lemma 1.14. *D is an $\mathcal{R}$–trivial semigroup.*

Proof. Suppose $a\mathcal{R}b$ (in D) and $a \neq b$. So, $a = bs$ and $b = at$ where $s, t \in D$. Thus, $a = ats$ implies $m(a) = m(a)m(ts)$. Hence, $m(a)m(ts) = m(a)(m(ts)^2$. Thus, $(m(ts))^2 = m(ts) \in C - U$ which is impossible. So, $a = b$.

Remark 1.15. $a\mathcal{R}b$ implies $a = b$ or $a, b \in S_{\text{Reg}} = H \times A \times B$ (notation of Lemmas 1.6 and 1.11) and $a = (g, i, j)$ and $b = (h, i, \ell)$, say, or $a, b \in D$.

Proof. Suppose $a\mathcal{R}b$ and $a \neq b$. Thus, $a = bs$ and $b = at$ where $s, t \in S_E$ by Lemma 1.12. If $a, b \in S_E$, then $a, b \in S_{\text{Reg}} = H \times A \times B$ by Lemma 1.11. So, as is easily checked, $a = (g, i, j)$ and $b = (h, i, \ell)$, say. By Lemma 1.13, $a \in D$ and $b \in S_E$ is impossible.

Lemma 1.16 will be used in the proof of Lemma 1.20 and Theorem 1.21.

Lemma 1.16. *Let $a \in S$. Then, $R_{ae} = aS_E$ for all $e \in E(S)$.*

Proof. Let $b \in aS_E$. Then, $b = as$ for some $s \in S_E$. So, there exists $t \in S$ such that $ts, st \in E(S)$. Thus, $bt = ast = ae$ where $e = st$. Furthermore, $b = (ae)s$ by Lemma 1.3. Hence, $b \in R_{ae}$. However, if $f \in E(S), (ae)f = af$ and $(af)e = ae$ by Lemma 1.3. So, $ae\mathcal{R}af$. Hence, $b \in R_{af}$. So, $b \in R_{ae}$ for all $e \in E(S)$. Thus, $aS_E \subseteq R_{ae}$ for all $e \in E(S)$. Conversely, let $b \in R_{ae}$. Thus, $b = ae$ or $b = aes$ for some $s \in S_E$ by Lemma 1.12. So, in either case, $b \in aS_E$. Hence, $R_{ae} = aS_E$ for all $e \in E(S)$.

Lemma 1.17 will be used in the proof of Theorem 1.21.

Lemma 1.17. *If $|cS_E| = 1$ for some $c \in S, S_E = W \dot{\bigcup} I$ where W and I are disjoint sets under the product: for $a, b \in W$ and $i, i_1 \in I$, $ab = a\alpha, ia = i, ai = a\alpha$, and $ii_1 = i$ where α is a mapping of W into I.*

Proof. Suppose $|cS_E| = 1$ for some $c \in S$. Then, for any $e \in E(S), ce = cz$ for all $z \in S_E$. Thus, by TC, $e = ez$. The structure of S_E is given by Lemma 1.11. We will use this lemma and its notation. Suppose $e = (1, i_0, j_0)$ where 1 is the identity of H. Hence, $(1, i_0, j_0) = (1, i_0, j_0)(g, i, j)$ where $(g, i, j) \in H \times A \times B$. Thus, $(1, i_0, j_0) = (g, i_0, j)$. So, $j = j_0$ and $g = 1$. Hence, $J = \{j_0\}$ and $H = \{1\}$. Let $a \in W$. Thus, $(1, i_0, j_0) = (1, i_0, j_0)a = (a\phi, i_0, a\beta)$. So, $a\phi = 1$ for all $a \in W$ and $a\beta = j_0$ for all $a \in W$. Define $\varphi(a) = a$ for $a \in W$ and $\varphi(1, i, j_0) = i$ for $i \in I$. It is easily checked that φ defines an isomorphism of S_E onto $W \cup I$ under the multiplication given in the statement of Lemma 1.17.

Lemma 1.18 will be used in the proof of Lemma 1.19.

Lemma 1.18. *Let $S_E = W \cup I$ under the multiplication given in the statement of Lemma 1.17. Let $b \in S$. Then, $bS_E = bi_0$ for any $i_0 \in I$.*

Proof. Since $ii_0 = ii$ for all $i \in I$, $bi_0 = bi$ by TC. Let $a \in W$. Since $ia = ii_0$, $ba = bi_0$ by TC. So, $bS_E = bi_0$.

Lemma 1.19 will be used in the proof of Lemma 1.20.

Lemma 1.19. *If S_E is as in Lemma 1.17, then $bS_E = R_{bi_0} = bi_0$ for $b \in S$ and $i_0 \in I$.*

Proof. Suppose $c\mathcal{R}bi_0$. Thus, using Lemma 1.12, either $c = bi_0$ or $c = bi_0s$ for some $s \in S_E$. But, if $s \in W, i_0s = i_0$, and, if $s \in I, i_0s = i_0$. Thus, $c = bi_0$. Hence, $R_{bi_0} = bi_0$. Thus, by Lemma 1.18, $bS_E = bi_0 = R_{bi_0}$.

Lemma 1.20 will be used in the proof of Theorem 1.21.

Lemma 1.20. *If $S_E = W \cup I$ under the multiplication given in the statement of Lemma 1.17, then S is $\mathcal{R}$-trivial.*

Proof. Let $b \in S$. Suppose $|R_b| > 1$. Then, there exists $c \neq b$ such that $c\mathcal{R}b$. So, by Lemma 1.12, there exists $s, t \in S_E$ such that $st, ts \in E(S)$ and $c = bs$ and $b = ct$. Thus, $c = cts$. Let $ts = e$. So, $c = ce$. Hence, using Lemma 1.16, $R_b = R_c = R_{ce} = cS_E$. Thus, $|cS_E| > 1$. This contradicts Lemma 1.19. So, $|R_b| = 1$ and S is $\mathcal{R}$-trivial.

Theorem 1.21. *Let S be a TC semigroup. Then S is $\mathcal{R}$-trivial if and only if*

$S_E = \phi$ or $S_E = W \cup I$ where W and I are disjoint sets under the multiplication $ab = a\alpha(a, b \in W)$, $ai = a\alpha(i \in I), ia = i$, and $ii_1 = i(i_1 \in I)$ where α is a mapping of W into I.

Proof. First suppose S is $\mathcal{R}$–trivial. Suppose $S_E \neq \phi$. By Lemma 1.16, $R_{be} = bS_E$ for $b \in S$ and $e \in E(S)$. So, $|bS_E| = |R_{be}| = 1$. Thus, S_E has the required structure by Lemma 1.17. Conversely, if $S_E = \phi, S = D$ by Lemma 1.13. D is an $\mathcal{R}$–trivial semigroup by Lemma 1.14. If $S_E \neq \phi$ and has the structure given in the statement of the theorem, S is $\mathcal{R}$–trivial by Lemma 1.20.

Remark 1.22. Using Theorem 1.21 and Lemma 1.11, S is $\mathcal{R}$–trivial iff $S_E = \phi$ or S_E is $\mathcal{R}$–trivial.

2. Semisimple TC Semigroups

In this section, we determine the structure of semisimple TC semigroups (Theorem 2.6). We also give a new proof (proof of Theorem 2.5) of our structure theorem for regular TC semigroups [11, Corollary 2.6]. This proof is based on Theorem 1.1. We show that if S is a semisimple TC semigroup, then C (notation of Remark 1.2) is an abelian group (Lemma 2.2) and that a TC semigroup is E–inversive iff C is an abelian group (Lemma 2.3). We note (Remark 2.7) that a TC semigroup is semisimple iff it is regular.

A semigroup S is called semisimple if every principal factor is 0–simple or simple.

Equivalently, S is semisimple iff $I^2 = I$ for every ideal I of S [2].

Remark 2.1. If S is a semigroup, let $J(a)$ denote the principal ideal generated by a and define $a\mathcal{J}b$ iff $J(a) = J(b)$. Let $\mathcal{J}_a$ denote the $\mathcal{J}$–class containing a. S is semisimple iff for $a \in S$, there exists $x, y \in \mathcal{J}_a$ such that $xy \in \mathcal{J}_a$.

Lemma 2.2 will be used in the proof of Lemma 2.4.

Lemma 2.2. *Let S be a semisimple TC semigroup. Then C is an abelian group.*

Proof. Let S be a semisimple TC semigroup. Suppose $a\mathcal{J}b(a, b \in S)$. Since $J(a) = J(b) = (J(b))^3, a = rbs$ for some $r, s \in S$ and since $J(b) = J(a) = (J(a))^3, b = r_1as_1$ for some $r_1, s_1 \in S$. So, $m(a) = m(b)m(rs)$ and $m(b) = m(a)m(r_1s_1)$. Thus, $m(a) = m(a)m(r_1s_1)m(rs)$. So, if $m(r_1s_1)m(rs) = u, m(a) = m(a)u$. Hence, $m(a)u = m(a)u^2$. Thus, $u^2 = u$. So, $u = 1$, the identity of C. Hence, $m(r_1s_1) \in U$ and $m(b) = m(a)v$ where $v \in U$. Thus, $\mathcal{J}_a \subseteq \{z | m(z) = m(a)v$ for some $v \in U\}$. Since S is semisimple, there exists $z_1, z_2 \in \mathcal{J}_a$ such that $z_1z_2 \in \mathcal{J}_a$. Thus, $m(z_1) = m(a)v_1, m(z_2) = m(a)v_2$, and $m(z_1z_2) = m(a)v_3$ where $v_1, v_2, v_3 \in U$. So, $m(a)v_3 = m(a)v_1m(a)v_2$. Hence, $v_3 = m(a)v_1v_2$ or $m(a) = v_3v_2^{-1}v_1^{-1} \in U$. Hence, $m(a) \in U$ for all $a \in X$. So, $U = C$.

Lemma 2.3 will be used in the proof of Lemma 2.4.

Lemma 2.3. *Let S be a TC semigroup. Then, S is E–inversive if and only if C is an abelian group.*

Proof. Lemma 2.3 is an easy consequence of Lemma 1.9.

Lemma 2.4 will be used in the proof of Theorem 2.6 and Remark 2.7.

Lemma 2.4. *Let S be a semisimple TC semigroup. Then S is a regular semigroup.*

Proof. Let $a \in S$. Thus, $J(a) = (J(a))^3$. Hence, there exists $r, s \in S$ such that $a = ras$. By Lemma 2.2 and Lemma 2.3, S is an E-inversive semigroup. So, by Lemma 1.10, $a \in S_{\mathrm{Reg}}$. Hence, $S = S_{\mathrm{Reg}}$ and, thus, S is a regular semigroup.

Theorem 2.5 will be used in the proof of Theorem 2.6 and Corollary 3.4.

Theorem 2.5. *S is a regular TC semigroup if and only if $S \cong H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a right zero semigroup.*

Proof. If S is a regular TC semigroup, $S \cong H \times A \times B$ by Lemma 1.7. To prove the converse, let $V = H \times A \times B, X_{(g,i,j)} = \{(g, i, j)\}$, and $\phi_{ij}(m, n) = (mn, i, j)$. Thus, $H \times A \times B$ is a TC semigroup by Theorem 1.1. An easy calculation shows $H \times A \times B$ is regular. It can also be shown by easy calculations that $H \times A \times B$ is a TC semigroup.

Theorem 2.6. *S is a semisimple TC semigroup if and only if $S \cong H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a right zero semigroup.*

Proof. Let S be a semisimple TC semigroup. $S \cong H \times A \times B$ by Lemma 2.4 and Theorem 2.5. Conversely, $H \times A \times B$ is a regular semigroup by Theorem 2.5. However, every regular semigroup is semisimple.

Remark 2.7. Using Lemma 2.4 and proof of Theorem 2.6, a TC semigroup is semisimple iff it is regular.

3. Quasi–Regular TC Semirings

In this section, we give a new proof (proof of Theorem 3.3) of our structure theorem for quasi–regular TC semigroups [12, Theorem 3.8]. Our new shorter proof is based on Theorem 1.1 while our original proof was independent of this theorem. We show (Corollary 3.2) that a TC semigroup is quasi–regular iff it is E–inversive. As a corollary to Theorem 3.3 (Corollary 3.4) we show that a semigroup is TC and E–inversive iff it is an inflation of a direct product of an abelian group and a rectangular band. Other techniques (using results of Yamada and Taylor) that may be used in the proofs of Theorem 3.3 and Corollary 3.4 are given in Remark 3.5.

An element $a \in S$, an arbitrary semigroup, is termed quasi–regular if $a^n \in S_{\text{Reg}}$ for some positive integer n. S is termed quasi–regular if each element of S is quasi–regular. Let S_Q denote the set of quasi–regular elements of S.

Proposition 3.1 will be used in the proof of Corollary 3.2 and Theorem 3.3.

Proposition 3.1. *Let S be a TC semigroup. $a \in S_E$ if and only if $a \in S_Q$.*

Proof. Let $a \in S_E$. Thus, $m(a) \in U$ by Lemma 1.9. Let $e \in E(S)$. Then, $ae = \phi_{i(a)j(e)}(m(a), 1) = \phi_{i(a)j(e)}(m(a)m(x)m(a), 1)$ where $m(x) = m(a)^{-1}$. So, $ae = \phi_{i(ae)j(ae)}(1, m(ae))$. Thus, $ae \in S_{\text{Reg}}$ by Lemma 1.5. Similarly, $ea \in S_{\text{Reg}}$. Thus, using Lemma 1.3, $a^2 = (ae)(ea)$. Hence, $a^2 \in S_{\text{Reg}}$ by Lemma 1.6. Conversely, let $a \in S_Q$. Then, there exists a positive integer n such that $a^n \in S_{\text{Reg}}$. So, there exists $x \in S$ such that $a^n x a^n = a^n$. Thus, $a^n x \in E(S)$. So, $a(a^{n-1}x) \in E(S)$. Hence, $a \in S_E$.

Corollary 3.2 will be used in the proof of Theorem 3.3.

Corollary 3.2. *Let S be a TC semigroup. Then, S is E–inversive if and only if S is a quasi–regular.*

Construction of Quasi–regular TC Semigroups.

Let H be an abelian group, A, B, and W be sets and ϕ, α, and β functions of W into H, A, and B respectively. Define a binary operation $\circ$ on the set $W \dot{\bigcup} (H \times A \times B)$ by the following rule.

For $a, b \in W$ and $(g,i,j),(h,k,s) \in H \times A \times B$,

$$\begin{aligned} a \circ b &= (a\phi b\phi, a\alpha, b\beta) \\ a \circ (g,i,j) &= (a\phi g, a\alpha, j) \\ (g,i,j) \circ a &= (g(a\phi), i, a\beta) \\ (g,i,j) \circ (h,k,\ell) &= (gh, i, \ell). \end{aligned}$$

Theorem 3.3. *S is a quasi–regular TC semigroup if and only if S is isomorphic to some $(W \cup (H \times A \times B), \circ)$.*

Proof. Let S be a quasi–regular TC semigroup. Then, $S \cong (W \cup (H \times A \times B), \circ)$ by Proposition 3.1 or Corollary 3.2 and Lemma 1.11. To prove the converse, we use Theorem 1.1 and its notation. Let $V = H \times A \times B$. Define $X_{(g,i,j)} = \{\{(g,i,j)\} \cup \{a \in W | a\phi = g, a\alpha = i, \text{ and } a\beta = j\}\}$. Define $\phi_{ij}(m,n) = (mn, i, j)$. It is easily checked that (1), (2), and (3) are valid. Let $W = X - (H \times A \times B)$. Let $m = \phi, i = \alpha$, and $j = \beta$. Then, as is easily checked, the multiplication given by (4) of Theorem 1.1

coincides with "$\circ$". Thus, $(W \cup (H \times A \times B), \circ)$ is a TC semigroup. It is easily seen that $(W \cup (H \times A \times B), \circ)$ is quasi-regular.

Let T be a semigroup. With each element $t \in T$, associate a set X_t containing t such that the sets $X_t (t \in T)$ are mutually disjoint. Let $S = \cup(X_t : t \in T)$ and let the product in T be extended to the product in S by defining $ab = st$ if $a \in X_s$ and $b \in X_t$ $(s, t \in T)$. Then, S is a semigroup which is called an inflation of T [2].

Corollary 3.4. *A semigroup S is TC and E–inversive if and only if S is an inflation of a direct product of an abelian group and a rectangular band.*

Proof. Let S be an E–inversive (quasi-regular) TC semigroup. We use Theorem 3.3 and its notation. Let $a\delta = (a\phi, a\alpha, a\beta)$ if $a \in W$ and $(g, i, j)\delta = (g, i, j)$. For $(g, i, j) \in H \times A \times B$, let $X_{(g,i,j)} = (g, i, j)\delta^{-1}$. Let $V = \cup(X_{(g,i,j)} : (g, i, j) \in H \times A \times B)$ and define the product in V by the rule if $a \in X_{(g,i,j)}$ and $b \in X_{(h,k,\ell)}$, $ab = (g, i, j)(h, k, \ell)$ (product in $H \times A \times B$). Then V is an inflation of $H \times A \times B$. Note, $X_{(g,i,j)} = \{\{(g, i, j)\} \cup \{a \in W | a\phi = g, a\alpha = i, \text{ and } b\beta = j\}$. It is easily checked that $V = S$ and products in V and S coincide.

Conversely, suppose S is an inflation of the direct product of an abelian group and a rectangular band. So, $S = \cup(X_t : t \in H \times A \times B)$. If $a \in X_{(g,i,j)}$, let $y \in X_{(g^{-1},i,j)}$. Thus, $ay = (e, i, j) \in E(S)$. So, S is E–inversive. By Theorem 2.5, $H \times A \times B$ is a TC semigroup. Thus, it is easily checked that S is a TC semigroup.

Remark 3.5. Let S be a TC and E–inversive semigroup. One may use a theorem of Yamada ([2, p. 98] or [14]) to show S is an inflation of the direct product of a group G and a rectangular band. Using Lemma of Taylor [10 Lemma 4], G is abelian.

If S is an inflation of $H \times A \times B$ (notation of Theorem 3.3), let $W = S - (H \times A \times B)$. If $a \in X_{(g,i,j)}$, let $a\phi = g$, $a\alpha = i$, and $a\beta = j$. Then, ϕ, α, and β define functions of W into H, A, and B respectively. It is easily verified that $S \cong (W \cup (H \times A \times B), \circ)$ (notation of Theorem 3.3).

4. Hamiltonian Varieties of Semigroups

In this section, we show (Theorem 4.5) that if γ is a Hamiltonian variety of semigroups, then $S \in \gamma$ implies $S \cong H \times A \times B$ where H is a periodic abelian group, I is a left zero semigroup, and J is a right zero semigroup. In Proposition 4.6, we show $H \times A \times B$ is a Hamiltonian semigroup. By theorem of McKenzie and Warne (Theorems 4.1 – 4.3), the structure of $S \in \gamma$ is given by Theorem 3.3 with H (notation of Theorem 3.3) periodic. We use this result, our determination of the congruences on a quasi–regular TC semigroup (Proposition 4.4) and the Hamiltonian property to establish Theorem 4.5. Proposition 4.6 is established using Proposition 4.4 and a lemma of Garcia.

An algebra A is called Hamiltonian if for every subalgebra B of A there is a congruence θ of A such that B is a congruence class of θ. Varieties composed entirely of algebras that possess the Hamiltonian property are called Hamiltonian.

An algebra A is said to have the congruence extension property (CEP) if for every subalgebra B of A, the restriction map from congruences of A to congruences of B is surjective.

Theorem 4.1 (McKenzie, [9, Theorem 2.3]) *Let γ be a Hamiltonian variety of algebras. Then $A \in \gamma$ implies A is a TC algebra and A possesses the congruence extension property (CEP).*

Remark. Kiss [5] proved that Hamiltonian varieties have CEP.

Theorem 4.2. (Warne, [11, Theorem 4.2]). *Let S be a TC semigroup. Then, S has the congruence extension property (CEP) if and only if S is periodic.*

Theorem 4.3. (Warne, [11, Theorem 2.11]) *S is a periodic TC semigroup if and only if S is isomorphic to some $(W \cup (H \times A \times B), \circ)$ (notation of Theorem 3.3) with H periodic.*

Let $S = (W \cup (H \times A \times B), \circ)$ (notation of Theorem 3.3) be a quasi-regular TC semigroup. If $x \in W$, let $\overline{x} = (x\phi, x\alpha, x\beta)$. If $v \in H \times A \times B$, let $\overline{v} = v$. If $v = (g, i, j)$, let $m(v) = g, i(v) = i$, and $j(v) = j$.

Proposition 4.4. *Let ρ be a congruence relation on a quasi-regular TC semigroup S. Then, there exists a subgroup N_ρ of H, an equivalence relation ρ_A on A, and an equivalence relation ρ_B on B such that $x\rho y$ iff $m(\overline{x})m(\overline{y})^{-1} \in N_\rho$, $(i(\overline{x}), i(\overline{y})) \in \rho_A$, and $(j(\overline{x}), j(\overline{y})) \in \rho_B$. Conversely, let N be a subgroup of G, δ be an equivalence relation on A, and λ be an equivalence relation on B. Then $\rho = \{(x, y) \in S \times S : m(\overline{x})m(\overline{y})^{-1} \in N,\ (i(\overline{x}), i(\overline{y})) \in \delta$ and $(j(\overline{x}), j(\overline{y})) \in \lambda\}$ is a congruence relation on*

S. Furthermore, $N = N_\rho, \delta = \rho_A$, and $\lambda = \rho_B$.

Proof. Let ρ be a congruence relation on S. Then, $\rho \cap (H \times A \times B)^2$ is a congruence relation on $H \times A \times B$. Thus, by [12, Lemma 4.1], there exists a subgroup N_ρ of H, an equivalence relation ρ_A on A, and on equivalence relation ρ_B on B such that $(g,i,j)(\rho \cap (H \times A \times B)^2)(h,k,s)$ iff $gh^{-1} \in N_\rho, (i,k) \in \rho_A$ and $(j,s) \in \rho_B$. Let $\rho^* = \rho \cap (H \times A \times B)^2$. Suppose $a\rho b (a, b \in W)$. Thus, $a \circ (e,i,j) \rho b \circ (e,i,j)$ where e is the identity of H. Then, $(a\phi, a\alpha, j)\rho^*(b\phi, b\alpha, j)$. Hence, $m(\overline{a})m(\overline{b})^{-1} \in N_\rho$ and $(i(\overline{a}), i(\overline{b})) \in \rho_A$. Since $(e,i,j) \circ a\rho(e,i,j) \circ b, (j(\overline{a}), j(\overline{b})) \in \rho_B$. The other cases are handled similarly. Conversely, by [12, Lemma 4.1], $\rho^* = \rho \cap (H \times A \times B)^2$ is a congruence relation on $H \times A \times B$. It is easily checked that for $x, y \in W, x \circ y = \overline{x}\,\overline{y}$ (juxtaposition denoting multiplication in $H \times A \times B$). It is also easily checked that if $x, y \in S$, $x\rho y$ iff $\overline{x}\rho^*\overline{y}$. Thus, ρ is a congruence relation on S. Clearly, $N = N_\rho, \delta = \rho_A$, and $\lambda = \rho_B$.

Theorem 4.5. *Let γ be a Hamiltonian variety of semigroups. Then, $S \in \gamma$ implies $S \cong H \times A \times B$ where H is a periodic abelian group, A is a left zero semigroup, and B is a right zero semigroup. So, S is a periodic semisimple TC semigroup.*

Proof. Let $S \in \gamma$. Then S is a periodic TC semigroup by Theorem 4.1 and Theorem 4.2. Hence, the structure of S is given by Theorem 4.3. We will use this theorem and its notation and Proposition 4.4 to show $S \cong H \times A \times B$. First, suppose $|W| \geq 1$. Then $V = H \times A \times B$ is a subsemigroup of S. Since S is a Hamiltonian semigroup, there exists a congruence ρ on S such that V is a ρ-class. Let N_ρ be the subgroup of H and $\rho_A(\rho_B)$ be the equivalence relation on $A(B)$ associated with ρ by Proposition

4.4. Let $a_0(b_0)$ be a fixed element of $A(B)$. Let $g \in H$. Then, since V is a ρ–class, $(g, a_0, b_0)\rho(e, a_0, b_0)$ where e is the identity of H. Thus, $g \in N_\rho$ by Proposition 4.4. Hence, $N_\rho = H$. Furthermore, $(e, a, b)\rho(e, a_0, b_0)$ for all $a \in A$ and $b \in B$. Thus, $(a, a_0) \in \rho_A$ and $(b, b_0) \in \rho_B$ for all $a \in A$ and $b \in B$. Hence, $\rho_A = A \times A$ and $\rho_B = B \times B$. Let $w_0 \in W$. Then $w_0\phi e \in N_\rho$, $(w_0\alpha, a_0) \in \rho_A$, and $(w_0\beta, b_0) \in \rho_B$. Thus, $w_0\rho(e, a_0, b_0)$ by Proposition 4.4. Hence, we have a contradiction. Thus, $W = \phi$ and $S = H \times A \times B$.

Proposition 4.6. *Let $S \cong H \times A \times B$ where H is a periodic abelian group, A is a left zero semigroup, and B is a right zero semigroup. Then, S is a Hamiltonian semigroup.*

Proof. Let T be a subsemigroup of $H \times A \times B$. Using a lemma of Garcia [3, Lemma 6.2], $T = G \times I \times J$ where G is a subgroup of H, I is a subsemigroup of A, and J is a subsemigroup of B. Define ρ to be the congruence on S determined by $N_\rho = G, \rho_A = (I \times I) \cup \Delta$, and $\rho_B = (J \times J) \cup \Delta$ by Proposition 4.4. Then T is ρ–class. For, suppose $(h, i, j) \in T$ and $(e, i_0, j_0) \in T$ where $i_0(j_0)$ is a fixed element of $I(J)$ and e is the identity of H. So, $he \in N_\rho$,$(i, i_0) \in \rho_A$, and $(j, j_0) \in \rho_B$. Thus, $(h, i, j)\rho(e, i_0, j_0)$. Thus, $T \subseteq \rho_{(e,i_0,j_0)}$. Let $(g, i, j)\rho(e, i_0, j_0)$. Then, $ge \in G, (i, i_0) \in \rho_A$, and $(j, j_0) \in \rho_B$. So, $g \in G$, $i \in I$, and $j \in J$. Thus, $\rho_{(e,i_0j_0)} \subseteq G \times I \times J = T$. Hence, $\rho_{(e,i_0j_0)} = T$.

References

1. S. Burris, and H.P. Sankappanavor, *A Course in Universal Algebra* (Springer-Verlag, New York, 1981).

2. A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, Math. Surveys, Amer. Math. Soc. 7, Vol. I (AMS, Providence, R.I., 1961), Vol. II, 1967.

3. Garcia, J.I., *The congruence extension property for algebraic semigroups*, Semigroup Forum 43 (1991), 1–18.

4. J.M. Howie, *An Introduction to Semigroup Theory* (Academic Press, London, 1976).

5. Kiss, E.W., *Each Hamiltonian variety has the congruence extension property*, Algebra Universalis 12(1981), 395–398.

6. McKenzie, R., *On Minimal, Locally Finite Varieties, with Permuting Congruence Relations*, Berkeley Manuscript, 1976.

7. McKenzie, R., *The number of non-isomorphic models in quasi-varieties of semigroups*, Algebra Universalis 16(1983), 195–203.

8. McKenzie, R., McNulty, G., and Taylor, W., *Algebras, Lattice, Varieties*, Vol. 1, The Wadsworth and Brooks/Cole Mathematical Series, 1987.

9. McKenzie, R., *Congruence extension, Hamiltonian, and abelian properties in locally finite varieties*, Algebra Universalis 28(1991), 589–603.

10. Taylor, W., *Some applications of the term condition,* Algebra Universalis 14(1982), 11–24.

11. Warne, R.J., *Semigroups obeying the term condition*, Algebra Universalis, 1993.

12. Warne, R.J., *On the structure of TC semigroups*, Proceedings of International Conference on Semigroups: Algebraic Theory and Applications to Formal Languages and Codes, Luino, Italy, June, 1992, to be published by World Scientific; Technical Report No., 137, Department of Mathematical Sciences, KFUPM, Dhahran, Saudi Arabia, April 1993.

13. Warne, R.J., *TC Semigroups and related semigroups*, Workshop on Semigroups, Formal Languages, and Combinatorics on Words, Kyoto, Japan, August 1992, Abstracts, 130–132; Technical Report No. 138, Department of Mathematical Sciences, KFUPM, Dhahran, Saudi Arabia, April, 1993.

14. Yamada, M., *A note on middle unitary semigroups*, Kodai Math. Sem. Rep. 7(1955), 49–52.