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Direct Criterion of Memoryless Stabilizability Of Delay Systems

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Abstract

State feedback controllers for delay systems are generally difficult and/or expensive to implement because of the infinite dimensionality of the state of this class of systems. In certain cases however, adequate stabilization can be achieved through the use of a memoryless feedback controller, namely, a controller based on the finite dimensional instantaneous state. This work delineates a sufficient condition which guarantees the memoryless stabilizability of delay systems.

1 Introduction

Consider a delay system described by

$$\begin{aligned}\mathcal{S}_d: \quad \dot{x}(t) &= A_0x(t) + \sum_{j=1}^{n_d} A_jx(t-r_j) + \int_{-r}^{0} A(\theta)x(t+\theta)d\theta \\ &\quad + B_0u(t) + \sum_{j=1}^{m_d} B_ju(t-h_j) + \int_{-h}^{0} B(\theta)u(t+\theta)d\theta \qquad (1.1)\end{aligned}$$

where $t > 0$ represents time, $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $0 = r_0 < r_1 < r_2, < \ldots, < r_{n_d} \leq r < \infty$, $0 = h_0 < h_1 < h_2, < \ldots, < h_{m_d} \leq h < \infty$; $A_j \in \mathbf{R}^{n\times n}$, $j = 0, 1, \ldots, n_d$, $B_j \in \mathbf{R}^{n\times m}$, $j = 0, 1, \ldots, m_d$, $A(\cdot)$ is an $\mathbf{R}^{n\times n}$ valued piecewise continous function on $[-r, 0]$ and $B(\cdot)$ is an $\mathbf{R}^{n\times m}$ valued piecewise continuous function on $[-h, 0]$. Let $\mathcal{C}([-r, 0]; \mathbf{R}^n)$ denote the class of $\mathbf{R}^n$-valued continuous functions on $[-r, 0]$ and $\phi \in$

$\mathcal{C}([-r,0];\mathbf{R}^n)$ represent the initial function of $\mathcal{S}_d$. Following Hale (1977), $x(t;\phi,u)$ is the solution of $\mathcal{S}_d$ if it coincides with ϕ on $[-r,0]$ and satisfies (1.1) for $t>0$. $\mathcal{S}_d$ is said to be stable if, for t sufficiently large, there exist $k>0$ and $\nu>0$ such that $\|x(t;\phi,0)\|<ke^{-\nu t}$

The problem of interest may therefore be posed as follows: If $\mathcal{S}_d$ is not stable, determine conditions under which $\mathcal{S}_d$ can be stabilized by a memoryless (instantaneous) state controller of the form $u(t)=-Kx(t)$ where $K\in\mathbf{R}^{m\times n}$. The presence of instrumentation and/or computational delays in a feedback closed system is noted. Thus, to make the notion of memoryless feedback control practically meaningful, we have incorporated control variable delays into the system model, (1.1), as a means of accounting for any instrumentation and/or computational delays.

Using Lyapunov functionals, conditions for memoryless stabilizability of various sub-classes of (1.1) have been obtained by Feliachi and Thowsen (1981), Kwon and Pearson (1977) and by Mori *et al.* (1983). Our approach is direct and employs Tonneli's Theorem (see, for example, Bartle (1966), p. 118) and Gronwall Inequality (see, for example, Mitrinovic *et al.* (1991)).

Theorem 1.1 *Assume that (A_0,B_0) is a controllable pair. Let $K\in\mathbf{R}^{m\times n}$ be such that $\|e^{(A_0-B_0K)t}\|<Me^{-\nu_0 t}$ where $1\le M<\infty$ and $\nu_0>0$. If*

$$e^{\nu_0 r}\left(\sum_{j=1}^{n_d}\|A_j\|+\int_{-r}^{0}\|A(\theta)\|d\theta\right)+\|K\|e^{\nu_0 h}\left(\sum_{j=1}^{m_d}\|B_j\|+\int_{-h}^{0}\|B(\theta)\|d\theta\right)<\frac{\nu_0}{M} \tag{1.2}$$

then $\mathcal{S}_d$ is memoryless stabilizable by

$$u(t)=-Kx(t) \tag{1.3}$$

Remark 1.1 *It is known that the property of controllability of a pair of matrices in $\mathbf{R}^{n\times n}\times\mathbf{R}^{n\times m}$ is generic (see Wonham (1979), p.44). This means that the pair $(A_0,B_0)\in\mathbf{R}^{n\times n}\times\mathbf{R}^{n\times m}$ in a randomly selected delay system, $\mathcal{S}_d$, will almost surely be completely controllable. Thus, the above theorem can be paraphrased as: If the delays* *and the delay coefficient matrices are sufficiently small, then S_d will be memoryless stabilizable.*

Proof. In view of the complete controllability of the pair (A_0, B_0), there is a $K \in \mathbf{R}^{m\times n}$ such that $\Re e\{\sigma(A_0 - B_0K)\} \subset (-\infty, -\nu_0)$ where $\sigma(\cdot)$ denotes the spectrum of $(\cdot)$. Put $(A_0 - B_0K) = PJP^{-1}$ where J is a Jordan matrix such that $\sigma(J) = \sigma(A_0 - B_0K)$ and P is a modal matrix of $(A_0 - B_0K)$. Then $\|e^{(A_0-B_0K)t}\| = \|Pe^{(A_0-B_0K)t}P^{-1}\| \le \|P\|\ \|P^{-1}\|e^{-\nu_0 t}$. Thus, one can choose $M = \|P\|\ \|P^{-1}\|$. With this choice, it is evident that $M \ge 1$. Under the memoryless feedback law, (1.3), S_d is transformed into

$$\begin{aligned}\overline{S}_d: \quad \dot{x}(t) &= (A_0 - B_0K)x(t) + \sum_{j=1}^{n_d} A_j x(t-r_j) - \sum_{j=1}^{m_d} B_j K x(t-h_j) \\ &\quad - \int_{-h}^{0} B(\theta)Kx(t+\theta)d\theta + \int_{-r}^{0} A(\theta)x(t+\theta)d\theta \qquad (1.4)\end{aligned}$$

To demonstrate stability of $\overline{S}_d$, we must show that for t sufficiently large, there exist $k > 0$ and $\nu > 0$ such that $\|x(t)\| < ke^{-\nu t}$ where $x(\cdot)$ is the solution of $\overline{S}_d$. To this end, the variation of constants formula is applied to $\overline{S}_d$ to obtain

$$\begin{aligned}x(t) = e^{(A_0-B_0K)t}x(0) &+ \sum_{j=1}^{n_d}\int_0^t e^{(A_0-B_0K)(t-\tau)}A_j x(\tau - r_j)d\tau \\ &- \sum_{j=1}^{m_d}\int_0^t e^{(A_0-B_0K)(t-\tau)}B_j K x(\tau - h_j)d\tau \\ &+ \int_0^t\int_{-r}^0 e^{(A_0-B_0K)(t-\tau)}A(\theta)x(\tau+\theta)d\theta d\tau \\ &- \int_0^t\int_{-h}^0 e^{(A_0-B_0K)(t-\tau)}B(\theta)Kx(\tau+\theta)d\theta d\tau\end{aligned}$$

Using the triangle inequality together with $\|e^{(A_0-B_0K)t}\| \le Me^{-\nu_0 t}$, the last equation becomes

$$\begin{aligned}\|x(t)\| \le Me^{-\nu_0 t}\|x(0)\| &+ \sum_{j=1}^{n_d}\int_{-r_j}^{t-r_j} Me^{-\nu_0(t-r_j-\tau)}\|A_j\|\ \|x(\tau)\|d\tau \\ &+ \sum_{j=1}^{m_d}\int_{-h_j}^{t-h_j} Me^{-\nu_0(t-h_j-\tau)}\|B_j\|\ \|K\|\ \|x(\tau)\|d\tau\end{aligned}$$

$$+ \int_0^t \int_{-r}^0 M e^{-\nu_0(t-\tau)} \|A(\theta)\| \, \|x(\tau+\theta)\| d\theta d\tau$$
$$+ \int_0^t \int_{-h}^0 M e^{-\nu_0(t-\tau)} \|B(\theta)\| \, \|K\| \, \|x(\tau+\theta)\| d\theta d\tau.$$

That is,

$$\begin{aligned} e^{\nu_0 t}\|x(t)\| \le M\|x(0)\| &+ \sum_{j=1}^{n_d} M e^{\nu_0 r_j} \|A_j\| \int_{-r_j}^{t-r_j} e^{\nu_0 \tau} \|x(\tau)\| d\tau \\ &+ \sum_{j=1}^{m_d} M e^{\nu_0 h_j} \|B_j\| \, \|K\| \int_{-h_j}^{t-h_j} e^{\nu_0 \tau} \|x(\tau)\| d\tau \\ &+ \int_0^t \int_{-r}^0 M e^{\nu_0 \tau} \|A(\theta)\| \, \|x(\tau+\theta)\| d\theta d\tau \\ &+ \int_0^t \int_{-h}^0 M e^{\nu_0 \tau} \|B(\theta)\| \, \|K\| \|x(\tau+\theta)\| d\theta d\tau \end{aligned}$$

Extending all the upper limits of integration to t yields

$$\begin{aligned} e^{\nu_0 t}\|x(t)\| \le M\|x(0)\| &+ \sum_{j=1}^{n_d} M e^{\nu_0 r_j} \|A_j\| \int_{-r_j}^{t} e^{\nu_0 \tau} \|x(\tau)\| d\tau \\ &+ \sum_{j=1}^{m_d} M e^{\nu_0 h_j} \|B_j K\| \int_{-h_j}^{t} e^{\nu_0 \tau} \|x(\tau)\| d\tau \\ &+ \int_0^t \int_{-r}^0 M e^{\nu_0 \tau} \|A(\theta)\| \, \|x(\tau+\theta)\| d\theta d\tau \\ &+ \int_0^t \int_{-h}^0 M e^{\nu_0 \tau} \|B(\theta)\| \, \|K\| \|x(\tau+\theta)\| d\theta d\tau \qquad (1.5) \end{aligned}$$

Now observe that

$$\begin{aligned} &\int_0^t \int_{-r}^0 M e^{\nu_0 \tau} \|A(\theta)\| \|x(\tau+\theta)\| d\theta d\tau \\ &\qquad = \int_0^t \int_{-r}^0 M \|A(\theta)\| e^{-\nu_0 \theta} e^{\nu_0(\theta+\tau)} \|x(\tau+\theta)\| d\theta d\tau \\ &\qquad = \int_{-r}^0 M \|A(\theta)\| e^{-\nu_0 \theta} \left\{ \int_0^t e^{\nu_0(\theta+\tau)} \|x(\tau+\theta)\| d\tau \right\} d\theta \\ &\qquad = \int_{-r}^0 M \|A(\theta)\| e^{-\nu_0 \theta} \left\{ \int_\theta^{t+\theta} e^{\nu_0 \tau} \|x(\tau)\| d\tau \right\} d\theta \\ &\qquad \le \int_{-r}^0 M \|A(\theta)\| e^{-\nu_0 \theta} d\theta . \int_{-r}^t e^{\nu_0 \tau} \|x(\tau)\| d\tau \end{aligned}$$

where Tonneli's Theorem has been employed to interchange the order of the integration. Similarly, with reference to (1.5),

$$\int_0^t \int_{-h}^0 M e^{\nu_0 \tau} \|B(\theta)\| \, \|K\| \, \|x(\tau+\theta)\| d\theta d\tau$$

$$= \int_0^t \int_{-h}^0 M\|B(\theta)\| \, \|K\| e^{-\nu_0\theta} e^{\nu_0(\theta+\tau)} \|x(\tau+\theta)\| d\theta d\tau$$
$$= \int_{-h}^0 M\|B(\theta)\| \, \|K\| e^{-\nu_0\theta} \left\{ \int_0^t e^{\nu_0(\theta+\tau)} \|x(\tau+\theta)\| d\tau \right\} d\theta$$
$$= \int_{-h}^0 M\|B(\theta)\| \, \|K\| e^{-\nu_0\theta} \left\{ \int_\theta^{t+\theta} e^{\nu_0\tau} \|x(\tau)\| d\tau \right\} d\theta$$
$$\leq \int_{-h}^0 M\|B(\theta)\| \, \|K\| e^{-\nu_0\theta} d\theta . \int_{-r}^t e^{\nu_0\tau} \|x(\tau)\| d\tau$$

Employing the last two estimates in (1.5) gives

$$\begin{aligned} e^{\nu_0 t}\|x(t)\| \leq M\|x(0)\| \;&+\; \sum_{j=1}^{n_d} M e^{\nu_0 r_j} \|A_j\| \int_{-r_j}^t e^{\nu_0\tau} \|x(\tau)\| d\tau \\ &+\; \sum_{j=1}^{m_d} M e^{\nu_0 h_j} \|B_j\| \, \|K\| \int_{-h_j}^t e^{\nu_0\tau} \|x(\tau)\| d\tau \\ &+\; \int_{-r}^0 M\|A(\theta)\| e^{-\nu_0\theta} d\theta \int_{-r}^t e^{\nu_0\tau} \|x(\tau)\| d\tau \\ &+\; \int_{-h}^0 M\|B(\theta)\| \, \|K\| e^{-\nu_0\theta} d\theta \int_{-h}^t e^{\nu_0\tau} \|x(\tau)\| d\tau \qquad (1.6) \end{aligned}$$

Break the integrals with upper limit, t, in the last inequality into two parts, one over the initial function, ϕ, and the other over the system response. For example,

$$\int_{-r}^t e^{\nu_0\tau} \|x(\tau)\| d\tau = \int_{-r}^0 e^{\nu_0\tau} \|\phi(\tau)\| d\tau + \int_0^t e^{\nu_0\tau} \|x(\tau)\| d\tau$$

etc. Then (1.6) becomes

$$e^{\nu_0 t}\|x(t)\| \leq k_0 + \int_0^t k_1 e^{\nu_0\tau} \|x(\tau)\| d\tau \qquad (1.7)$$

where

$$\begin{aligned} k_0 = M\|x(0)\| \;&+\; M e^{\nu_0 r} \sum_{j=1}^{n_d} \|A_j\| \int_{-r_j}^0 e^{\nu_0\tau} \|\phi(\tau)\| d\tau \\ &+\; M e^{\nu_0 h} \|K\| \sum_{j=1}^{m_d} \|B_j\| \int_{-h_j}^0 e^{\nu_0\tau} \|\phi(\tau)\| d\tau \\ &+\; M \int_{-r}^0 e^{-\nu_0\theta} \|A(\theta)\| d\theta \int_{-r}^0 e^{\nu_0\tau} \|\phi(\tau)\| d\tau \\ &+\; M\|K\| \int_{-h}^0 e^{-\nu_0\theta} \|B(\theta)\| d\theta \int_{-h}^0 e^{\nu_0\tau} \|\phi(\tau)\| d\tau \end{aligned}$$

and

$$\frac{k_1}{M} = e^{\nu_0 r}\left(\sum_{j=1}^{n_d}\|A_j\| + \int_{-r}^{0}\|A(\theta)\|d\theta\right) + \|K\|e^{\nu_0 h}\left(\sum_{j=1}^{m_d}\|B_j\| + \int_{-h}^{0}\|B(\theta)\|d\theta\right) \quad (1.8)$$

Application of Gronwall Inequality to (1.7) then gives $e^{\nu_0 t}\|x(t)\| \leq k_0 e^{k_1 t}$ or $\|x(t)\| \leq k_0 e^{-(\nu_0 - k_1)t}$. By assumption (1.2), $k_1 < \nu_0$. Therefore $\overline{S}_d$ is stable to complete the proof.

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