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TC SEMIGROUPS AND INFLATIONS

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Abstract

An algebra A satisfies TC (the term condition) if $p(a,\tilde{x}) = p(a,\tilde{y})$ iff $p(b,\tilde{x}) = p(b,\tilde{y})$ for any $a,b \in A$, $\tilde{x},\tilde{y} \in A^n$ and any $n+1$–ary term p. TC algebras have been extensively studied. We previously determined the structure of all TC semigroups. We use this result to show that if S is a TC semigroup then $S_E = \{a \in S | ax \text{ is an idempotent}\}$ is an inflation of S_{Reg} (the set of regular elements of S) and $S_{\text{Reg}} \cong H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a right zero semigroup. As consequences of this result, we show that S is a semisimple TC semigroup iff $S \cong H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a right zero semigroup and that, if S is a TC semigroup, then S is an $\mathcal{R}$–trivial semigroup iff $S_{\text{Reg}} = \phi$ or is an $\mathcal{R}$–trivial semigroup iff $S_E = \phi$ or is an inflation of a left zero semigroup.

The term condition for algebras (TC) was introduced by McKenzie [4] and algebras obeying TC (also called abelian algebras) have been extensively studied. A semigroup satisfies TC if and only if (c1) $xy = xz$ implies $uy = uz$ (c2) $yx = zx$ implies $yu = zu$ (c3) $y_1xy_2 = z_1xz_2$ implies $y_1uy_2 = z_1uz_2$. TC semigroups were first considered by Taylor [7] and McKenzie [5]. In [5], McKenzie characterized TC semigroups of finite exponent and posed the problem of characterizing all TC semigroups. In [8, Theorem 1.13], we determined the structure of all TC semigroups, and in [8, Theorem 2.11], we gave a more detailed structure theorem for periodic TC

semigroups. In [8, Corollary 2.6], we gave a structure theorem for TC regular semigroups. In [9, Theorem 2.1], we specialized [8, Theorem 1.13] to obtain the structure of reversible TC semigroups $S(aS \cap bS \neq \phi$ and $Sa \cap Sb \neq \phi$ for all $a, b \in S)$. In [9, Theorem 3.8], we determined the structure of quasi-regular TC semigroups. In [9, Theorem 4.2], we showed that a TC semigroup S has the congruence extension property (CEP) if and only if S is periodic (a semigroup S has CEP if for every subsemigroup T of S and congruence relation ρ on T, there is a congruence relation $\overline{\rho}$ on S such that $\overline{\rho} \cap (T \times T) = \rho$). In [10, Theorem 3.4], we determined the structure of a class of semigroups S that satisfies (c1)$'$ (if $x \in S$ there exists $u^2 = u \in S$ such that (c1) is valid for $y, z \in S^1$ (S with appended identity)) and $(c2)'$ and additional conditions.

We show that if S is a TC semigroup, then $S_E = \{a \in S | ax \text{ is an idempotent}\}$ is an inflation of S_{Reg} (the set of regular element of S) and $S_{\text{Reg}} \cong H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a right zero semigroup (Theorem 12). Using this theorem, we show that S is a semisimple TC semigroup if and only if $S \cong H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a aright zero semigroup (Theorem 13). As another consequence of Theorem 12, we show that if S is a TC semigroup, then S is an $\mathcal{R}$-trivial semigroup iff $S_{\text{Reg}} = \phi$ or is an $\mathcal{R}$-trivial semigroup iff $S_E = \phi$ or S_E is an inflation of a left zero semigroup (Theorem 14). As an immediate consequence of Theorem 12, if S is a TC semigroup, then S is quasi-regular ($a \in S$ implies $a^n \in S_{\text{Reg}}$ for some positive integer n) iff S is E-inversive ($S_E = S$) (so, the structure of S is given by Theorem 12) (Theorem 15).

For universal algebra concepts not defined here we refer the reader to [1] or [6]. For algebraic semigroup concepts not defined here, we refer the reader to [2], [3], or [6].

A semigroup S is termed $\mathcal{R}$–trivial if each $\mathcal{R}$–class ($\mathcal{R}$ is Green's relation) of S consists of a single element. If S is a semigroup, $E(S)$ will denote the set of idempotents of S and S_{Reg} will denote the set of regular element of S. $\dot{\bigcup}$ will denote disjoint unions, ϕ denotes the empty set, and $\cong$ means "isomorphic to".

We first state our structure theorem [8, Theorem 1.13] for TC semigroups (Theorem 1) and give some consequences of this theorem (Remark 2 and Lemmas 3–11). These results will be needed to prove Theorems 12–14).

Construction of TC Semigroups

Let G be an abelian group, I be a left zero semigroup, and J be a right zero semigroup. Let V be a subsemigroup of $G \times I \times J$. Let $(X_v : v \in V)$ be a collection of pairwise disjoint sets and let $X = \cup(X_v : v \in V)$. For $v = (m, i, j)$ and $x \in X_v$ define $m(x) = m, i(x) = i$, and $j(x) = j$. Let $M_i = \{m : (m, i, j) \in V \text{ for some } j\}$ and $N_j = \{m : (m, i, j) \in V \text{ for some } i\}$. For $(i, j) \in Pr_I V \times Pr_J V$, let ϕ_{ij} be a function from $M_i \times N_j \to X$ such that

1. $\phi_{ij}(m, n) \in X_{(mn,i,j)}$

2. $\phi_{ij}(mk, n) = \phi_{ij}(m, kn)$ for $k \in UM_i$

3. $\phi_{ij}(m, kn) = \phi_{ij}(p, kq)$ implies $\phi_{ij}(m, sn) = \phi_{ij}(p, sq)$ for $k, s \in UM_i$.

Let (X, V, ϕ) denote X under the multiplication

4. $xy = \phi_{i(x)j(y)}(m(x), m(y))$.

Theorem 1 or Remark 2 will be used in the proofs of Lemmas 3–8, Lemmas 10–11, Theorem 12 and Theorem 13.

Theorem 1. *S is TC semigroup if and only if $S \cong (X, V, \phi)$ for some X, V, and ϕ.*

Remark 2. Let $C = (m(a) : a \in X)$. Then, using (4) and (1) of Theorem 1, m is a homomorphism of X onto C. Hence, C is an abelian cancellative semigroup. Furthermore $\varphi(a) = (m(a), i(a), j(a))$ defines a homomorphism of $X = S$ onto V. Let U denote the group of units of C and 1 denote the identity of U (of C).

We often use Theorem 1 and Remark 2 and their notation without explicit mention.

In Lemmas 3–11, S will denote a TC semigroup. Lemma 3 will be used in the proof of Lemma 4 and Theorem 12.

Lemma 3. $S_{Reg} = \{\phi_{i(a)j(a)}(1, m(a)) : m(a) \in U\}$.

Proof. Suppose $a \in S_{\text{Reg}}$. Then $(ax)a = a$ for some $x \in S$. So, $a = \phi_{i(a)j(a)}(1, m(a))$. Since $m(ax) \in E(C)$, $m(a) \in U$. Conversely, if $m(a) \in U$, let $m(x) = (m(a))^{-1}$. So,

$$\phi_{i(a)j(a)}(1, m(a)) \cdot \phi_{i(a)j(x)}(1, m(x))\ \phi_{i(a)j(a)}(1, m(a)) = \phi_{i(a)j(a)}(1, m(a)).$$

Lemma 4 is [8, Lemma 2.4] but a different proof using Theorem 1 is given here. Lemma 4 will be used in the proof of Lemma 5.

Lemma 4. *$S_{Reg} = \phi$ or S_{Reg} is a regular subsemigroup of S.*

Proof. If $m(a), m(b) \in U$, let $x = \phi_{i(a)j(a)}(1, m(a))$ and $y = \phi_{i(b)j(b)}(1, m(b))$. Thus $xy = \phi_{i(a)j(b)}(m(a), m(b))$. Let $m(x) = (m(a))^{-1}$. Then, $\phi_{i(a)j(b)}(m(a), m(b)) = \phi_{i(a)j(b)}(m(a)m(x)m(a), m(b)) = \phi_{i(ab)j(ab)}(1, m(ab))$. Since $m(ab) = m(a)m(b) \in U$, $xy \in S_{\text{Reg}}$ by Lemma 3.

Lemma 5 will be used in the proof of Lemma 10 and Theorem 12.

Lemma 5. *If $S_{Reg} \neq \phi$, $\varphi(a) = (m(a), i(a), j(a))$ is an isomorphism of S_{Reg} onto $H \times A \times B$ where $H = (m(a) : a \in S_{Reg})$ is an abelian group, $A = (i(a) : a \in S_{Reg})$ is a left zero semigroup and $B = (j(a) : a \in S_{Reg})$ is a right zero semigroup.*

Proof. Since H is a regular subsemigroup of C by Remark 2 and Lemma 4, H is an abelian group. Also, $A \subseteq I$ and $B \subseteq J$. By Remark 2, if $a \in S_{\text{Reg}}$, $\varphi(a) = (m(a), i(a), j(a))$ defines a homomorphism of S_{Reg} into $H \times A \times B$. We next show φ is "onto". Let $(x, y, z) \in H \times A \times B$. Thus, $x = m(a), y = i(b)$, and $z = j(c)$ for some $a, b, c \in S_{\text{Reg}}$. There exists $r, s \in S(= X)$ such that $brb = b$ and $csc = c$. Thus, $(x, y, z) = (m(brasc), i(brasc), j(brasc)) = \varphi(brasc)$ with $brasc \in S_{\text{Reg}}$ by Lemma 4. We next show φ is one–to–one. Suppose $\varphi(a) = \varphi(b)$ where $a, b \in S_{\text{Reg}}$. Suppose $axa = a$ and $byb = b$. So, $a = (ax)a = \phi_{i(a)j(a)}(1, m(a))$

and $b = (by)b = \phi_{i(b)j(b)}(1, m(b))$. Hence, $a = b$.

Remark. If $a, b \in S_{\text{Reg}}$ and as $b \in X_v$, then $\varphi(a) = \varphi(b)$. Thus, $a = b$ by Lemma 5.

Lemma 6 will be used in the proof of Lemma 7.

Lemma 6. *$E(S) \neq \phi$ if and only if C has an identity element 1 and, then $E(S) = (\phi_{ij}(1,1) : 1 \in M_i, 1 \in N_j)$.*

Proof. Suppose C has an identity element 1. Then $m(e) = 1$ for some $e \in S$. Let $t = \phi_{i(e)j(e)}(m(e), m(e))$. Then, $tt = \phi_{i(e)j(e)}(m(e), m(e)) = t$. So, $t \in E(S)$. Conversely, suppose $E(S) \neq \phi$. Let $a \in E(S)$. Thus, $aa = \phi_{i(a)j(a)}(m(a), m(a)) = a$. Hence, $(m(a))^2 = m(a)$ and, thus, $m(a)$ is the identity of C. Furthermore, $a = \phi_{i(a)j(a)}(1, 1)$ with $1 \in M_{i(a)}$ and $1 \in N_{j(a)}$. Let $t = \phi_{ij}(1,1)$ where $1 \in M_i$ and $1 \in N_j$. Then, $tt = \phi_{ij}(1,1)\phi_{ij}(1,1) = \phi_{ij}(1,1) = t$ since $\phi_{ij}(1,1) \in X_{(1,i,j)}$.

An element a in an arbitrary semigroup S is termed E–inversive if there exists $b \in S$ such that $ab \in E(S)$. We will denote the set of E–inversive elements of S by S_E. Following [2], a semigroup S is termed E–inversive if each element of S is E–inversive. Let $a \in S_E$. Then, there exists $y \in S$ such that $ay, ya \in E(S)$. If $ax \in E(S)$, just let $y = xax$.

Lemma 7 will be used in the proof of Lemma 11, Theorem 12, and Theorem 13.

Lemma 7. *$a \in S_E$ if and only if $m(a) \in U$.*

Proof. Let $a \in S_E$. Then, there exists $b \in S$ such that $ab \in E(S)$. So, $m(a)m(b) = m(b)m(a) = 1$. Hence, $m(a) \in U$. Conversely, let $m(a) \in U$. So, $m(a)m(b) = m(b)m(a) = 1 = m(e)$ for some $e \in S$. Thus, $ab = \phi_{i(a)j(b)}(m(a), m(b)) = \phi_{i(a)j(b)}(m(a), m(b)m(e)) = \phi_{i(a)j(b)}(m(a)m(b), m(e)) = \phi_{i(a)j(b)}(1, 1)$ (Note $ab \in X_{(m(e), i(a), j(b))}$. Thus, $1 = m(e) \in M_{i(a)}$ and $1 = m(e) \in N_{j(b)}$). Hence, $ab \in E(S)$ by Lemma 6.

Lemma 8 will be used in the proof of Lemma 9.

Lemma 8. *If $x, y \in S$ and $e \in E(S)$, then $xey = xy$.*

Proof. $(xe)y = \phi_{i(x)j(y)}(m(x), m(y)) = xy$.

Lemma 9 will be used in the proof of Lemma 10.

Lemma 9. *$S_E = \phi$ or S_E is an E–inversive semigroup with $S_E^2 \subseteq S_{Reg}$.*

Proof. Suppose $S_E \neq \phi$. Let $a, b \in S_E$. Then, there exists $x, y \in S$ such that $xa, by \in E(S)$. Hence, using Lemma 8, $ab(yx)ab = a(by)(xa)b = ab$. So, $ab \in S_{\text{Reg}}$. If $a \in S_E$, there exists $y \in S$ such that $ay, ya \in S_E$. Thus, $y \in S_E$.

Lemma 10 will be used in the proof of Lemma 11 and Theorem 12.

Lemma 10. *$S_E = \phi$ or $S_E = W \dot{\bigcup} H \times A \times B$ where W, A, and B are sets and H is an abelian group under the following product: if $a, b \in W$ and $(g, i, j), (h, k, \ell) \in H \times A \times B$, then $ab = (a\phi b\phi, a\alpha, b\beta), (g, i, j)a = (g(a\phi), i, a\beta)$, $a(g, i, j) = (a\phi g, a\alpha, j)$*

and $(g,i,j)(h,k,\ell) = (gh,i,\ell)$ *where* ϕ, α, *and* β *are functions of* W *into* H, A, *and* B *respectively and* $S_{Reg} = H \times A \times B$.

Proof. Assume $S_E \neq \phi$. Thus, $S_{\text{Reg}} = \phi$ and $S_{\text{Reg}} \subseteq S_E$. By Lemma 5, $\varphi(a) = (m(a), i(a), j(a))$ $(a \in S_{\text{Reg}})$ defines an isomorphism of S_{Reg} onto $H \times A \times B$ (notation of Lemma 5). Let $a \in S_E$. Thus, there exists x and $y \in S$ such that $ax, ya \in E(S)$. Hence, $\varphi(axaya) = (m(axaya), i(axaya), j(axaya)) = \varphi(a)$. Since $axaya \in S_{\text{Reg}}$ by Lemma 9, φ defines a homomorphism of S_E onto $H \times A \times B$. We will repeatedly use the fact that $u, v \in S_E$ imply $uv \in S_{\text{Reg}}$ (Lemma 9). Let $W' = S_E - S_{\text{Reg}}$. So, $S_E = W' \dot{\bigcup} S_{\text{Reg}}$. There exists a one–to–one mapping θ of W' onto a set W with $W \cap \varphi(S_{\text{Reg}}) = \phi$. Let $\overline{S_E} = W \dot{\bigcup} \varphi(S_{\text{Reg}})$. Define $\delta(x) = \theta(x)$ if $x \in W'$ and $\delta(x) = \varphi(x)$ if $x \in S_{\text{Reg}}$. Then, δ is a 1–1 mapping of S_E onto $\overline{S_E}$. Define a product on $\overline{S_E}$ by the rule $x \circ y = \delta(\delta^{-1}(x)\delta^{-1}(4))$. Then, $(\overline{S_E}, o)$ is a groupoid isomorphic to S_E under δ. Define $a\phi = m\delta^{-1}(a), a\alpha = i\delta^{-1}(a)$, and $a\beta = j\delta^{-1}(a)$ for $a \in W$. Thus, ϕ, α, and β are functions of W into H, A, and B respectively. If $a, b \in W$, $a \circ b = \delta(\delta^{-1}(a)\delta^{-1}(b)) = (m(\delta^{-1}(a)\delta^{-1}(b)), i(\delta^{-1}(a)\delta^{-1}(b)), j(\delta^{-1}(a)\delta^{-1}(b))) = (a\phi b\phi, a\alpha, b\beta)$. If $(g,i,j) \in H \times A \times B$, there exists $c \in S_{\text{Reg}}$ such that $m(c) = g, i(c) = i$ and $j(c) = j$ by Lemma 5. Thus, $a \circ (g,i,j) = \delta(\delta^{-1}(a)\delta^{-1}(g,i,j)) = \delta(\delta^{-1}(a)c) = (m(\delta^{-1}(a)c), i(\delta^{-1}(a)c), j(\delta^{-1}(a)c)) = (a\phi g, a\alpha, j)$. Similarly, $(g,i,j) \circ a = (g(a\phi), i, a\beta)$ and $(g,i,j) \circ (h,k,\ell) = (gh,i,\ell)$ where $(h,k,\ell) \in H \times A \times B$. We may find an isomorphism λ of S onto a semigroup $\overline{S}$ with subsemigroup $\overline{S_E}$ such that $\lambda | S_E = \delta$ and $\overline{S_E} = \overline{S_E}$. So, we identify $\overline{S}$ with S and $\overline{S_E}$ with S_E. Clearly,

$S_{\text{Reg}} = H \times A \times B$.

Lemma 11 will be used in the proof of Theorem 14.

Lemma 11. *$a\mathcal{R}b$ implies $a = b$ or $a = bs$ and $b = at$ where $s, t \in S_E$ and $st, ts \in E(S)$.*

Proof. Suppose $a \neq b$. Thus, there exists $s, t \in S$ such that $a = bs$ and $b = at$. Hence, $a = ats$. So, $m(a) = m(a)m(ts)$. Thus, $m(a)m(ts) = m(a)(m(ts))^2$. Hence, $m(ts) \in E(C)$. So, $m(t)m(s) = m(s)m(t) = 1$. Thus, $s, t \in S_E$ by Lemma 7. Furthermore, $ats = a(ts)^2$. Hence, $(ts)^2 = (ts)^3$ by TC. Since $s, t \in S_E$, $ts = (g, i, j) \in H \times A \times B$ by Lemma 10 (notation of Lemma 10). So, $g^2 = g^3$ and $g = e$, the identity of H. Thus, $ts \in E(S)$. Similarly, $st \in E(S)$.

Let T be a semigroup. With each element $t \in T$ associate a set P_t containing t such that the sets $P_t (t \in T)$ are mutually disjoint. Let $S = U(P_t : t \in T)$ and let the product in T be extended to a product in S by defining $ab = st$ if $a \in P_s$ and $b \in P_t$ $(s, t \in T)$. Then, S is a semigroup which is called an inflation of T [2].

Theorem 12 will be used in the proof of Theorem 13, Theorem 14, and Theorem 15.

Theorem 12. *Let S be a TC semigroup. Then, $S_E = \phi$ or S_E is an inflation of $S_{Reg} = V_{Reg} = H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a right zero semigroup.*

Proof. Assume $S_E \neq \phi$. We first use Lemma 10 to show S_E is an inflation of $S_{\text{Reg}} = H \times A \times B$ (notation of Lemma 10). Let $P_{(g,i,j)} = \{(g,i,j)\} \cup \{a \in W : a\phi = g, a\alpha = i, \text{ and } a\beta = j\}$. So, by Lemma 10, $S_E = U(P_{(g,i,j)} : (g,i,j) \in H \times A \times B)$ is an inflation of S_{Reg}. By Lemma 5, $\varphi(a) = (m(a), i(a), j(a))$ defines an isomorphism of S_{Reg} onto $\varphi(S_{\text{Reg}})$. Using Lemma 7, $V_{\text{Reg}} = ((m(a), i(a), j(a)) : a \in S_E)$. So, $\varphi(S_{\text{Reg}}) \subseteq V_{\text{Reg}}$. Let $(m(a), i(a), j(a)) \in V_{\text{Reg}}$. So, $a \in S_E$. Thus, by Lemma 3, $\phi_{i(a)j(a)}(1, m(a)) \in S_{\text{Reg}}$. Hence, $\varphi(\phi_{i(a)j(a)}(1, m(a))) = (m(a), i(a), j(a))$. So, $\varphi(S_{\text{Reg}}) = V_{\text{Reg}}$.

A semigroup S is called semisimple if every principal factor of S is 0–simple or simple. Equivalently, S is semisimple iff $I^2 = I$ for every ideal $I$of S [2]. If S is a semigroup, let $J(a)$ denote the principal ideal generated by a and define $a\mathcal{J}b$ if $J(a) = J(b)$ ($\mathcal{J}$ is a Green's relation). Let $\mathcal{J}_a$ denote the $\mathcal{J}$–class of S containing a. S is semisimple iff for $a \in S$, there exists $x, y \in \mathcal{J}_a$ such that $xy \in \mathcal{J}_a$.

Theorem 13. *S is a semisimple TC semigroup if and only if $S \cong H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a right zero semigroup.*

Proof. Let S be a semisimple TC semigroup. We will show that $S = S_E$ and then apply Theorem 12. Suppose $a\mathcal{J}b$ $(a, b \in S)$. Since $J(a) = J(b) = J(b)^3$, $a = rbs$ for some $r, s \in S$. Similarly, $b = r_1 a s_1$ for some $r_1, s_1 \in S$. So $m(a) = m(b)m(rs)$ and $m(b) = m(a)m(r_1 s_1)$. Thus, $m(a) = m(a)m(r_1 s_1)m(rs)$. So, if $m(r_1 s_1)m(rs) = u$, $m(a) = m(a)u$. Hence, $m(a)u = m(a)u^2$. Thus, $u^2 = u$. So, $u = 1$, the identity of C. Hence, $m(r_1 s_1) \in U$ and $m(b) = m(a)v$ where $v \in U$. Thus, $\mathcal{J}_a \subseteq \{z | m(z) = m(a)v$

for some $v \in U\}$. Since S is semisimple, there exists $z_1, z_2 \in \mathcal{J}_a$ such that $z_1z_2 \in \mathcal{J}_a$. Thus, $m(z_1) = m(a)v_1, m(z_2) = m(a)v_2$, and $m(z_1z_2) = m(a)v_3$ where $v_1, v_2, v_3 \in U$. So, $m(a)v_3 = m(a)v_1m(a)v_2$. Hence, $v_3 = m(a)v_1v_2$ or $m(a) = v_3v_2^{-1}v_1^{-1} \in U$. Hence, $m(a) \in U$ for all $a \in S$. Thus, $S = S_E$ by Lemma 7. Let $a \in S_E$. Thus, as above, there exists $r, s \in S_E$ such that $a = ras$. So, $a \in S_{\text{Reg}}$ by Theorem 12. So, $S = S_{\text{Reg}}$. Thus, by Theorem 12, $S \cong H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a right zero semigroup. By easy calculations, $H \times A \times B$ is a regular TC semigroup. Hence, $H \times A \times B$ is a semisimple TC semigroup.

Theorem 14. *Let S be a TC semigroup. S is an $\mathcal{R}$–trivial semigroup iff $S_{Reg} = \phi$ or is an $\mathcal{R}$–trivial semigroup iff $S_E = \phi$ or is an inflation of a left zero semigroup.*

Proof. Suppose S_{Reg} is an $\mathcal{R}$–trivial semigroup. Then, using Theorem 12, S_E is an inflation of $S_{\text{Reg}} \cong A$ (notation of Theorem 12). Next, we show S is an $\mathcal{R}$–trivial semigroup. Let $b \in S$ and suppose $|R_b| > 1$. Then, there exists $c \neq b$ such that $c\mathcal{R}b$. So, by Lemma 11, there exists $s, t \in S_E$ such that $st, ts \in E(S)$ and $c = bs$ and $b = ct$. Thus, $c = cts$. Let $ts = e$. So, $c = ce$. Thus, $R_b = R_c = R_{ce}$. Let $x\mathcal{R}ce$. Thus, by Lemma 11, $x = ce$ or $x = cer$ where $r \in S_E$. So, by Theorem 12, $x = ce$ in either case. So, $|R_b| = |R_{ce}| = 1$, a contradiction. So, S is an $\mathcal{R}$–trivial semigroup. Clearly, if S is an $\mathcal{R}$–trivial semigroup, $S_{\text{Reg}} = \phi$ or is an $\mathcal{R}$–trivial semigroup. If $S_{\text{Reg}} = \phi$, S is an R–trivial semigroup by Lemma 11. If S_E is an inflation of a left zero semigroup A, then $A = S_{\text{Reg}}$ and hence S_{Reg} is an $\mathcal{R}$–trivial semigroup.

A semigroup S is termed quasi–regular if $a \in S$ implies there exists a positive integer n such that $a^n \in S_{\text{Reg}}$.

Theorem 15. *Let S be a TC semigroup. S is a quasi–regular semigroup iff S is an E–inversive semigroup iff S is an inflation of $H \times A \times B$ where H is an abelian group, A is a left zero semigroup, and B is a right zero semigroup.*

Proof. Let S be a quasi–regular semigroup. Then, clearly, S is an E–inversive semigroup. If S is an E–inversive semigroup, then S is an inflation of $H \times A \times B$ by Theorem 12. If S is an inflation of $H \times A \times B$, it is easily seen that S is a quasi–regular semigroup.

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