King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 157

December 1993

Modulational Instability in a Defocussing Coupled Non-Linear Schroedinger System

Otic C. Wright

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

Modulational Instability in a Defocussing Coupled Non-linear Schroedinger System

Otis C. Wright
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran, 31261
Kingdom of Saudi Arabia

The plane wave modulation equations of an integrable coupled non-linear Schroedinger system, with a *trigonal* spectral curve, are placed in Riemann invariant form. In the defocussing case there is a boundary between modulational stability and instability, in marked contrast to the scalar integrable NLS which is always stable. The presence of the instability is consistent with the experimental and numerical results of Rothenberg [5] for a near integrable defocussing coupled NLS system arising in the propagation of ultra-fast signals in an optical fiber.

1 Introduction

In this report a two-timing modulational ansatz is applied to the simplest solutions of an integrable coupled nonlinear Schroedinger (CNLS) system to obtain a coupled system of two eikonal and two transport equations. If this system of four equations is hyperbolic then it governs the slow-scale evolution of the modulational ansatz. However if the system is elliptic then a modulational instability is present. It is shown that these modulation equations can be placed in Riemann invariant form due to the integrability of the underlying system. There is a *trigonal* spectral curve associated with the Lax pair of differential operators which integrates the CNLS system (see Krichever [1], Manakov [2], and McKean [3] for work on trigonal systems). The branch points of this trigonal spectral curve are the Riemann invariants of the modulation equations. Thus the results of Flaschka, Forest, and McLaughlin [6](KdV), Forest and Lee [7](scalar NLS), and Forest and McLaughlin [8](sinh-gordon and sine-gordon) are extended, in the special case of plane waves, from two-sheeted spectral curves to a non-trivial three-sheeted spectral curve of genus zero.

The Riemann invariant speeds are constructed explicitly from differentials on the *trigonal* spectral curve. This leads to the discovery of an explicit boundary between regions of hyperbolicity and ellipticity for the defocussing modulation equations . This is in contrast to the *scalar* integrable defocussing NLS equation whose modulation equations are *always* hyperbolic.

Recent studies, see Rothenberg [5], in the propagation of ultra-fast laser pulses (terahertz frequencies) have observed the instability of a near-integrable defocussing NLS system. This report shows that a modulational instability persists in the integrable *defocussing* case and so it may be a useful qualitative model. Further studies, extending the work done by Ercolani, Forest, and McLaughlin [4] on instabilities of integrable equations, are in preparation, see Muraki, Wright, McLaughlin, Forest, and David [9].

2 Spectral Theory

Consider the coupled system

$$i(p_t + \delta p_x) + \frac{1}{2}p_{xx} - \sigma(|p|^2 + K|q|^2)p = 0$$

$$i(q_t - \delta q_x) + \frac{1}{2}q_{xx} - \sigma(K|p|^2 + |q|^2)q = 0$$

In the work of Rothenberg [5], p and q represent the orthogonal components of an electric field propagating in a glass fiber. The coefficient δ represents an asymmetry in the fiber leading to different group velocities in the two polarizations; this effect is called birefringence. The usual Kerr nonlinearity in the refractive index of the glass leads to $K = \frac{2}{3}$. The defocussing (+1) and focussing (−1) cases are distinguished by $\sigma = \pm 1$.

The above system becomes integrable if $\delta = 0$ and $K = 1$, in this sense we consider it to be near-integrable and henceforth work with the integrable case only.

The CNLS system

$$ip_t + \frac{1}{2}p_{xx} - \sigma(|p|^2 + |q|^2)p = 0$$

$$iq_t + \frac{1}{2}q_{xx} - \sigma(|p|^2 + |q|^2)q = 0$$

is integrable in the sense that it is equivalent to the compatibility condition of the following Lax pair of third order differential operators

$$\begin{aligned} \psi_x &= L\psi \\ i\psi_t &= B\psi \end{aligned} \tag{1}$$

where

$$\begin{aligned} L &= iED + N \\ B &= iEL + \tfrac{1}{2}D(N_x - N^2) \end{aligned} \tag{2}$$

and

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 0 & p & q \\ \sigma p^* & 0 & 0 \\ \sigma q^* & 0 & 0 \end{pmatrix}. \tag{3}$$

The Lax pair can also be placed in matrix form as

$$\begin{aligned} Q_x &= [L, Q] \\ iQ_t &= [B, Q] \end{aligned} \tag{4}$$

where $[A, B] = AB - BA$ and Q is a 3×3 "squared eigenfunction" matrix.

The importance of the Floquet multiplier curve of L, the spatial operator in the Lax pair, has long been known in studies of periodic or quasi-periodic solutions of completely integrable KdV-like equations. However it is the characteristic polynomial of the squared eigenfunction matrix Q which underlies the previous work on modulations of integrable equations with two-sheeted spectral curves.

Indeed the Floquet multiplier curve of L and the characteristic polynomial of the squared eigenfunction matrix Q have the same branch points when considered as sheeted coverings of the Riemann sphere. The branch points contain the modulational information of the solutions of the integrable system. Since the characteristic polynomial of the squared eigenfunction matrix is an algebraic curve by construction, it is easier to work with it than the transcendental Floquet multiplier curve. For a further discussion of the squared eigenfunction matrix, see the Appendix.

3 Plane Wave Modulations

The simplest non-trivial solutions of the integrable CNLS are plane waves

$$p = p_0 e^{i(kx - \omega t)}$$

$$q = q_0 e^{i(kx - \omega t)}$$

where the dispersion relation is given by

$$\omega = \frac{1}{2}k^2 + \sigma(|p_0|^2 + |q_0|^2)$$

The modulation equations come from the two-timing modulational ansatz

$$p = p(\epsilon) e^{iS_p(\epsilon)/\epsilon}$$

$$q = q(\epsilon) e^{iS_q(\epsilon)/\epsilon}$$

where $p(\epsilon)$, $q(\epsilon)$, $S_p(\epsilon)$, and $S_q(\epsilon)$ are slowly varying real functions; viz. they vary on a slow time and space scale $T = \epsilon t$ and $X = \epsilon x$. The generalized wavenumbers are denoted by $(S_p)_x = k_p(\epsilon)$ and $(S_q)_x = k_q(\epsilon)$.

The eikonal and leading order transport equations arising from this ansatz produce a coupled system of four equations. If we change variables according to

$$
\begin{aligned}
k &= \frac{1}{2}(k_p(\epsilon) + k_q(\epsilon)) \\
d &= \frac{1}{2}(k_p(\epsilon) - k_q(\epsilon)) \\
r &= p(\epsilon)^2 + q(\epsilon)^2 \\
s &= p(\epsilon)^2 - q(\epsilon)^2
\end{aligned}
$$

we obtain *the modulation equations*:

$$
\vec{v}_T + \begin{pmatrix} k & d & \sigma & 0 \\ d & k & 0 & 0 \\ r & s & k & d \\ s & r & d & k \end{pmatrix} \vec{v}_X = 0
$$

where $\vec{v} = (k, d, r, s)^T$.

It is the above quasilinear system of four equations which will be placed in Riemann invariant form. In general this is only possible for 2×2 systems of quasilinear partial differential equations. Flaschka, Forest, and McLaughlin [6] showed that this is possible for the modulation equations of N-phase quasiperiodic solutions of the KdV equation. Forest and Lee [7] extended that result to the integrable scalar NLS equation. In both cases the Riemann invariants are branch points of a two-sheeted Riemann surface. We will show that the same is true in the present case even though the genus zero curve is *three-sheeted.*

Even though this analysis is restricted to the genus zero case (viz. modulation of plane waves only), the construction of the correct differentials on the Riemann surface to obtain the Riemann speeds should set up the machinery required for studying higher phase modulations.

4 Riemann Invariants

We seek a change of variables which will place the modulation equations in Riemann invariant form, viz. diagonalize the system. It turns out that the Riemann invariants are the branch points of the characteristic polynomial of the squared eigenfunction matrix solution of the Lax pair whose potentials are plane wave solutions of the CNLS system. The modulation of the plane waves is equivalent to the slow time and space variations of the spectral curve via its branch points.

The plane wave solutions correspond to a linear-in-E ansatz for the squared eigenfunction matrix Q:

$$
Q = iE \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & p & q \\ \sigma p^* & \frac{ik_p}{2} & 0 \\ \sigma q^* & 0 & \frac{ik_q}{2} \end{pmatrix}
$$

The characteristic polynomial (viz. the spectral curve) of Q (normalized to have zero trace) is

$$\lambda^3 + a(E)\lambda + b(E) = 0$$
$$a(E) = \frac{1}{12}(2E+k)^2 + \frac{d^2 - \sigma r}{4}$$
$$b(E) = \frac{-i}{108}(2E+k)^3 + (\frac{\sigma r i}{24} + \frac{i d^2}{12})(2E+k) - \frac{i\sigma d s}{8}$$

Considering λ as a function of E produces a three sheeted covering of genus zero of the E sphere. Generically the covering has four branch points, which degenerate to two if $d \to 0$ since two of them "pinch-off" at the point over infinity.

The four branch points of the characteristic polynomial are the zeros of the discriminant:

$$\begin{aligned}\Delta(E) &= 4a^3(E) + 27b^2(E) \\ &= \frac{d^2}{16}(2E+k)^4 - \frac{\sigma d s}{16}(2E+k)^3 + (-\frac{d^4}{8} - \frac{5}{16}\sigma r d^2 + \frac{r^2}{64})(2E+k)^2 + \\ &+ \frac{27\sigma d s}{48}(d^2 + \frac{\sigma r}{2})(2E+k) + \frac{(d^2-\sigma r)^3}{16} - \frac{27 d^2 s^2}{64}\end{aligned}$$

The four zeros of $\Delta(E)$ are locations of a square root type of branch point in the Riemann surface of the characteristic polynomial. *The roots of Δ are the Riemann invariants of the modulation equations.*

By constructing two canonical differentials on the Riemann surface of the characteristic polynomial we can show the equivalence of the modulation equations to a diagonal system in which the new variables are the branch points and the speeds are functions of the branch points.

Let Ω_1 and Ω_2 be the unique differentials of the second kind which are holomorphic everywhere except at the three points over infinity, where they have the following asymptotic behaviour:

$$\begin{aligned}\Omega_1 &=_{\infty_1} \frac{i}{2\zeta^2} + O(1) \\ &=_{\infty_2} \frac{i}{2\zeta^2} + O(1) \\ &=_{\infty_3} -\frac{i}{\zeta^2} + O(1)\end{aligned}$$

$$\begin{aligned}\Omega_2 &=_{\infty_1} \frac{-i}{\zeta^3} + O(1) \\ &=_{\infty_2} \frac{-i}{\zeta^3} + O(1) \\ &=_{\infty_3} \frac{2i}{\zeta^3} + O(1)\end{aligned}$$

where $\zeta = 1/E$ is a local coordinate at $E = \infty$. This parallels the construction in Forest and Lee [7] for the scalar NLS case. Note that at $E = \infty$ there is a triple point which is not branched, there being three distinct function elements there, but two function elements have identical singular parts, thus Ω_1 and Ω_2 also have identical singularities at two of the points over ∞.

The explicit expressions for these two differentials are

$$\Omega_1 = \frac{(\lambda - \frac{i}{3}E - \frac{ik}{6})(E + \frac{k}{2}) + \frac{id^2}{4} + \frac{i\sigma r}{8}}{3\lambda^2 + a} dE \tag{5}$$

$$\Omega_2 = -2\frac{(\lambda - \frac{i}{3}E - \frac{ik}{6})(E^2 + \frac{k}{2}E - \frac{3\sigma r}{8}) + (\frac{id^2}{4} + \frac{i\sigma r}{8})E - \frac{3\sigma ids}{16}}{3\lambda^2 + a} dE \tag{6}$$

In order to prove that the modulation equations are equivalent to a diagonal system of four equations in which the branch points are the Riemann invariants, it is necessary to know some higher order terms in the expansions of the two differentials at infinity.

Explicit expansion shows that near $\frac{1}{\zeta} = E = \infty$,

$$\Omega_1 =_{\infty_{1,2}} \frac{i}{2\zeta^2} \pm \frac{3i\sigma}{16}(r \mp s) - \frac{i\sigma(r \mp s)}{8d}(\frac{3\sigma(r \pm s)}{8} \mp \frac{3kd}{2} + \frac{3d^2}{2})\zeta + O(\zeta^2)$$

$$\Omega_1 =_{\infty_3} \frac{-i}{\zeta^2} - \frac{3i\sigma r}{8} + \frac{3i\sigma}{8}(kr + ds)\zeta + (-\frac{9i\sigma kds}{16} - \frac{9i\sigma rd^2}{32} - \frac{9ir^2}{32} - \frac{9i\sigma rk^2}{32})\zeta^2 + \\ + \frac{1}{16}(3\sigma irk^3 + 9i\sigma k^2ds + 9i\sigma rkd^2 + 3i\sigma d^3s + 9ikr^2 + 9irds)\zeta^3 + O(\zeta^4)$$

$$\Omega_2 =_{\infty_{1,2}} \frac{-i}{\zeta^3} - \frac{3i\sigma}{32}(r \mp s)(\mp k + d) + \\ \pm \frac{3i\sigma}{16}\frac{(r \mp s)}{d^2}(\frac{\sigma kd}{4}(\mp r - s) + k^2d^2 \mp 2kd^3 + d^4 + \frac{\sigma rd^2}{2})\zeta + O(\zeta^2)$$

$$\Omega_2 =_{\infty_3} \frac{2i}{\zeta^3} - \frac{3i\sigma}{8}(rk + ds) + (\frac{3i\sigma k^2r}{8} + \frac{3i\sigma kds}{4} + \frac{3ir^2}{16} + \frac{3\sigma ird^2}{8})\zeta + \\ - \frac{1}{32}(27i\sigma dsk^2 + 18ikr^2 + 27i\sigma rkd^2 + 9i\sigma sd^3 + 18idsr + 9\sigma irk^3)\zeta^2 + \\ + \frac{1}{32}(6i\sigma d^4r + 36i\sigma d^2k^2r + 6i\sigma k^4r + 18id^2r^2 + 27ik^2r^2 + 6i\sigma r^3 + \\ + 24i\sigma d^3ks + 24i\sigma dk^3s + 54idkrs + 9id^2s^2)\zeta^3 + O(\zeta^4)$$

With these expansions and with the aid of the Riemann-Roch theorem we can now prove the central result:

Theorem 1 *The modulation equations are equivalent, for generic initial data, to the following Riemann invariant system of diagonal quasilinear p.d.e.'s:*

$$\frac{\partial E_j}{\partial T} - 2\frac{(\lambda_j - \frac{i}{3}E_j - \frac{ik}{6})(E_j^2 + \frac{k}{2}E_j - \frac{3\sigma r}{8}) + (\frac{id^2}{4} + \frac{i\sigma r}{8})E_j - \frac{3\sigma ids}{16}}{(\lambda_j - \frac{i}{3}E_j - \frac{ik}{6})(E_j + \frac{k}{2}) + \frac{id^2}{4} + \frac{i\sigma r}{8}}\frac{\partial E_j}{\partial X} = 0 \tag{7}$$

where $j = 1,2,3,4$ (without summation) correspond to the four branch points E_j which are the roots of $\Delta(E) = 0$. λ_j is the root of

$$\lambda^3 + a(E)\lambda + b(E) = 0$$

corresponding to the double root at the branch point $E = E_j$.

In particular the Riemann invariant speeds are real if all the branch points are real but are generically complex if any of the branch points are complex.

Proof:

Consider the differential

$$\Omega = \frac{\partial \Omega_1}{\partial T} + \frac{\partial \Omega_2}{\partial X}$$

which is holomorphic everywhere on the Riemann surface except at the branch points where it has second order poles whose coefficients are precisely the four equations in the Riemann invariant system. Note that the residues at each of these poles are automatically zero as can be seen from applying the chain rule and noting that each term coming from differentiating with respect to a branch point is itself a differential with only one pole, whose residue is automatically zero.

Thus, if the Riemann invariant equations are satisfied then $\Omega \equiv 0$. Calculating the expansion of Ω over $E = \infty$ show that the first two terms in the three expansions imply that

$$\frac{\partial r}{\partial T} + \frac{\partial}{\partial X}(rk + sd) = 0$$

$$\frac{\partial s}{\partial T} + \frac{\partial}{\partial X}(rd + sk) = 0$$

$$r(\frac{\partial k}{\partial T} + k\frac{\partial k}{\partial X} + d\frac{\partial d}{\partial X} + \sigma\frac{\partial r}{\partial X}) + s(\frac{\partial d}{\partial T} + d\frac{\partial k}{\partial X} + k\frac{\partial d}{\partial X}) = 0$$

$$-\frac{(r^2 - s^2 - 4\sigma r d^2)}{d^2}(\frac{\partial d}{\partial T} + d\frac{\partial k}{\partial X} + k\frac{\partial d}{\partial X}) + 4\sigma s(\frac{\partial k}{\partial T} + d\frac{\partial d}{\partial X} + k\frac{\partial k}{\partial X} + \sigma\frac{\partial r}{\partial X}) = 0$$

which, in general, imply the modulation equations.

Conversely, if the modulation equations are satisfied then explicit calculation of the expansions at ∞ shows that the divisor of Ω is greater than or equal to

$$D = \infty_1^2\,\infty_2^2\,\infty_3^4\,E_1^{-2}\,E_2^{-2}\,E_3^{-2}\,E_4^{-2}$$

where $\infty_1, \infty_2, \infty_3$ are the three points over infinity, ∞_1 and ∞_2 having the same singular term. The Riemann-Roch theorem implies that the dimension of the space of meromorphic differentials having divisor greater than or equal to D is zero. Therefore $\Omega \equiv 0$. Thus the Riemann invariant equations must be satisfied by the four branch points.

Finally, if all the E_j are real than $i\lambda_j$ are double roots of a real polynomial and consequently λ_j must be purely imaginary. Therefore the Riemann invariant speeds are real if all the branch points are real.

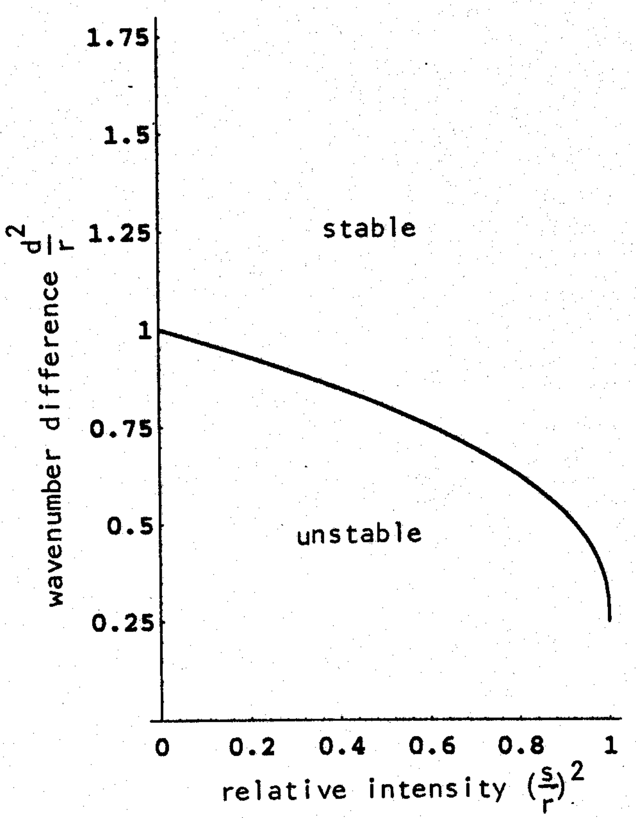

Figure 1: Regions of Modulational Stability and Instability

Corollary 1 *The defocussing modulation equations may be either hyperbolic or elliptic, depending on the values of the modulating variables.*

Proof:

In the defocussing case the modulation variables (k, d, r, s) can be chosen so that there are four real branch points (hence real speeds and hyperbolicity) or two real branch points and two complex branch points (hence complex speeds and ellipticity). To demonstrate the existence of both elliptic (unstable) and hyperbolic (stable) regions, we note that the speeds of the Riemann invariants satisfy the characteristic polynomial of the matrix defining the quasilinear system of modulation equations. In the defocussing case this matrix is:

$$\begin{pmatrix} k & d & 1 & 0 \\ d & k & 0 & 0 \\ r & s & k & d \\ s & r & d & k \end{pmatrix}$$

Calculating the discriminant of the characteristic polynomial of the above matrix reveals the transition boundary between regions of modulational stability and instability. The horizontal axis represents the relative intensity of the two channels, $x = (\frac{s}{r})^2$, and the vertical axis represents the normalized wavenumber difference, $y = \frac{d^2}{r}$. The explicit expression for the boundary is

$$x = \frac{1}{27}(-64y^2 + 48y + 15 + \frac{1}{y})$$

For a fixed relative intensity and large wavenumber difference there are four real branch points and the coupled system is modulationally stable but as the wavenumber difference decreases, at the transition boundary two branch points leave the real axis and become a complex conjugate pair, implying complex speeds and modulational instability. Eventually, when the wavenumber difference equals zero, the complex conjugate pair of branch points pinches at infinity and the modulational equations decouple. See Figure 1.

Remark: The focussing system is always modulationally unstable.

Acknowledgements

The author would like to thank David McLaughlin for suggesting this problem, and David Muraki and Greg Forest for many interesting conversations.

5 Appendix: The Squared Eigenfunction Matrix

If infinitely many linearly independent Q, parametrized by E, satisfy Equation (4) for the same potentials (by which we mean the entries in the matrix N) then those potentials must satisfy the CNLS system. Thus special classes of solutions can be generated by making a polynomial in E ansatz for Q, where E plays the role of a spectral parameter. This polynomial ansatz underlies the second order operator work done by Flaschka, Forest and McLaughlin [6] and by Forest and Lee [7].

Also, for fixed potentials, if Ψ is a matrix solution of Equation (1) and if Φ is a matrix solution of the adjoint problem, then for any constant matrix C, $\Psi C \Phi$ is a solution of Equation (4).

Now consider periodic potentials p and q with period ℓ for which solutions of the Lax pair exist. Let $M(x,t)$ be the fundamental matrix solution of the Lax pair. Suppose we construct a $Q(x,t)$, in terms of certain potentials p and q, which satisfies the matrix Lax pair. Then Q and M are related by

$$Q(x,t) = M(x,t)Q(0,0)M^{-1}(x,t), \tag{8}$$

because of the uniqueness of solutions to ordinary differential equations.

Thus the characteristic polynomial of $Q(x,t)$ is an integral of the CNLS flow. Moreover, if Q is a polynomial in the spectral parameter E then the characteristic polynomial of Q is an algebraic curve.

On the other hand, the multiplier curve for the spatial Lax operator is a transcendental curve given by

$$\det(\rho - M(\ell, t)) = 0 \tag{9}$$

where ρ is the Floquet multiplier.

These fundamental objects are related by the following:

Theorem 2 *If the entries of Q have the same period as the potentials, and if the characteristic polynomial of Q is irreducible, then its desingularisation is conformally equivalent to the desingularised Floquet multiplier curve.*

Proof: By periodicity,

$$Q(0,t)M(\ell,t) = M(\ell,t)Q(0,t) \tag{10}$$

Irreducibility insures that for all but isolated E the two matrices share the same eigenspace because for all but isolated E there are three distinct roots of the characteristic polynomials. By considering analytic function elements one obtains conformal equivalence of the desingularized curves.

Remarks: Arbitrary choices of $Q(0,0)$ will not produce a Q with periodic elements. In practice we can construct Q and the potentials together according to some ansatz because of the integrability of the system, then by uniqueness we see the relation between Q and the fundamental matrix of the Lax operators for those potentials.

In the case of periodic plane wave potentials the multiplier curve can be explicitly mapped onto an algebraic curve via the mapping

$$\rho(E) \mapsto \exp(i\mu(E)\ell), \tag{11}$$

where $\mu(E)$ is a root of the characteristic polynomial of Q (modulo a linear change of variable). Since the mapping is x, t independent and the characteristic polynomial of Q is independent of x and t, we conclude that the multiplier curve is itself an integral of the flow.

References

[1] Krichever, I. M., "Spectral theory of two-dimensional periodic operators and its applications," Russian Math. Surveys, 44(2):145-225 (1989).

[2] Manakov, S. V., ZETP, 65, 505 (1973).

[3] McKean, H. P., "Boussinesq's equation on the Circle", CPAM, 34, 599-691 (1981).

[4] Ercolani, N., Forest, M. G., McLaughlin, D. W., "The Origin and Saturation of Modulational Instabilities," Physica D (1986) 18:472-474.

[5] Rothenberg, J. E., "Observation of the Buildup of Modulational Instability from Wave Breaking," Optical Letters 16(1): 18-20, 1991.

[6] Flaschka, H., M.G. Forest, and D. W. McLaughlin, "Multiphase Averaging and the Inverse Spectral Solution of the Korteweg-de Vries Equation," C. P. A. M., 33, 1980, 739-784.

[7] Forest, M.G., and J. E. Lee, "Geometry and Modulation Theory for the periodic Nonlinear Schroedinger equation," Oscillation Theory, Computation, and Methods of Compensated Compactness; Dafermos, et al, eds., I.M.A. in Math. and its Appl., 2:35-70, Springer-Verlag, NY, 1986.

[8] Forest, M. G., and D. W. McLaughlin, "Modulations of Sinh-Gordon and Sine-Gordon Wavetrains," Studies in Applied Mathematics, 68:11-59 (1983).

[9] Muraki, D., O. C. Wright, D. W. McLaughlin, M. G. Forest, D. David, "Instabilities and Oscillations in a Coupled Nonlinear Schroedinger System," in preparation.

[10] Whitham, G. B., "Non-linear dispersive waves," Royal Society of London, Proceedings, Ser. A 283, 1965, 238-261.