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Pythagorean Theorems and Mumpsimus

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Quite recently two historians of science have offered some didactic observations on their subject. Unguru[1] has argued the need to study the early history of mathematics within a cultural context, and Bruins[2] has pointedly shown the necessity for distinguishing between those early mathematical manuscripts which are "research papers" and those which are "instructive essays." The point of the former's argument is that the history of mathematics deserves more, than to be viewed as a linear progression whose terms are successively less anachronistic. The latter quite rightly asserts, that to assess mathematical progress - say at the time of al-Kashī - we must know whether his Miftāh al-Hisāb is intended as a text for beginners, or is the author's final statement on numerical methods.[3]

In the same vein, we would like to offer some personal observations on a fear that must strike the hearts of many who work in the history of science - the fear of "mumpsimus." Both of us have always had the fear, but we owe R.J.Gillings[4] a debt for having given us its name. In case the word "mumpsimus" is new to you, it means simply: "an adherence to exposed, but customary errors."[5] Sometimes the harsher perjorative "bigoted" preceeds adherence, but we rather like the weaker definition, and perhaps should offer it for the shorter word "m-u-m- simus!"

In any case let us give a few illustrations of our meaning from the history, in particular some involving Pythagoras' right triangle theorem (Elements I,47). Between 80 and 100 years ago, it was well known that Pythagoras discovered this theorem after conversations with certain Egyptian rope stretchers, whom the cognoscenti knew as "harpedonapti"! Moreover, others knew that Pythagoras' theorem had emerged in China in the Chou Pei Suan Ching dating to at least 1100 B.C., and in India in the Śulvasutras a few centuries later. In the last 50 years these, and many other traditions relating to Pythagoras' theorem have been challenged and revised. We now distinguish cultures which studied "Pythagorean triples" distinct from Pythagorean geometry. The works of Neugebauer and Sachs[6] have called our attention to the Babylonians; Gillings[7] and R.A.Parker[8] to the Egyptians and possible influences of Babylonians on Egyptians, and the works of Needham,[9] Kaye[10] and Pingree[11] have revised our notions on the antiquity of Chinese and Indian texts. However, in spite of this work, the old stories and traditions crop up again in various recent histories of science and mathematics - ever persistent victories for "mumpsimus."[12]

Let's take one more example, one which is much closer to our own area of interest: Thābit ibn Qurra's generalization of Pythagoras' theorem. This example has the additional charm of isolating features in the history of mathematics which call "mumpsimical tendencies" into play - namely gaps in the historical record, hypothetical proof-reconstructions, problems in determining an author's sources, and so on.

Thābit, as is well known, was born in Harran (Mesopotamia) around 836, and worked there as a money changer until going to Baghdad with Mūsā ibn Shākir, who was impressed by Thābit's linguistic talents.[13] While in Baghdad, he earned a considerable reputation for Arabic translations of Greek mathematical texts, including parts of Apollonius' Conics, the Elements, the Almagest, and Nicomachus' Arithmetic. The translations are clear and precise, and in some cases, became part of the Medieval Latin tradition, owing to retranslation by Gerard of Cremona. In addition, Thābit made a number of original mathematical contributions; a rule for construction of amicable numbers, theories for composition of ratios, a computation equivalent to $\int_0^1 \sqrt{x}\, dx$, an attempt to establish Euclid's Fifth postulate, a remarkable treatise on cylindrical sections, which uses notions akin to equi-affine transformations, and establishes propositions equivalent (in modern language) to reduction of elliptic integral of a more general type to those of first kind.[14] Of course, all this information has become relatively general knowledge only in the last century. As late as 1857, Thābit's entry in the Cambridge University Library catalogue made him a 13th century Jew of unknown origin.[15]

The particular contribution of Thābit's that interests us here is his Risāla fi'l-Ḥujja al-Mansūba ilā Suqrāṭ fi'l-murabbaʿ wa-quṭrihi (Treatise on the proof attributed to Socrates on the square and its diagonal), which survives in two manuscripts, Ayasofia 4832/5 and Cairo, Dār, riyāḍ. 40m. The first dates from approximately the 5th century A.H. and the second is dated 1159 A.H.[16] Aydin Sayili was the first to call attention to the Ayasofia manuscript in a long article in Turkish[17] in 1958, followed by an English summary in 1960.[18]

In this preliminary report, we shall refer only to Sayili's Arabic transcription of the Ayasofia manuscript. This manuscript is apparently a copy of a letter of Thābit to a friend who has expressed the wish for a proof of the general Pythagorean theorem that would be as immediately convincing as that described in Plato's Meno for isosceles right triangles.[19] Thābit's answer is to offer two such proofs, and three generalizations of the Pythagorean theorem. The style of the manuscript proceeds in the very best mathematical fashion - from very detailed, to very sketchy! It is this last element which will occupy most of our attention. To be fair, we must note, as Thābit does, that his discussion is ever increasing in its generality.

Thābit's first two proofs follow figures 1 and 2 respectively, and depend, in the Socratic tradition, on dissection and composition. (In figures 1 and 2 the Δ ABC is right, with the right angle at C).

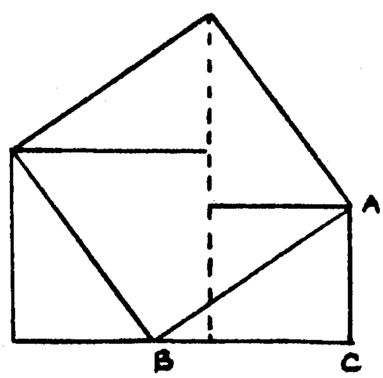

Figure 1

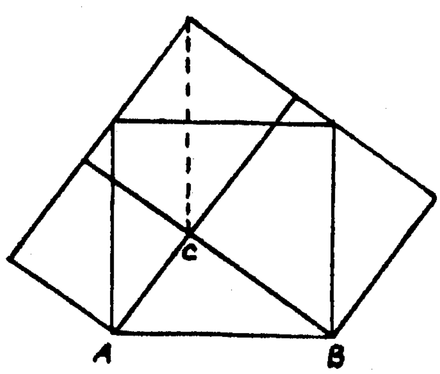

Figure 2

Thābit's proof is clear and careful in each case. As Sayili[20] points out the first proof is also documented in al-Nayrīzī's 10th century commentary on Euclid, and is there attributed to Thābit. Heath,[21] in a minor slip, implies that Thābit's first proof may have been suggested by Pappus' generalization of Pythagoras' theorem.[22] But an easy check shows this is not the case. Thabit's second proof has no mention in al-Nayrīzī, though it has been rediscovered, in various forms several times in the last 80 years.

Thabit then mentions Euclid's own generalization of the theorem: Elements VI, 31: "In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle."[23] As Thābit notes, this removes the restriction of only having squares inscribed on the sides; then he goes on to consider the case most interesting to us, namely, removal of the right angle condition. He lets Δ ABC be arbitrary, and breaks his discussion into cases. First, if the vertex angle at A is acute, (figure 3), draw AC', AB' to

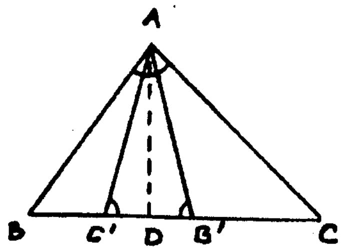

Figure 3

base AB making ∡ AC'C = ∡ AB'B = ∡ A. Of course C', B' or both may lie on $\overline{BC}$ extended, and Thabit shows due concern for their relative locations. This care is quite usual in Thabit's geometric constructions, as anyone will testify who has studied, for example Thābit's Qawl fī Taṣḥīḥ masā'il al-jabr bi 'l-barāhī al-handasiyya,[24] which sets al-Khwārīzmī's discussion of solutions of quadratics on a firm Euclidean basis. The cases are enumerated: 1) If $\measuredangle A < 60^\circ$, [then both $\measuredangle B$ and $\measuredangle C$ could exceed 60°, with both B',C' outside $\overline{BC}$, or, for example, $\measuredangle B < 60^\circ$, $\measuredangle C > 60^\circ$, giving C' inside, B' outside $\overline{BC}$];[25] 2) If $\measuredangle B$ or $\measuredangle C$ is 90°, [one or both of B' and C' outside $\overline{BC}$]; 3) If $\measuredangle A = 60^\circ$, [only one of B' or C' may be outside $\overline{BC}$]; 4) If $60^\circ < \measuredangle A < 90^\circ$, [one of B' or C' must be outside $\overline{BC}$]; 5) If $\measuredangle A = 90^\circ$, B' = C' = D. Note that since he chose to work with $\measuredangle A$, his case 2) is redundant. However, he omits cases 3) and 4), possibly because this part of the manuscript is merely intended as a sketch, and because the main idea has already been conveyed.

Thābit then looks at $\measuredangle A$ obtuse, and constructs AC', AB' making $\measuredangle AC'B = \measuredangle AB'C = \measuredangle A$; in effect, B' and C' are interchanged in the diagram (figure 4), and there's only

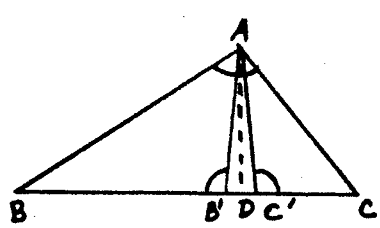

Figure 4

one case, namely $BB' + CC' < BC$, since both B' and C' lie inside $\overline{BC}$. Thābit then remarks that the generalization he has in mind is that $(BB' + CC')BC = \overline{AB}^2 + \overline{AC}^2$, and that the proof follows from Euclid. Since none is actually offered, however, a number of modern commentators have supplied their own –

usually involving the law of cosines, and gone on to assert that Thâbit probably used it himself, in the form of Elements II, 12 or II,13.[26] This would seem more likely if Thābit's drawing included the dotted line AD perpendicular to BC, but it doesn't. Boyer, who states the theorem but omits mentioning Thâbit's concern for the locations of B', C', remarks that a proof is easily supplied by similar triangles.[27] This is certainly true; for example, in figure 4, Δ BAC $\sim$ Δ BB'A $\sim$ Δ AC'C. Hence $\frac{BB'}{AB} = \frac{AB}{BC}$ and $\frac{CC'}{AC} = \frac{AC}{BC}$. Thus $(BB' + CC')BC = (AD)^2 + (AC)^2$, as Thābit claims. In fact, an examination of Thābit's various cases shows that such an argument is generally applicable. Moreover, as Thābit was adept at handling composite ratios - witness his manuscript on them: Kitāb fī Ta'līf al-nisab - such a proof reconstruction is possible. In addition, Boyer gives a nice drawing showing where the "pieces" (BB')(BC), (CC')(BC), etc. fit (figure 5).

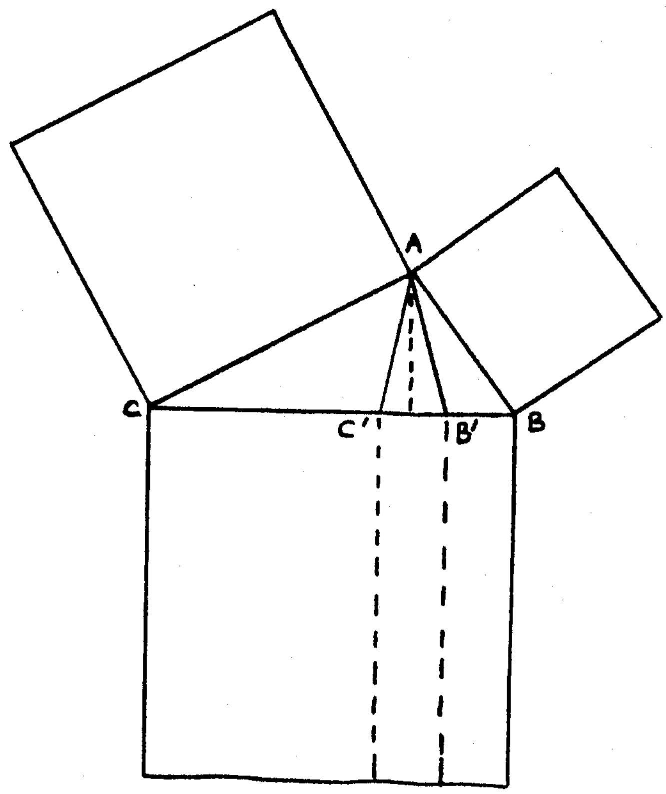

Figure 5

But there is another course open. Recall that Thābit had just been discussing Euclid's generalization, Elements VI,31. The diagram for this is (figure 6).

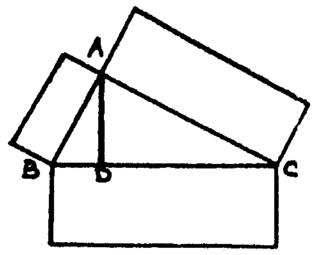

Figure 6

Here the triangle is right, and the bounding figures are similar rectangles, similarly placed, and as Thābit undoubtedly saw, the proof "works" because there are three similar triangles, something he merely has to duplicate to get his theorem. In fact, if one works through VI,31 with Thābit's diagram (figure 4), it seems most likely that Thābit would have used the three triangle similarity in the form

$$\frac{BB'}{BC} = \frac{(BB')^2}{(AB)^2} \quad \text{and} \quad \frac{CC'}{BC} = \frac{(CC')^2}{(AC)^2},$$

i.e., following Euclid's prescription [VI, 19, porism] that, if for example, $\frac{BB'}{AB} = \frac{AB}{BC}$, then BB' is to BC as the similarly placed similar figure on BB' is to the similarly placed similar figure on AB.

This latter reconstruction has the benefit of providing a rather more immediate possible motivation for Thābit's theorem, as well as being closer to a definite Euclidean model. Of course, the second consideration is not particularly decisive in Thābit's case, since he was quite capable of departures from Euclid, e.g. Maqāla fī burhān al-musādara 'l-mashhūra min Uqlīdis, and Maqāla fī anna 'l-khaṭṭayn idhā ukhrija 'alā zawiyatayn aqal min qā'imatayn iltaqayā, two treatises on the parallel postulate.

Of course, one may rightly ask: what value is such a reconstruction anyway? Well, some have certainly been important for mathematics itself, e.g. Fermat's reconstruction of Apollonius' Plane Loci, to say nothing of everyone else's attempt to reconstruct Fermat's own "too small for the margin proof." But, in general reconstructions are of value in forcing the historian of mathematics to examine mathematics of a given time within the context of that time, and in allowing him to separate flashes of genius from routine mechanics.

Thābit closes his "instructive essay" with the remark that one might also wish to deal with general triangles, and bounding figures that are not squares. He immediately supplies the theorem: the sum of similarly placed similar figures on two sides of a triangle is equal to a figure whose ratio to a similar figure similarly placed on the third side as BB' + CC' is to BC. No proof is offered, but it's immediate that one follows in a similar fashion to the proof reconstructed above.

Of course Pappus (fl. AD 300-350) had included an elegant generalization of Elements I, 47 in his Collection, which allows arbitrary triangles and the similar figures to be parallelograms.[28] The form of this generalization, and the circle of ideas in its proof are quite different from Thābit's.

In fact, as far as is known, the Collection never became a part of the Arabic mathematical tradition, and very few of Pappus' other works were rendered into Arabic. The only one of these with purely mathematical content is, possibly, the Tafsīr al-maqāla al-ʿāšira min Kitab Uqlīdis fī maqālatain (Commentary on Euclid, X) translated by Abū ʿUthman al-Dimīshqī (Fl. ca 908-932).[29] So, as Sayili quite rightly observes, Thābit makes no mention of Pappus' generalization.

Still, there is a detail worth noting. If one takes Pappus' generalization, restricts the triangle to be right angled, and the similar figures to be squares, the resulting figure is the same as Thābit's second "Socratic" construction. Lest one places too much significance in this, let's also note that Thābit's generalization itself has been independently rediscovered at least three times in less than 150 years: by Wallis (1065), and by Ozanam (1790), possibly through Clairaut,[30] and as I noted earlier, the second "Socratic" proof itself has been rediscovered several times in the last 80 years.

And finally, two disquieting observations, and a suggestion. Let's not forget that the translatorium in which Thābit worked also produced translations from Indian sources as well. Also some mathematical notions, even specific examples from texts do get transmitted from one language to another even though the texts themselves may never be translated. This point was driven home to us in some studies we're currently doing on the mathematical sources of Ibn Sīnā's Kitāb al-Shīfa, and an earlier study on Arabic work on the parallel postulate.[31]

The suggestion is simply a reiteration of one of Gilling's: perhaps even the name Pythagorean theorem is a "mumpsimus."[32]

NOTES

1. Unguru (47).
2. Bruins (4).
3. Actually Bruins remarks in this paper are concerned with Archimedes' "Measurement of the circle." The references to al-Kāshī, equally apt, appear in a paper he read at the 2nd International Congress on the History of Arabic Science, Aleppo, Syria, 1979.
4. Gillings (18), 1.
5. See entry "mumpsimus" in Webster's Third International Dictionary.
6. See Neugebauer and Sachs (31), and also (5) - (10), (17), (20), (23) - (24), (29)-(30), (35), (46), (48)-(49).
7. Gillings (18), (19).
8. Parker (31)-(32).
9. Needham (28), 21 ff. See also Mikami (27).
10. Kaye (26). See also (1), (14)-(15).
11. Specifically, see Pingree's Bibliography of Sanskrit Science.
12. Compare with van der Waerden (49), I, 6.
13. Rosenfeld and Grigorian (36). See also Suter (45), 34-38; Sarton (37), 599-600; Daffa' (12), 101-104, and (13), 93-104.
14. See note 13.
15. Steinschneider (44), 331.
16. Sezgin (41), 269.
17. Sayili (38).

18. Sayili (39).

19. Plato (34), 82B-85B. As Gow rightly points out [A Short History of Greek Mathematics, 175], Plato is less interested in the Pythagorean theorem, than in the chain of reasoning. See also Heath (21), 297-303.

20. Sayili (39), 45-46, note 2.

21. Heath (22), I, 365.

22. Hultsch (25), I, 177. (Book IV, prop.1)

23. Heath (22), II, 268-269.

24. See, for example P. Luckey, "Ṭābit b. Qurra über den geometrischen Richtigkeitsnachweis der Auflősung der quadratischen Gleichungen," Berichte uber die Verhandl. der sachs. Akad. d. Wiss., Leipzig, math.- phys. Kl 93(1941), 93-114.

25. Brackets in 1)-5) are editorial additions.

26. Sayili (39), and R. Schloming, "Thābit ibn Qurra and the Pythagorean Theorem," Math. Teacher <u>63</u> (1970), 519-528.

27. See C. Boyer, A History of Mathematics. Wiley, N.Y., 1968, especially, 259. But see also Scriba (40), 62. Actually, all references to the Elements used by Thābit, including figures, can be traced to the Isḥāq-Thābit translation (Bodleian, Nicoll 279), rather than to Heath. But Heath is used as a convenience.

28. Hultsch (25), I, 177.

29. Bulmer-Thomas (11), 299-300. Also see Sezgin (41), 174-176; Brockelmann (3), passim; Steinschneider (42), (43); Dodge (16).

30. Scriba (40) and Boyer (2).

31. These are both joint works. The second will appear in the Proc. Internat. Cong. History of Science (UNESCO), and the first is only in manuscript.

32. Gillings (18), 1.

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