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Abstract

In this paper, we prove that for a commutative semiperfect ring, the algebraic compactness of certain reduced powers forces its local factors to be valuation rings. We then use this to derive some characterizations of perfect rings. In particular, we show that a result, originally proved by Jensen and Zimmermann–Huisgen for artinian rings, holds in fact for perfect rings.

1. Introduction

Several authors have studied the algebraic compactness of reduced products in the context of representation theory of algebras and have derived characterizations of various types of rings. In many instances, it is shown that the algebraic compactness of certain reduced products of modules, which, regardless of the structure of the underlying ring, are always $\aleph_0$–compact, forces the ring to have specific properties. In [6] for example, Jensen and Lenzing prove that if R is an uncountable commutative noetherian local ring which is either complete or with an uncountable residue field and if R has algebraically compact reduced products, then R is a principal ideal ring or has Krull dimension 1. Analogous results for finite–dimensional algebras were established in [6] and also by Okoh [11]. In [7], Jensen and Zimmermann–Huisgen prove that a commutative artinian ring, all of whose local summands are uncountable is a principal ideal ring if and only if every $\aleph_1$–generated module has a non–trivial countable reduced power which is algebraically compact. Their proof relies on an adaptation to p–functors of a classical argument of Chase (see [1, 2, 15]), and also makes use of a result of Warfield [12, 13] on commutative artinian rings. In this paper, we show by using a more direct argument, that the stated result can in fact be extended to commutative perfect rings. More generally, we give, in the case of semiperfect rings, a necessary condition for certain filter sums to have algebraically compact reduced powers. The idea here is to modify a constructive argument given by Jensen and Zimmermann–Huisgen in [7] so as to conduct the proofs in a filter–theoretic rather than a ring–theoretic context. This will provide a different and a shorter approach to the artinian case since neither Chase's lemma nor Warfield's theorem are used. Furthermore, in our results, algebraic compactness is replaced by the more specific $\aleph$–compactness (see Fuchs [4]) allowing for a further refinement of Jensen and Zimmermann–Huisgen's result.

Throughout this paper, all rings are associative with 1 and all modules are unital. A theory of ordinals is assumed where each cardinal is an initial ordinal. For each cardinal κ, κ^+ denotes the infinite successor cardinal of κ and, for any set S, $|S|$ is the cardinality of S. Finite cardinals are denoted by $\aleph_{-1}$.

2. Notation and Preliminary Results

Definitions. Let M be an R–module and let κ be an infinite cardinal

1. M is *κ–compact* if every finitely solvable system of κ equations over M is solvable. If M is λ–compact for all cardinals λ, M is algebraically compact.
2. A submodule N of M is *κ–pure* in M if every system of less than κ equations over N which is solvable in M is also solvable in N. In particular $\aleph_0$–purity coincides with the usual purity (in the sense of Cohn [3]).

Remark. It can be shown (see for example [4] or [7]), that M is algebraically compact if and only if it is $(|R| + \aleph_0)$-compact.

Next we give some basic facts on filters and introduce a notation that will prove useful in Section 3.

Definitions. Let φ be a filter on a non-empty set I and let κ be an infinite cardinal or $\aleph_{-1}$. Denote by I_φ the set $I \setminus \bigcap_{X \in \varphi} X$ and by φ_κ the filter arising from φ by adding κ intersections.

1. φ is said to be *proper* if $\phi \notin \varphi$. Clearly, $\mathcal{P}(I)$ is the only non-proper filter on I.

2. φ is *κ-complete* if, for each cardinal $\lambda < \alpha$, φ is closed under λ intersections, i.e. $\varphi = \varphi_\lambda$ for all cardinals $\lambda < \kappa$.

3. φ is called a *principal* filter if $I \setminus I_\varphi \in \varphi$. We then say that φ is generated by the set $\bigcap_{X \in \varphi} X$. It is easy to see that φ is principal if and only if $\varphi = \varphi_\lambda$ for all cardinals λ. If $I_\varphi = I$, φ is said to be a *free* filter.

4. If κ is infinite, the set $\{X \subseteq I : |I \setminus X| < \kappa\}$ is a free $cf(\kappa)$-complete filter on I, which we denote by $I(\kappa)$. Note that when $I = \kappa$, this filter is simply the generalized Fréchet filter on κ.

5. Let $\{M_i\}_{i \in I}$ be a family of R-modules. The set $\{m \in \prod_{i \in I} M_i : \{i \in I : m(i) = 0\} \in \varphi\}$ is a submodule of $\prod_{i \in I} M_i$ denoted by $\sum_\varphi M_i$. It is easy to check that it is λ-pure in $\prod_{i \in I} M_i$ if φ is λ-complete for some cardinal λ. Clearly, $\sum_{I(\aleph_0)} M_i = \bigoplus_{i \in I} M_i$ and $\sum_{\mathcal{P}(I)} M_i = \prod_{i \in I} M_i$. As usual, the reduced product $\prod_{i \in I} M_i / \sum_\varphi M_i$ is written $\prod_{i \in I} M_i / \varphi$. If ψ is a filter on I containing φ, then $\sum_\varphi M_i \subseteq \sum_\psi M_i$ and the filter quotient $\sum_\psi M_i / \sum_\varphi M_i$ is also denoted $\sum_\psi M_i / \varphi$.

Proposition 1. *Let φ be a filter on a set I such that $|I_\varphi| = \kappa$. Then φ is non–principal if and only if $\varphi \subset \varphi_\kappa$. Moreover, φ_κ is principal (generated by $I \backslash I_\varphi$).*

Proof. If φ is principal, then clearly $\varphi = \varphi_\kappa$. Assume now that φ is non–principal, then for each $x \in I_\varphi$, we have $I\backslash\{x\} \in \varphi$ (for, otherwise, every X in φ would contain x contradicting that $x \in I_\varphi$). Hence, $I\backslash I_\varphi = \bigcap_{x \in I_\varphi} I\backslash\{x\} \in \varphi_\kappa$ which means that φ_κ is principal and that $I\backslash I_\varphi \in \varphi_\kappa \backslash \varphi$.

Proposition 2. *Let $\{M_i\}_{i\in I}$ be a family of R–modules and let φ be a filter on I, where $|I| = \kappa$. Then,*

(i) $\sum_{\varphi_\kappa} M_i = \prod_{i\in I} M_i'$, *where* $M_i' = \begin{cases} M_i & \text{if } i \in I_\varphi \\ 0 & \text{otherwise} \end{cases}$

(ii) If, in addition, $\{i \in I : M_i \text{ is } \kappa\text{–compact}\} \in \varphi$ and φ is κ–complete, the reduced product $\prod_{i\in I} M_i/\varphi$ is κ–compact.

Proof. By Proposition 1 (and as $\kappa \geq |I_\varphi|$), φ_κ is principal, so $m \in \sum_{\varphi_\kappa} M_i$ if and only if $\bigcap_{X\in\varphi} X \subseteq z(m)$, i.e. if and only if $m \in \prod_{i\in I} M_i'$. This proves (i). To prove (ii), consider the pure exact sequence

$$0 \to \sum_{\varphi_\kappa} M_i/\varphi \to \prod_{i\in I} M_i/\varphi \to \prod_{i\in I} M_i/\varphi_\kappa \to 0.$$

The first term is κ–compact by [9, Corollary 3], and the third is κ–compact because, as φ_κ is principal, $\prod_{i\in I} M_i/\varphi_\kappa \cong \prod_{i\in I\backslash I_\varphi} M_i$ and $I\backslash I_\varphi \subseteq \{i \in I : M_i \text{ is } \kappa\text{–compact}\}$. By [9, Lemma], $\prod_{i\in I} M_i/\varphi$ is κ–compact.

Let us finally state the following result. It is required in Section 3 and its proof is straightforward.

Proposition 3. *Let κ be a cardinal and let I be a two-sided principal ideal of a κ-compact ring R. Then the R-module R/I is κ-compact.*

(Note. Using the definition of κ-compactness, we can in fact obtain that every finitely presented module over a commutative κ-compact ring is itself κ-compact (cf [14, Theorem 6]).)

3. The Main Results

Let R be a commutative local ring which is not a valuation ring, so that there are elements u, v in R with $u \notin (v)$ and $v \notin (u)$. Let J be the maximal ideal of R and H be a subset of $R \backslash J$, whose elements are distinct modulo J. For each $h \in H$, set $M_h = R/(r_h)$ where $r_h = u - hv$.

Lemma 1. *With the above notation, for any filter φ on H and any proper filter ψ on a set I, if $\varphi_{|I|}$ is non-principal, then the reduced power M^I/ψ where $M = \sum_\varphi M_h$, is not $|H_\varphi|$-compact. If, in addition, each M_h is κ-compact for some $\kappa \geq |H_\varphi|$, then the following statements are equivalent:*

(i) M is κ-compact.

(ii) M is $|H_\varphi|$-compact.

(iii) φ is principal.

Proof. Set $P = \prod_{h \in H} M'_h$, where $M'_h = \begin{cases} M_h & \text{if } h \in H_\varphi \\ 0 & \text{otherwise} \end{cases}$.

It is clear that P is isomorphic to $\prod_{h \in H_\varphi} M_h$ (P is in fact the filter sum $\sum_{\varphi_{|H|}} M_h$).

Denote by q_h ($h \in H$) the canonical composition $P \xrightarrow{\text{proj}} M'_h \xrightarrow{\text{inc}} P$, let $\mu \in P$ be given by

$$\mu(h) = \begin{cases} 1 + (r_h) & \text{if } h \in H_\varphi \\ 0 & \text{otherwise} \end{cases},$$

and consider the system

$$x + r_h y_h = d(q_h(v\mu)) \qquad (h \in H_\varphi) \tag{1}$$

with unknowns $x, (y_h)_{h \in H_\varphi}$, where $d : M \to M^I/\psi$, is the diagonal map. The right-hand sides of (1) are in M^I/ψ, since, for each $h \in H_\varphi$, there exists $X \in \varphi$ with $X \subseteq H \backslash \{h\} \subseteq z(q_h(v\mu))$ and this means $q_h(v\mu) \in M$. Also, if (1)′ is the subsystem obtained from (1) by restricting h to a finite subset $\{h_1, h_2, \ldots, h_n\}$ of H_φ, and if w_{ij} $(1 \le i, j \le n, \quad i \ne j)$ are elements of R with $w_{ij}(h_i - h_j) = 1$ (recall that when $i \ne j$, $h_i - h_j \notin J$ and so $h_i - h_j$ is a unit of R), then $x = \sum_{j=1}^{n} dq_{h_j}(v\mu)$, $y_{h_i} = \sum_{j=1}^{n} w_{ij} dq_{h_j}(\mu)$ $(1 \le i \le n)$ is easily seen to be a solution of (1)′ in M^I/ψ. Thus, if M^I/ψ is $|H_\varphi|$-compact, there exist elements $\overline{((x_{it})_{t \in H})_{i \in I}}$, $\overline{((y^h_{it})_{t \in H})_{i \in I}}$ $(h \in H_\varphi)$ in M^I/ψ such that, for each $h \in H_\varphi$, the set $A_h = \{i \in I : (x_{it})_{t \in H} + r_h (y^h_{it})_{t \in H} = q_h(v\mu)\}$ is in ψ. Clearly $A_h \subseteq A'_h$, where $A'_h = \{i \in I : x_{ih} = \mu(h)\}$, and therefore $A'_h \in \psi$ for each $h \in H_\varphi$. Furthermore, for each $i \in I, (x_{it})_{t \in H} \in M$, i.e. the set $B_i = \{h \in H : x_{ih} = 0\} \in \varphi$ and $\bigcap_{i \in I} B_i \in \varphi_{|I|}$. Suppose by way of contradiction, that $\varphi_{|I|}$ is non-principal. Then $\bigcap_{i \in I} B_i \nsubseteq \bigcap_{X \in \varphi} X$, and so there exists $h_0 \in \left(\bigcap_{i \in I} B_i\right) \cap H_\varphi$, which implies that $A'_h = \{i \in I : v\mu(h_0) = 0\} \in \psi$, i.e. $\{i \in I : v \in (r_{h_0})\}$, which is empty, is in ψ, contradicting that ψ is proper. This proves the first part of the proposition. Next, it is obvious that (i) $\Rightarrow$ (ii). Also, if M is $|H_\varphi|$-compact, then so too is the isomorphic copy M^I/ψ where I is a singleton and $\psi = \{I\}$. By the first part, $\varphi_{|I|} = \varphi$ is principal and (iii) follows. Finally, if φ is principal, then $\varphi = \varphi_{|H|}$, giving $M = P$, which is κ–compact, and so (iii) $\Rightarrow$ (i).

Remarks.

1. Although there is no apparent restriction on the size of the set I, Lemma 1 is effective only if $|I| < |H_\varphi|$. This is so because $\varphi_{|H_\varphi|}$ is always principal by Proposition 1.

2. A closer look at the proof above shows that the commutativity of the ring R can in fact be replaced by the following less stringent condition: "For each $h \in H$, $vh \in Rv$". This is the case in particular for left duo rings.

Lemma 1 immediately yields

Corollary 1. *Let R be a commutative semiperfect ring and suppose that it possesses a local factor which is not a valuation ring and with residue field K. Then for any non–empty subset H of K, there exists a family $\{M_h\}_{h\in H}$ of cyclically presented R–modules such that given any filter φ on H and any set I with $\varphi_{|I|}$ non–principal, the reduced power $(\sum_\varphi M_h)^I/\psi$ is not $|H_\varphi|$–compact for any proper filter ψ on I. If, in addition, each M_h is algebraically compact, then the following are equivalent:*

(i) $\sum_\varphi M_h$ is algebraically compact

(ii) $\sum_\varphi M_h$ is $|H_\varphi|$–compact

(iii) φ is principal.

The following appears as Theorem 4.1 in [6]:

Suppose that R is a commutative noetherian local ring with uncountable residue field and ψ a proper filter on N containing all cofinite subsets of N. If R is neither *a discrete valuation ring nor an artinian principal ideal ring, there is a sequence of R–modules M_n of finite length, necessarily algebraically compact, such that $\prod_n M_n/\psi$ is not algebraically compact.*

In this direction, we can use Lemma 1 to obtain a similar result, without the restriction on the filter ψ.

Corollary 2. *Let R be a commutative noetherian local ring with residue field R/J, let κ be an uncountable cardinal such that $|R/J| \geq \kappa$ and let ψ be a proper filter on N. If R is neither a discrete valuation ring nor an artinian principal ideal ring, then there exists a κ–sequence of R–modules $\{M_h\}_{h\in H}$ of finite length, necessarily $\sum$–algebraically compact, such that $\left(\bigoplus_{h\in H} M_h\right)^N \Big/ \psi$ is not κ–compact.*

We now prepare for the main results. First, we need.

Definition. We say that a commutative semiperfect ring is *serial* if each one of its local ring factors is a valuation ring.

A valuation ring need not be a principal ideal ring; however, we have

Lemma 2. *Let R be a commutative perfect ring. Then R is a principal ideal ring if and only if it is serial.*

Proof. Since the local factors of R are perfect and since each one of them is a principal ideal ring exactly when R is a principal ideal ring, we may assume that R is local. Suppose first that R is a principal ideal ring, and hence artinian. Let $J = (v)$ be the Jacobson radical of R. Then, as J is T–nilpotent, for any non–zero non–unit element x of R, there exist a unit $r(x)$ of R and a natural number $n(x)$ such that $x = r(x) \cdot v^{n(x)}$, i.e. $(x) = (v^{n(x)})$. If y is a non–zero non–unit element of R, then clearly either $(x) \subseteq (y)$

or $(y) \subseteq (x)$ according as $n(x) \geq n(y)$ or $n(x) \leq n(y)$. This proves that R is a valuation ring. Conversely, let R be a valuation ring. It is enough from the first part, to show that J is principal. Assume not, and let $\{a_i : i \in I\}$ be a generating set for J, with $a_i \neq 0$ for each $i \in I$. Choose any $i_1 \in I$, then there exists $i_2 \in I$ with $(a_{i_1}) \subset (a_{i_2})$, so that $a_{i_1} = r_1 a_{i_2}$ for some $r_1 \in J$. Similarly, there exist $i_3 \in I$ and $r_2 \in J$ with $a_{i_2} = r_2 a_{i_3}$. Continuing in this way, we obtain elements a_{i_n}, r_n $(n \in N)$ in J such that $a_{i_1} = r_1 r_2 \cdots r_n a_{i_{n+1}}$ for each $n \in N$. Since J is T–nilpotent, $a_{i_1} = 0$, a contradiction.

Proposition 4. *Let R be a commutative perfect ring, let $K_1, K_2, \ldots, K_n$ be the residue fields of its local ring factors and let κ be an infinite cardinal such that $\kappa \leq \min\{|K_i| : 1 \leq i \leq n\}$. Then the following statements are equivalent.*

(i) R is not a principal ideal ring.

(ii) R is not a serial ring.

(iii) There exists a κ–generated R–module M such that for every set I with $|I| < \kappa$ and every proper filter ψ on I, the reduced power M^I/ψ is not κ–compact.

(iv) There exists a countably generated R–module which is not $\aleph_0$–compact.

Proof. (i) $\Leftrightarrow$ (ii) follows from Lemma 2. Also, if R is a principal ideal ring then R is artinian and so every R–module is algebraically compact. Hence (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i). Suppose now that R is not serial and that K_1 is the residue field of a local summand of R which is not a valuation ring. Let H be a subset of K_1 with $|H| = \kappa$ and let φ be the filter $H(\aleph_0)$ of all cofinite subsets of H. Clearly, for every set I with $|I| < \kappa$, $\varphi_{|I|}$ is non–principal and we have $H_\varphi = H$, since $H(\aleph_0)$ is free. This means that the preceding corollary applies and there exists a module M (namely, $\bigoplus_{h \in H} M_h$)

satisfying (iii). If we now choose I to be any non-empty finite set and $\kappa = \aleph_0$, we obtain that M is not $\aleph_0$-compact, and so (iv) holds as well.

Remarks

1. Although the hypotheses in Proposition 4 concern the cardinality of the residue fields K_i rather than that of the local rings R_i, it is clear that the proposition generalizes (1) $\Leftrightarrow$ (2) of [7, Theorem 1]. In fact, if R is a commutative semiprimary ring with local factors R_i $(1 \leq i < n)$ and if its Jacobson radical is generated by less than $\min\{|R_i| : \ 1 \leq i \leq n\}$ elements (e.g. if R is artinian), then, by using an argument similar to the proof of [7, Lemma 2] on the Loewy length of the R_i's, we infer that the condition "$\kappa \leq \min\{|K_i| : 1 \leq i \leq n\}$" in Proposition 4 can be replaced by the weaker "$\kappa \leq \min\{|R_i| : 1 \leq i \leq n\}$"

2. It is not difficult to show that if H, H' are subsets of K_1 (in the proof above) with $|H| = |H'| = \kappa$, then $\bigoplus_{h \in H} M_h \cong \bigoplus_{h \in H'} M_h$ if and only if $H = H'$. This means that there are at least 2^κ non-isomorphic κ-generated modules M satisfying condition (iii) of Proposition 4.

3. Let R be a commutative ring such that all countably generated R-modules are $\aleph_0$-compact. As $R^{(N)}$ is $\aleph_0$-compact, it follows from [8] that R is $\sum$-algebraically compact, and therefore R is perfect. By Proposition 4, this means that R is an artinian principal ideal ring and so $p.gl.\dim R = 0$ or, equivalently, R has finite representation type (see for example [5, 16]).

If, in Proposition 4, the ring R, as a module over itself, is κ-compact, we can sharpen the proposition in the following way.

Proposition 5. *Let R and κ be as in Proposition 4 and assume moreover that R is*

κ–compact (as an R–module) and that κ is regular. Then the following statements are equivalent.

(i) R is not a principal ideal ring.

(ii) There exists an R–module M such that for every set I with $|I| < \kappa$ and every $|I|$–complete proper filter ψ on I, the reduced power M^I/ψ is $|I|$–compact but is not κ–compact.

(iii) There exists an R–module which is λ–compact for all $\lambda < \kappa$ but which is not κ–compact.

Proof. We need only prove that (i) implies (ii) and (iii). Assume therefore that R is not a principal ideal ring, and let K_1 be as in the proof of Proposition 4. Choose a subset H of K_1 with $|H| = \kappa$ and let $\{M_h\}_{h \in H}$ be a family of cyclically presented R–modules satisfying Corollary 1. Since R is κ–compact, so too is each M_h by Proposition 3. Put $\varphi = H(\kappa)$; then, as κ is regular, φ is a free κ–complete filter and $\varphi_{|I|} = \varphi$ is non–principal for any set I with $|I| < \kappa$. We thus obtain by Corollary 1, that M^I/ψ, where $M = \sum_\varphi M_h$, is not κ–compact for any proper filter ψ on I, and that M is not κ–compact either. Furthermore, M is a κ–pure submodule of $\prod_{h \in H} M_h$ and hence is λ–compact for all $\lambda < \kappa$. By Proposition 2, this implies that if ψ is $|I|$–complete, the reduced product M^I/ψ is $|I|$–compact.

Remark. Modules which are λ–compact but not algebraically compact have already been discussed in [4]. However, the construction given there (over principal ideal domains) used a result which does not extend to weakly inaccessible cardinals (see

[10]).

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