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1. Introduction and Preliminaries

In this paper we introduce the notion of a regular near–ring module by extending the usual elementwise definition of a regular near–ring to arbitrary near–ring modules. We characterize these modules in terms of certain restricted injectivity properties (Proposition 2.9). Using this characterization we deduce several characterizations of regular near–rings (Theorem 2.10). We also determine a characterization of strictly semisimple near–rings among near–rings with no nonzero nilpotent elements (Theorem 2.13). Throughout, R will denote a right near–ring with identity 1 such that $x0 = 0$ for all $x \in R$, and all modules (that is, near–ring modules) over R are left unital. The symbol ${}_RM$ will denote a left near–ring module M over R, and the term *R–subgroup of M* will mean a subgroup of $(M, +)$ which is closed under left R–multiplication. By an *R–submodule of M* we shall mean a normal subgroup A of $(M, +)$ satisfying $r(m+a)-rm \in A$ for all $r \in R$, $m \in M$, $a \in A$. Submodules of ${}_RR$ are *left ideals* of R. If A is a left ideal of R and $AR = \{ar \mid a \in A, r \in R\} \subseteq A$, then A is an *ideal* of R. For any subset B of an R–module M the set $\{r \in R | rB = (0)\}$ is called the *left annihilator* of B, denoted by $\ell(B)$. If $B = \{x\}$ we write $\ell(x)$ instead of $\ell(\{x\})$. For all subsets B of ${}_RM$, $\ell(B)$ is a left ideal of R. If B is an R–subgroup, $\ell(B)$ is an ideal. Near–ring homomorphisms and R–homomorphisms (that is, near–ring module homomorphisms) are defined in the usual manner. The set of all R–homomorphisms between left R–modules M and N will be denoted by $\mathrm{Hom}_R(M, N)$. An R–homomorphism $f : M \to N$ is *normal* if $f(M)$ is an R–submodule of N. An exact sequence $M \xrightarrow{f} N \longrightarrow 0$ *splits* if there exists a normal $g : N \to M$ such that $fg = 1_N$. The short exact sequence (s.e.s.) $0 \to L \to M \to N \to 0$ *splits* if the sequence $M \to N \to 0$ splits. The exact sequence $0 \to L \xrightarrow{f} M$ splits if there exists $g : M \to L$ such that $gf = 1_L$. The exact sequence $0 \to L \xrightarrow{h} M$ splits if and only if the s.e.s. $0 \to L \xrightarrow{h} M \xrightarrow{f} N \to 0$ splits [2, Lemma 2.1]. If L and N are submodules of M such that $M = L + N$ and $L \cap N = (0)$ we write $M = L \oplus N$. A left R–module M is *monogenic* if there exists $a \in M$ such that $Ra = M$; M is *right cancellative* if for each nonzero $m \in M$ and $r, r' \in R$ the identity $rm = r'm$ implies $r = r'$. Thus R is *right cancellative* if $_RR$ is right cancellative. An R–module M is called *irreducible* if it has no proper nonzero R–subgroups. M is *simple* if it contains no proper nonzero R–submodules. M is *semisimple* (also called *completely reducible*) if M is a direct sum of simple submodules. A near–ring R is called *semisimple* if $_RR$ is semisimple. Therefore, a near–ring R is semisimple if and only if R is the direct sum of simple left ideals. A module M is said to be the *semi–direct sum* of its R–subgroups A and B, and write $M = A \dot{+} B$ if A is an R–submodule, $M = A + B$ and $A \cap B = (0)$. In this case B is called a *semi–direct summand* of M. Then every $m \in M$ can be expressed uniquely as $a + b$ for some $a \in A$, $b \in B$. Moreover the canonical projection $p : M \to B$ given by $p(a + b) = b$ is an R–homomorphism (see Mason [2, p. 46]). R is called *strictly semisimple* if R is a direct sum of irreducible left ideals. As noted in [2, Theorem 3.5], R is strictly semisimple if and only if every R–subgroup of a monogenic R–module is a semi–direct summand (equivalently, if L_1, L_2 are R–subgroups of R and $0 \to L_1 \to L_2$ is exact, then it splits). A near–ring R is *regular* if for each $a \in R$, $\exists\ x \in R$ such that $a = axa$. It can be shown that R is regular if and only if, for all $a \in R$, $\exists$ an idempotent $e = e^2 \in R$ such that $Ra = Re$ [3, p. 346]. The *socle* of R is the sum of all minimal left ideals of R and is denoted by $\mathrm{soc}(R)$. For undefined terms and notations used in the sequel, we refer to Pilz [3].

2. Results

Definition 2.1. Let Q be a left R–module. An element $x \in Q$ is *R–divisible* in Q if for each (nonzero) $r \in R$, $\exists\ y \in Q$ such that $x = ry$; Q is called *R–divisible* if and only if $rQ = Q$ for all nonzero $r \in R$.

Remark. The additive group of rational numbers, considered as a module over the near–ring Z of integers, is Z–divisible. More generally, in the category of groups, any group which is *n–injective* in the sense of Mason [2, Def. 2.5(a)] is Z–divisible.

Proposition 2.2. *Let Q be an R–divisible module. Then for each monogenic R–subgroup I of R and the R–homomorphism $\phi : I \to Q$, there exists an R–homomorphism $\overline{\phi} : R \to Q$ which extends ϕ.*

Proof. Let $I = Ra$ for some $a \in R$. Suppose $\phi(a) = x$ for $x \in Q$. For each $r \in R$, we have $\phi(ra) = r\phi(a) = rx$. Since Q is R-divisible, there exists $y \in Q$ such that $x = ay$. Define $\overline{\phi} : R \to Q$ by $\overline{\phi}(r) = ry$ for $r \in R$. Clearly $\overline{\phi}$ is an R–homomorphism. Moreover, $\overline{\phi}(1) = y$. Thus $\overline{\phi}(ra) = ray = rx = \phi(ra)$. Hence $\overline{\phi}$ extends ϕ.

The above proposition motivates the following definition.

Definition 2.3. Let Q be a left module over a near–ring R. Then Q is called *P–injective* if for each monogenic R–subgroup I of R and the R–homomorphism $\phi : I \to Q$, there exists an R–homomorphism $\overline{\phi} : R \to Q$, which extends ϕ. More generally, for any fixed left R–module M, Q is called *PM–injective* if each R–homomorphism ϕ from a monogenic R–subgroup of M to Q extends to an R–homomorphism $\overline{\phi} : M \to Q$. Thus PR–injective modules are P–injective. We shall say that R is *self P–injective* if ${}_RR$ is P–injective.

Proposition 2.4. *Let R be a right cancellative near–ring. Then the following assertions are equivalent:*

(1) Q is a P–injective left R–module.

(2) Q is R–divisible.

Proof. (1) $\Rightarrow$ (2): Let $x \in Q$ and $(0 \neq)r \in R$. Define $\phi : Ra \to Q$ by $\phi(ra) = rx$. Clearly ϕ is a well–defined R–homomorphism, since R is right cancellative. Hence ϕ extends to an R–homomorphism $\overline{\phi} : R \to Q$, since Q is P–injective. Thus we have $x = \phi(a) = \overline{\phi}(a) = \overline{\phi}(a.1) = a\overline{\phi}(1)$. As $\overline{\phi}(1) \in Q$, x is R–divisible. Hence Q is R–divisible.

(2) $\Rightarrow$ (1): This is Proposition 2.2.

Definition 2.5. A left R–module M is called *pre–regular* if for each $a \in M$, there exists an R–homomorphism $f \in \mathrm{Hom}_R(Ra, R)$ such that $a = f(a) \cdot a$. We shall say that R is *pre–regular* if ${}_RR$ is pre–regular.

Proposition 2.6. *Let R be a near–ring with a unique nonzero idempotent. For a left R–module M the following conditions are equivalent:*

(1) M is pre–regular.

(2) M is right cancellative.

Proof. (1) $\Rightarrow$ (2): Suppose for $r, r' \in R$ and $(0 \neq)$ $m \in M$, we have $rm = r'm$. Since M is pre–regular, so there exists an R–homomorphism $f : Rm \to R$ such that $m = f(m) \cdot m$. This implies that $f(m) = f(f(m) \cdot m) = f(m) \cdot f(m)$. Hence $f(m)$ is a nonzero idempotent element of R, and so by the uniqueness of the nonzero idempotent element, it follows that $f(m) = 1$. Hence the equation $rm = r'm$ implies that $r = r \cdot 1 = r \cdot f(m) = f(rm) = f(r'm) = r'f(m) = r' \cdot 1 = r'$, that is, $r = r'$. Therefore, M is right cancellative.

(2) $\Rightarrow$ (1): Let ${}_RM$ be a right cancellative module. We show that M is pre–regular. Let $m \in M$. Define an R–homomorphism $g : Rm \to R$ by $g(rm) = r$ for $r \in R$. As $g(m) = g(1 \cdot m) = 1$, it follows that $m = 1 \cdot m = g(m) \cdot m$. Hence M is pre–regular.

Corollary. *R is a pre–regular near–ring with a unique nonzero idempotent if and only if R is right cancellative.*

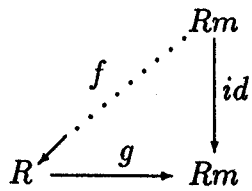
Proposition 2.7. *Let R be a d.g. (that is, distributively generated) near–ring. Then the following assertions for a left R–module M are equivalent:*

(1) M is pre–regular.

(2) Each monogenic R–subgroup of M is projective (in the usual categorical sense [1]).

Proof. (1) $\Rightarrow$ (2): Let $m \in M$. We show that the monogenic R–subgroup Rm of M is projective. Define $g : R \to Rm$ by $g(r) = rm$ for $r \in R$. Since M is pre–regular, there exists $f \in \mathrm{Hom}_R(Rm, R)$ such that $m = f(m) \cdot m$. Thus for $r \in R$, we have $f(rm) \cdot m = (rf(m))m = r(f(m) \cdot m) = rm$. Hence $g(f(rm)) = f(rm) \cdot m = rm$. Thus $gof = id_{Rm}$. Hence Rm is a retract of R. Moreover, since $1 \in R$, ${}_RR$ is free and hence projective (cf. [1]). As retracts of projective objects are projective, so Rm is projective.

(2) $\Rightarrow$ (1): Let $m \in M$. We define $g : R \to Rm$ by setting $g(r) = rm$ for $r \in R$. Clearly g is a surjective R–homomorphism. By hypothesis, Rm is projective. Hence there exists $f \in \mathrm{Hom}_R(Rm, R)$ such that the following diagram is commutative.



Thus $g \circ f = id$. Hence $m = (g \circ f)(m) = g(f(m)) = f(m) \cdot m$. Therefore M is pre–regular.

Definition 2.8. A left R–module M is called *regular* if for each $m \in M$, there exists an R–homomorphism $g \in \mathrm{Hom}_R(M, R)$ such that $m = g(m) \cdot m$. Thus if ${}_RR$ is regular, then for each $a \in R$, there exists $g \in \mathrm{Hom}_R(R, R)$ such that $a = g(a) \cdot a = g(a \cdot 1) \cdot a = ag(1) \cdot a \in aRa$. Hence R is regular in the usual sense. An obvious consequence of this definition is that a submodule of a regular module is regular. Thus every left ideal of a regular near–ring is regular as an R–module (but not necessarily as a near–ring).

Proposition 2.9. *For a left R–module M, the following assertions are equivalent:*

(1) M is regular.

(2) M is pre–regular and R is PM–injective.

Proof. (1) $\Rightarrow$ (2): Assume that M is a regular module. Then for each $m \in M$, there exists $g \in \mathrm{Hom}_R(M, R)$ such that $m = g(m) \cdot m$. In particular, there exists $f \in \mathrm{Hom}_R(Rm, R)$ such that $m = f(m) \cdot m$. Hence M is pre–regular. We now show that R is PM–injective. For $m \in M$, let there be an R–homomorphism $f : Rm \rightarrow R$. Since M is regular, there exists an R–homomorphism $g : M \rightarrow R$ such that $m = g(m) \cdot m$. Define $\overline{f} : M \rightarrow R$ by setting $\overline{f}(a) = g(a) \cdot f(m)$ for $a \in M$. It is easily checked that $\overline{f}$ is an R–homomorphism. Moreover $\overline{f}(rm) = r(\overline{f}(m)) = r(g(m) \cdot f(m)) = r(f(g(m) \cdot m)) = r(f(m)) = f(rm)$. Thus $\overline{f}$ is an extension of f. Hence R is PM–injective.

(2) $\Rightarrow$ (1): Assume that M is pre–regular and R is PM–injective. We show that M is regular. Since M is pre–regular, so for each $m \in M$, there exists $f \in \mathrm{Hom}_R(Rm, R)$ such that $m = f(m) \cdot m$. On the other hand, since R is PM–injective, so there exists an R–homomorphism $g : M \rightarrow R$ which extends f. Hence $m = f(m) \cdot m = g(m) \cdot m$, showing that M is regular.

Taking $M = {}_RR$ in the above proposition, we obtain the following characterization of regular near–rings.

Corollary. *A near–ring R is regular if and only if R is pre–regular and self P–injective.*

We now prove the following characterization theorem for regular near–rings.

Theorem 2.10. *The following conditions are equivalent:*

(1) Every left R-module M is P–injective.

(2) ${}_RR$ is completely P–injective (that is, every homomorphic image of R, considered as an R–module, is P–injective).

(3) Every monogenic left R–module $M = Rm$ is P–injective.

(4) Every monogenic R–subgroup Ra of ${}_RR$ is P–injective.

(5) R is regular.

(6) R is pre–regular and self P–injective.

Proof. Since the equivalence of the last two conditions has just been proved, we only need to verify the following implications: (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (1).

(1) ⇒ (2): Obvious.

(2) ⇒ (3): Rm is the image of R under the epimorphism $f : R \to Rm$, defined by $f(r) = rm$, for all $r \in R$.

(3) ⇒ (4): Put $m = a$.

(4) ⇒ (5): Let $i : Ra \to Ra$ be the identity map. As Ra is a P–injective R–module, there exists an R–homomorphism $g : R \to Ra$ such that $a = g(a) = g(a \cdot 1) = ag(1)$, and $g(1) \in Ra$. Hence we can write $g(1) = xa$ for some $x \in R$. Hence $a = axa$, and so R is regular.

(5) ⇒ (1): Let M be a left R–module. We show that M is P–injective. For $a \in R$, consider an R–homomorphism $f : Ra \to M$. Since R is regular, so $\exists\ e = e^2 \in R$ such that $Ra = Re$. Define $\overline{f} : R \to M$ by $\overline{f}(r) = f(re)$. Clearly $\overline{f}$ is an R–homomorphism. Moreover, $\overline{f}(re) = f(re \cdot e) = f(re)$. Hence $\overline{f}$ is an extension of f. Hence M is P–injective.

Proposition 2.11. *Let R be a near–ring with no nonzero nilpotent elements and M be a maximal left ideal of R. If R/M is a regular left R–module, then M is a monogenic left R–module generated by a central idempotent.*

Proof. Consider the element $(1 + M) \in R/M$. Since R/M is regular, so there exists $f \in \mathrm{Hom}_R(R/M, R)$ such that $(f(1+M))(1+M) = (1+M)$. This means that $f(1+M)-1 \in \ell(1+M)$. Note that $\ell(1+M) = \{r \in R | r(1+M) = M\} = M$. Hence $f(1+M)-1 = m \in M$.

For any $b \in M, f(b(1+M)) = b(f(1+M)) = b(m+1)$. But $f(b(1+M)) = f(b+M) = f(M) = 0$. Hence $b(m+1) = 0$. Now $[(m+1)b]^2 = (m+1)b(m+1)b = (m+1)(b(m+1))\cdot b = 0$. Since R has no nonzero nilpotent elements, it follows that $(m+1)b = 0$. Thus $mb + b = 0$, that is, $b = -mb = (-m)b$ for all $b \in M$. Thus $-m$ is an idempotent element. Since R has no nonzero nilpotent elements, it follows from [3, Prop. 9.43, p. 304] that $-m$ is a central idempotent. Hence $b = b(-m)$ for all $b \in M$. This implies that $M = R(-m)$. Hence M is monogenic left R–module generated by a central idempotent.

Proposition 2.12. *Let R be a near–ring with no nonzero nilpotent elements. If each maximal left ideal M of R is a monogenic left R–module generated by a central idempotent, then $R = soc(R)$.*

Proof. Let S be the socle of R. Suppose $R \neq S$. Then since $1 \in R$, there exists a maximal left ideal M of R containing S. By the assumption, $M = Re$, for some central idempotent $e \in M$. Hence by [3, Prop. 3.47, p. 93], we can write $R = Re \oplus \ell(e)$. This implies that $R/Re \cong \ell(e)$ is a simple left R–module, and hence $\ell(e)$ is a minimal left ideal of R. Consequently, $\ell(e) \subseteq S \subseteq M$. Since $1 - e \in \ell(e)$ and $e \in M$, it follows that $1 = (1-e) + e \in M$, *a* contradiction. Hence $R = S$ as we were to show.

Corollary. *Let R be a near-ring with no nonzero nilpotent elements. If for each maximal left ideal M of R the left R–module R/M is regular, then R is the direct sum of a finite number of simple left ideals.*

Proof. The preceding two propositions show that $R = \text{soc}(R)$. By [3, Thm. 2.48, p. 55] this means that R is a direct sum of simple left ideals with finite number of summands, since R contains an identity (see [3], Th. 2.30).

The assumptions of the above corollary are satisfied if each simple left R–module is regular (since for each maximal left ideal M of R, the left R–module R/M is simple) or if each monogenic left R–module is regular (since for each maximal left ideal M of R the left

R–module R/M is monogenic, with generator $1+M$ where 1 is the identity of R). Moreover in the latter case we have a stronger consequence.

Theorem 2.13. *Let R be a near–ring with no nonzero nilpotent elements. Then the following conditions are equivalent:*

(1) Each monogenic left R–module is regular.

(2) R is strictly semisimple.

Proof. (1) $\Rightarrow$ (2): By the previous corollary, R is semisimple. On the other hand, since ${}_RR$ is a monogenic R–module generated by 1, it is regular. Therefore R is a regular near-ring. As R has no nonzero nilpotent elements, it follows from ([3], Thm. 9.158 and Corollary 9.159) that R-subgroups of R coincide with left ideals of R. Hence R is strictly semisimple.

(2) $\Rightarrow$ (1): Let m be an element of a monogenic left R–module M. Define $f : R \to Rm$ by $f(r) = rm$ for $r \in R$. Clearly f is an R-homomorphism. Since the sequence $0 \to \ell(m) \xrightarrow{i} R$, where i denotes the canonical inclusion, is exact and since R is strictly semisimple, it follows from [2, Thm. 3.5(d)] that the above sequence splits. Hence the s.e.s. $0 \to \ell(m) \xrightarrow{i} R \xrightarrow{f} Rm \to 0$ splits by [2, Lemma 2.1]. Hence there exists a normal $g : Rm \to R$ such that $fg = 1_{Rm}$. Moreover since Rm is an R-subgroup of the monogenic R-module M and R is strictly semisimple, it follows from [2, Thm. 3.5(b)] that Rm is a semi-direct summand of M. Hence we can write $M = A \dot{+} Rm$ for some R-submodule A of M. Moreover, the projection map $\pi : M \to Rm$ defined by $\pi(a+b) = b$ for $a \in A$ and $b \in Rm$ is an R-homomorphism [2]. Let $\alpha = g\pi$. Then $\alpha : M \to R$ is an R-homomorphism, and $m = id_{Rm}(m) = fg(m) = f(g(m)) = g(m)\cdot m = g(\pi(0+m))\cdot m = [g\pi(0+m)]\cdot m = [\alpha(0+m)]\cdot m = \alpha(m)\cdot m$. Hence M is regular. This completes the proof.

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