Generalized Bruck-Reilly Extensions

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Abstract

Let $T$ be a monoid with group of units $U$. Let $\theta$ be a homomorphism of $T$ into $U$ ($\theta^0$ denoting the identity automorphism of $T$). Let $P$ and $K$ be disjoint sets and let $\gamma$ be a homomorphism of $T$ into $G_K$, the full transformation group on $K$. Let $S = ((I^0 \times \{0\}) \times (T \times P)) \cup ((I^0 \times N) \times (T \times K))$ where $N(I^0)$ is the natural numbers (non-negative integers) under the multiplication $((n,k),(g,p))((r,s),(h,q)) = ((n+r-t,k+s-t),(g\theta^t-h\theta^{k-t},x))$ where $t = \min(k,r)$ and $x = q$ or $p(h\theta^{k-t}r^{-1}\gamma)$ according to whether $k \leq r$ or $k > r$. If $|P| = |K| = 1$, $S$ is the Bruck–Reilly extension of $T$ determined by $\theta$. So, we call $S$ the generalized Bruck–Reilly extension of $T$ determined by $P, K, \theta$, and $\gamma$. We characterize $S$ in case $T$ is a group or a finite chain of groups generalizing the corresponding results for Bruck–Reilly extensions. We explicitly describe the group congruences on $S$, describe the minimum such congruence $\beta$, give the structure of $S/\beta$, and determine $\vee$ and $\wedge$ in $[\beta, S \times S]$, the lattice of group congruences on $S$.

1. Introduction.

Let $T$ be a monoid with group of units $U$. Let $\theta$ be a homomorphism of $T$ into $U$ ($\theta^0$ denoting the identity automorphism of $T$). Let $P$ and $K$ be disjoint sets and let $\gamma$ be a homomorphism of $T$ into $G_K$, the full transformation group on $K$. Let $S = ((I^0 \times \{0\}) \times (T \times P)) \cup ((I^0 \times N) \times (T \times K))$ where $N(I^0)$ is the natural
numbers (non-negative integers) under the multiplication

\[
((n, k), (g, p))((r, s), (h, q)) = \begin{cases} 
((n + r - k, s), (g \theta^{r-k}h, q)) & \text{if } k \leq r \\
((n, k + s - r), (g(h \theta^{k-r}), p(h \theta^{k-r-1}\gamma)) & \text{if } k > r
\end{cases}
\]

Let \( S_1 = (I^0)^2 \times T \) under the multiplication \((n, k), (g, s), (h, q) = ((n + r - t, k + s - t), g \theta^{r-t}h \theta^{k-t}) \) where \( t = \min(k, r) \). Howie [5] calls \( S_1 \) the Bruck–Reilly extension of \( T \) determined by \( \theta \) and writes \( S_1 = BR(T, \theta) \) (Howie actually uses triples \( I^0 \times T \times I^0 \) and uses \( \max(k, r) \)). Since if \(|P| = |K| = 1, S \cong S_1\), we call \( S \) the generalized Bruck–Reilly extension of \( T \) determined by \( P, K, \theta, \gamma \) and write \( S = gBR(T, P, K, \theta, \gamma) \).

Howie [5] also noted that constructions characterizing bisimple \( \omega \)-inverse semigroups and simple \( \omega \)-inverse semigroups are special Bruck–Reilly extensions. If \(|U| = 1, BR(T, \theta) = C \circ T \), the Bruck product of \( C \), the bicyclic semigroup, and \( T \) [14]. The congruences on a Bruck–Reilly extension have been studied by Piochi [11] and Rankin [12].

Our first major result (Theorem 3.9) characterizes \( S = gBR(T, P, K, \theta, \gamma) \) when \( T \) is a finite chain of groups. Before stating Theorem 3.9, we will need a definition.

Let \( d \) be a positive integer. A semigroup \( S \) with \( d \) \( \mathcal{D} \)-classes is called an \( \omega dPK \) semigroup of \( E(S) \) (the set of idempotents of \( S) \cong ((\{0\} \times Y \times P) \cup (N \times Y \times K) \) where \( Y = \{0, 1, \ldots, d - 1\} \) under the multiplication \((k, i, p)(r, j, q) = (k, i, p), (r, j, q) \) or \((r, \max\{i, j\}, q) \) according to whether \( k > r, k < r, \) or \( r = k \). We show (Theorem 3.9) that a semigroup \( S \) is isomorphic to a generalized Bruck–Reilly extension of a finite chain \((0 > 1 > \cdots > d - 1) \) of groups determined by \( P \) and \( K \) if and only if \( S \) is a regular simple \( \omega dPK \) semigroup. As a corollary to Theorem 3.9, we show (Corollary
that a semigroup $S$ is isomorphic to a Bruck–Reilly extension of a finite chain of groups if and only if $S$ is a simple $\omega$–inverse semigroup (Corollary 3.10 is due to Koçin [3, 9]; see also Howie [5] and Munn [8]). In Theorem 3.12, we characterize $S = gBR(T, P, K, \theta, \gamma)$ when $T$ is a group. Before stating this result, we will need a definition. A bisimple semigroup $S$ is termed $E$–bisimple if $E(S) = \cup(E_k : k \in I^0)$, $E_i \cap E_j = \phi$ if $i \neq j$, each $E_i$ is a right zero semigroup, and $e \in E_i, f \in E_j$ and $i > j$ imply $e < f$ [16]. We show (Theorem 3.12) that a semigroup $S$ is isomorphic to a generalized Bruck–Reilly extension of a group if and only if $S$ is an $E$–bisimple semigroup. (Theorem 3.12 is due to Warne [16].) The proof in [16] is of considerable length. The proof here just combines Theorem 3.9 and Lemma 3.11). As a corollary to Theorem 3.12, we show (Corollary 3.13) that a semigroup $S$ is isomorphic to a Bruck–Reilly extension of a group if and only if $S$ is a bisimple $\omega$–inverse semigroup (Corollary 3.13 is due to Reilly [13]. See also Howie [5] and Warne [14]).

In Theorem 4.6, we determine the group congruences on $S = gBR(T, P, K, \theta, \gamma)$. Before stating Theorem 4.6, we will need the definition of an “admissible triple” and the congruences determined by it. Let $H$ be a normal subgroup of $U$, let $x \in U$, and $k \in I^0$ such that (i) $g \in H$ if and only if $g\theta \in H(g \in U)$, (ii) $(x\theta)H = xH$, and (iii) $(g\alpha^k)H = (x^{-1}gx)H$. We call $[H; x, k]$ an admissible triple and define a relation $\sigma = \sigma[H; x, k]$ on $S$ by $((m, n), (g, s))\sigma((p, q), (h, t))$ if and only if $(n - m) - (q - p) = ak$ and $g\alpha^{n+k}(h\alpha^{-2})^{-1} \in x^aH$ for some integer $a$. We show (Theorem 4.6) that if $[H; x, k]$ is an admissible triple, then $\sigma[H; x, k]$ is a group congruence on $S$,
and conversely every group congruence on $S$ can be written in the form $\sigma[H; x, k]$ for some admissible triple $[H; x, k]$ (Theorem 4.6 generalizes the determination of the group congruences on $BR(U, \theta)$ and $BR(T, \theta)$ by Ault [1, 9] and Piochi [11] respectively.) In Lemma 4.10, we note that $S = gBR(T, P, K, \theta, \gamma)$ has a minimum group congruence $\beta$. In Theorem 4.11, we show $S/\beta \cong V$, the maximal group homomorphic image of $BR(U, \theta)$. $V$ is described in Theorem 4.7. (This description was originally given in [15].) (We showed $V$ is the maximal group homomorphic image of $gBR(U, P, K, \theta, \nu)$ in [19]). In Theorem 4.11, we also show that $x\beta y(x, y \in S)$ if and only if $xc = yc$ for some $c \in E(S)$. Finally, if $[H; x, k]$ and $[K; y, \ell]$ are admissible triples, we determine $\sigma[H; x, k] \lor \sigma[K; y, \ell]$ (Theorem 4.12) and $\sigma[H; x, k] \land \sigma[K; y, \ell]$ (Theorem 4.13) (These results generalize corresponding results of Petrich for $BR(U, \theta)$ [10].)

If $S$ is a semigroup, $E(S)$ will denote the set of idempotents of $S$. For terms not defined here, see [3] or [5] – for example, Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$ and $\mathcal{D}$, natural partial ordering of $E(S)$, semilattice of semigroups, simple semigroup, bisimple semigroup, regular semigroup, inverse semigroup, band, ideal, inner right translation, group of units, group congruence, maximal group homomorphic image, right group.

2. Preliminaries.

Let $S = gBR(T, P, K, \theta, \gamma)$. We first show $S$ is a semigroup (Theorem 2.1).
In Lemma 2.2, we determine $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{D}$ on $S$, determine $E(S)$ and the natural partial order on $E(S)$, show $S$ is a regular semigroup iff $T$ is a regular semigroup, and show $S$ is a simple semigroup. In Theorem 2.3, we show that if $T$ is a finite chain $(0 > 1 > 2 > \cdots > d-1)$ of groups, then $S$ is a regular simple $\omega dPK$ semigroup. As a corollary to Theorem 2.3, we show that if $T$ is a finite chain $(0 > 1 > \cdots > d-1)$ of groups, then $BR(T, \theta)$ is a simple $\omega$-inverse semigroup with $d-\mathcal{D}$-classes (Corollary 2.4). Theorem 2.3 and Corollary 2.4 are used in the proofs of Theorem 3.9 and Corollary 3.10 respectively.

**Theorem 2.1.** $S = (T, K, P, \theta, \gamma)$ is a semigroup.

**Proof.** Let $\phi_k = \theta$ for each $k \in I^0$ and define, for $i < j$, $\phi_{ij} = \phi_i \phi_{i+1} \cdots \phi_{j-1}$ and $\phi_{ii} = i_T$, the identity automorphism on $T$. So $\{\phi_{ij} : i, j \in I^0, \ i < j\}$ is a collection of homomorphisms of $T$ into $U$.

It is easily seen that

1. (a) $\phi_{ij} \phi_{jk} = \phi_{ik}$ for $i \leq j \leq k$, (b) $\phi_{ij} = \theta^{j-i}$.

2. Define $\gamma_{ji} = \theta^{i-1-j} \gamma$ for $i, j \in I^0$ with $i > j$. Thus $\{\gamma_{ji} : i, j \in I^0, i > j\}$ is a collection of homomorphisms of $T$ into $G_K$. It is easily seen that $\phi_i \gamma_{i+1,k} = \gamma_{i,k}$ from which it follows that

3. $\phi_{ij} \gamma_{jk} = \gamma_{ij}$ for $i \leq j < k$.

Let $X = ((T \times P \times \{0\}) \cup ((T \times K \times N)$ under the multiplication
(3) \((g, n, i)((h, m), j)\) = \(\begin{cases} 
(g(h \phi_{ji}), n(h \gamma_{ji}, i), i) & \text{if } i > j, \\
((g \phi_{ji} h, m), j) & \text{if } i \leq j.
\end{cases}\)

Using 1(a) and (2) and routine calculations, \(X\) under (3) is a semigroup.

Define

(4) \[
\begin{cases}
(g, n, 0, 0) \alpha = (g \theta, n_0, 0) & (n_0 \in P) \\
(g, n, 1) \alpha = (g, n_0, 0) \\
(g, n, k) \alpha = (g, n, k - 1) & \text{if } k > 1
\end{cases}
\]

(5) \[
(g, n, k) \beta = \begin{cases} 
(g, n, k + 1) & \text{if } k > 0 \\
(g, n_1, 1) & \text{if } k = 0 \quad (n_1 \in K).
\end{cases}
\]

Using (3), (2), (1(b)), and routine calculations, \(\alpha\) and \(\beta\) are endomorphisms of \(X\).

Let \(T_0 = T \times P \times \{0\}\) and \(T_n = T \times K \times \{n\}\) if \(n > 0\). Then

(6) \(\alpha : T_k \to T_{k - \min(k, 1)}\) and \(\beta : T_k \to T_{k+1}\) for \(k \in I^0\).

Let \(e_0 = (e, n_0, 0)\) and \(e_k = (e, n_1, k)\) for \(k \geq 1\) where \(e\) is the identity of \(T\). It is easily checked that \(e_j \in E(T_j)\), the set of idempotents of \(T_j\), for \(j \in I^0\). Let \(\rho_{e_j}\) denote inner right translation of \(T\) determined by \(e_j\) (i.e. \(x\rho_{e_j} = xe_j\) for all \(x \in T\)).

Using (3), (4), (5), (1(b)), and the fact \(e_{\gamma_{ji}}\) is the identity of \(G_K\), we obtain

(7) \(\alpha^n \beta^n = \rho_{e_n}\) for \(n \geq 1\) and \(\beta \alpha = \rho_{e_0}\).

Similarly, we obtain

(8) \(\rho_{e_0} \alpha = \alpha \rho_{e_0}; \rho_{e_0} \beta = \beta \rho_{e_0}\).

Define
(9) $\alpha_{(n,r)} = \alpha^r \beta^n \ (n, r \in I^0)$ with $\alpha^0 = \beta^0 = \rho_{e_0}$.

Using (7) and (8), it is easy to show that $(n, r) \to \alpha_{(n,r)}$ is an anti-homomorphism of $C$, the bicyclic semigroup $((C = I^0 \times I^0$ under the multiplication $(n, r)(p, q) = (n+p-\min(r, p), \ r+q-\min(r, p))$ into $\text{End} \ T$, the semigroup of endomorphisms of $T$.

Using (9), (6), and (3), we obtain

(10) $T_k \alpha_{(r, s)} \subseteq T_{k+r-\min(k, s)}$.

Let $V = \{(n, k), g_k) : (n, k) \in C, g_k \in T_k\}$ under the multiplication

(11) $((n, k), g_k) \cdot ((r, s), h_s) = ((n, k)(r, s), g_k, \alpha_{(s, r)} h_s)$ where juxtaposition denotes multiplication in $C$ and $X$.

Using the multiplication on $C$, (10), and (3), $(V, (11))$ is a groupoid.

Using the fact that $(r, s) \to \alpha_{(r, s)}$ is an anti-homomorphism of $C$ into $\text{End} \ T$, it is easily established that $(V, (11))$ is a semigroup.

Using (3), (11), (9), (7), and (5), we obtain

(12) $((n, k), (g, p, k))((r, s), (h, q, s)) = \begin{cases} (n, k)(r, s), (g, p, k)\alpha^r s h(q, s)) & \text{if } r > s \\ ((n, k)(r, s), (g, p, k)\beta^r s h(q, s)) & \text{if } s > r \\ (n, k)(r, s), (g, p, k)(h, q, s)) & \text{if } s = r \end{cases}$

where juxtaposition denotes multiplication in $C$ and $X$.

Using (12), (4), (5), 1(b), the definition of $\gamma_{i,i}$ and (3), we obtain

(13) $((n, k), (g, p, k))((r, s), (h, q, s)) = \begin{cases} ((n + r - k, s), (g^p q^{r-k} h, q, s)) & \text{if } k \leq r \\ ((n, k + s - r), (g(h^{k-r}p^{r-1} \gamma), k + s - r)) & \text{if } k > r \end{cases}$
We may write $V = ((n, k), (g, p, k)) : (n, k) \in C, (g, p, k) \in T_k$. So, $(V, (13))$ is a semigroup.

Since $((n, k)(g, p, k)) \varphi = ((n, k), (g, p))$ defines an isomorphism of $(V, (13))$ onto $S$ under the multiplication given in the statement of the theorem, $S = (T, K, P, \theta, \gamma)$ is a semigroup.

**Lemma 2.2.** Let $S = (T, K, P, \theta, \gamma)$ be the generalized Bruck-Reilly extension of $T$ determined by $K, P, \theta$, and $\gamma$. Then

(a) $((n, k), (g, p)) \mathcal{R}((r, s), (h, q))$ if and only if $n = r$ and $g \mathcal{R} h$ (in $T$)

(b) $((n, k), (g, p)) \mathcal{L}((r, s), (h, q))$ if and only if $k = s, p = q$, and $g \mathcal{L} h$ (in $T$)

(c) $((n, k), (g, p)) \mathcal{D}((r, s), (h, q))$ if and only if $g \mathcal{D} h$ (in $T$)

(d) $E(S) = (((0, 0), (f, p)) : p \in P, f \in E(T)) \cdot (((n, n), (g, q)) : n \in N, q \in K, g \in E(T))$

(e) Let $c, f \in E(T)$. Then $((k, k), (c, p)) < ((r, r), (f, q))$ if and only if $k > r$ or $k = r, p = q$, and $c < f$.

(f) $S$ is a regular semigroup if and only if $T$ is a regular semigroup.

(g) $S$ is a simple semigroup.

**Proof.** Utilizing the definition of $S = (T, K, P, \theta, \gamma)$ and Theorem 2.1, the proof of (a) – (f) is by routine calculations. To prove (g), let $((n, k), (g, p)), ((r, s), (h, q)) \in S$. 8
Let \( t \in K \). Then we obtain \(((r, n + 1), ((g\theta)^{-1}, t))((n, k), (g, p))((k + 1, s), (h, q)) = ((r, s), (h, q))\).

**Theorem 2.3.** If \( T \) is a finite chain \((0 > 1 > 2 > \cdots > d - 1)\) of groups, then \( S = gBR(T, P, K, \theta, \gamma) \) is regular simple \( \omega dPK \) semigroup.

**Proof.** \( S \) is a semigroup by Theorem 2.1. By [3, Theorem 4.11], we may write \( T = \bigcup \{G_i : i \in Y = \{0, 1, 2, \ldots, d-1\}\} \) where \( \{G_i : i \in Y\} \) is a collection of pairwise disjoint groups with the multiplication on \( T \) given by \( g_i g_j = g_i \varphi_{i, \max(i, j)} g_j \varphi_{j, \max(i, j)} \) where \( g_i \in G_i, g_j \in G_j \) and \( \varphi_{i,k} \) \((k > i)\) is a homomorphism \( G_i \) into \( G_k \) such that \( \varphi_{i,k} \varphi_{k,t} = \varphi_{i,t} \) if \( i < k < t \) and \( \varphi_{i,t} \) denotes the identity automorphism on \( G_i \). Using this multiplication it is easily checked that \( T \) is a regular semigroup with \( \{G_0, G_1, \ldots, G_{d-1}\} \) as \( D \)-classes. Thus, \( S \) is a regular semigroup with \( d \) \( D \)-classes by Lemma 2.2(f) and Lemma 2.2(c) respectively. Let \( e_i \) denote the identity of the group \( G_i \). Then, \( e_0 \) is the identity of \( T \). It is easily checked \( G_0 \) is the group of units of \( T \). Using Lemma 2.2(d) and the multiplication on \( T \), it is routine to verify that \(((k, k), (e, p)) \varphi = (k, i, p)\) defines an isomorphism of \( E(S) \) onto the band \((0 \times Y \times P) \cup (N \times Y \times K) \) where \( Y = \{0, 1, \ldots, d - 1\} \) under the multiplication \((k, i, p)(r, j, q) = (k, i, p), (r, j, q), \) or \((r, \max\{i, j\}, q)\) according to whether \( k > r, k < r, \) or \( r = k \). \( S \) is simple by Lemma 2.2(g). So \( S \) is a regular simple \( \omega dPK \) semigroup.

A semigroup \( S \) is termed an \( \omega \)-semigroup if \( E(S) \cong I^0 \) under the reverse of the
usual order.

As an immediate consequence of Theorem 2.3, we obtain the following corollary due to Howie [5, p 163–164].

**Corollary 2.4.** If $T$ is a finite chain $(0 > 1 > \cdots > d - 1)$ of groups, then $S = BR(T, \theta)$ is a simple $\omega$–inverse semigroup with $d$ $D$–classes.

**Proof.** Just take $|P| = |K| = 1$ in the statement of Theorem 2.3.

3. $gBR(T, P, K, \theta, \gamma)$ where $T$ is a finite chain of Groups.

In this section, we characterize $gBR(T, P, K, \theta, \gamma)$ where $T$ is a finite chain of groups (Theorem 3.9). As a Corollary to Theorem 3.9, we characterize $BR(T, \theta)$ when $T$ is a finite chain of groups (Corollary 3.10). In Theorem 3.12, we characterize $gBR(T, P, K, \theta, \gamma)$ when $T$ is a group. As a corollary to Theorem 3.12, we characterize $BR(T, \theta)$ when $T$ is a group (Corollary 3.13).

Let $S$ be a regular simple $\omega dPK$ semigroup. The major part of this section will be devoted to proving $S \cong gBR(H, P, K, \theta, \gamma)$ for some $\theta, \gamma$ where $H$ is a finite chain $\{0 > 1 > \cdots > d - 1\}$ of groups (Theorem 3.8). We will show (Remark 3.1) that any $\omega dPK$ simple regular semigroup is a simple $\omega\mathcal{L}$–unipotent semigroup (defined below) (equivalently, by Theorem 3.2, an $\omega\mathcal{Y}\mathcal{L}$–unipotent semigroup with $Y = (0 > 1 > \cdots > d - 1)$ (definition given below). We gave the structure of $\omega\mathcal{Y}\mathcal{L}$–unipotent semigroups in [18] (given here as Theorem 3.3). Using Theorems
3.2 and 3.3, we obtain a structure theorem for simple \( \omega-L \)-unipotent semigroups (Theorem 3.4). From Theorem 3.4, we obtain another structure theorem for simple \( \omega-L \)-unipotent semigroups (Corollary 3.5) which forms a basis for our proof of Theorem 3.8. Corollary 3.5 gives the structure of simple \( \omega-L \)-unipotent semigroups modulo \( \omega \)-chains of right groups \( T \). By Note 3.6 and Lemma 3.7, we give an explicit description of \( T \) in the case \( S \) is an \( \omega dPK \) regular simple semigroup. Lemma 3.7 is crucial in the proof of Theorem 3.8. First, we will need some definitions.

A semigroup \( X \) is termed \( L \)-unipotent if each principal left ideal of \( X \) has a unique idempotent generator. Equivalently [17], the \( L \)-unipotent semigroups are precisely the regular semigroups \( X \) such that \( E(X) \) is a semilattice \( \Lambda \) of right zero semigroups \( (E_{\delta} : \delta \in \Lambda) \).

If \( \Lambda = I^0 \times Y \) where \( I^0 \) is the non-negative integers and \( Y \) is a semilattice with greatest element, with \((k, \delta) \land (s, \eta) = (k, \delta), (s, \eta), \) or \((k, \delta \land \eta)\) according to whether \( k > s, s > k, \) or \( s = k \), we call \( \Lambda \) an \( \omega Y \)-semilattice. We term \( X \) and \( \omega Y-L \)-unipotent semigroup of (1) \( \Lambda \) is an \( \omega Y \)-semilattice, (2) \( E(D_{\delta}) = \cup(E(n, \delta) : n \in I^0) \) where \((D_{\delta} : \delta \in Y)\) is the collection of \( D \)-classes of \( S \).

If \( \Lambda \) is order isomorphic to \( I^0 \) under the reverse of the usual order (no further condition), we term \( X \) an \( \omega-L \)-unipotent semigroup.

**Remark 3.1.** We next note that a regular simple \( \omega dPK \) semigroup is a simple
$\omega$–$\mathcal{L}$–unipotent semigroup.

Let $S$ be an $\omega dPK$ simple regular semigroup. Let $E_{(0,i)} = \{(0, i, p) : p \in P\}$ and if $k > 0$ let $E_{(k,i)} = \{(k, i, p) : p \in K\}$. Then $E_{(k,i)}E_{(r,j)} \subseteq E_{(k,i)}E_{(r,j)}$ or $E_{(k,\max(i,j))}$ according to whether $k > r, r > k$, or $r = k$. We note $I^0 \times Y$ where $Y = (0 > 1 > 2 > \cdots > d - 1)$ is an $\omega Y$–semilattice with $(k, i) \land (r, j) = (k, i), (r, j)$, or $(k, \max(i, j))$ according to whether $k > r, r > k$, or $r = k$. It is easily checked that $\varphi(k, i) = i + kd$ defines an isomorphism of the $\omega Y$–semilattice $I^0 \times Y$ onto $I^0$ under reverse of the usual order. So if we let $E_{(k,i)} = E_{i+kd}$, then $S$ is a simple $\omega$–$\mathcal{L}$–unipotent semigroups.

**Theorem 3.2** (Warne, [18, Theorem 7.7]) $S$ is a simple $\omega$–$\mathcal{L}$–unipotent semigroup if and only if $S$ is an $\omega Y$–$\mathcal{L}$–unipotent semigroup with $Y$ the finite chain $0 > 1 > \cdots > d - 1$ where $d$ is a positive integer.

So, if $S$ is an $\omega dPK$ regular simple semigroup, then $S$ is an $\omega Y$–$\mathcal{L}$–unipotent semigroup with $Y = (0 > 1 > \cdots > d - 1)$.

**Theorem 3.3.** (Warne, [18, Theorem 6.1]). Let $S$ be an $\omega Y$–$\mathcal{L}$–unipotent semigroup. Then there exists $T = \cup(T_{(k,\delta)} : k \in I^0, \delta \in Y$, a semilattice with greatest element $\delta_0)$, an $\omega Y$–semilattice $\Lambda = I^0 \times Y$ of right groups $(T_{(k,\delta)} : k \in I^0$ and $\delta \in Y)$ and a homomorphism $(n, r) \to \alpha_{(n, r)}$ of $C$, the bicyclic semigroup, into $\text{End } T$, the semigroup of endomorphism of $T$, such that

(1) for each $k \in I^0$, there exists $e_{(k,\delta_0)} \in E(T_{(k,\delta_0)})$ such that $g\alpha_{(k,k)} = ge_{(k,\delta_0)}$ for
each \( g \in T \).

(2) For each \( k, r, s \in I^0 \),

\[
T_{(k, \delta)}^{(n, k)} \equiv \begin{cases} 
T_{(s, \delta_0)} & \text{if } r > k \\
T_{(k+s-r, \delta)} & \text{if } k \geq r.
\end{cases}
\]

Furthermore, \( S \cong \{((n, k), g_{k\delta}) : g_{k\delta} \in T_{(k, \delta)}, n, k \in I^0, \delta \in Y \} \) under the multiplication

\[
((n, k), g_{k\delta})((r, s), h_{s\delta}) = ((n, k)(r, s), g_{k\delta}h_{s\delta}) \quad \text{where juxtaposition denotes multiplication in } C \text{ and } T.
\]

Conversely, let \( T = \bigcup T_{(k, \delta)} : k \in I^0, \delta \in Y, \) a semilattice with greatest element \( \delta_0 \) be an \( \omega Y \)-semilattice \( \Lambda = I^0 \times Y \) of right groups \( T_{(k, \delta)} : k \in I^0, \delta \in Y \) and let \( (n, r) \to \alpha_{(n, r)} \) be a homomorphism of \( C \) into \( \text{End } T \) such that (1) and (2) are valid. Then \( \{((n, k), g_{k\delta}) : g_{k\delta} \in T_{(k, \delta)}, k \in I^0, \delta \in Y \} \) under the multiplication (3) is an \( \omega Y \)-\( \mathcal{L} \)-unipotent semigroup.

Let \( T \) be a semigroup which is a disjoint union of right groups \( \{T_k : k \in I^0 \} \) such that \( T_kT_r \subseteq T_{\max(k, r)} \). We call \( T \) an \( \omega \)-chain of right groups.

Using Theorems 3.2 and 3.3, we obtain

**Theorem 3.4.** Let \( S \) be a simple \( \omega \)-\( \mathcal{L} \)-unipotent semigroup. Then there exists a positive integer \( d \), on \( \omega \)-chain of right groups \( \{T_{i+kd} : k \in I^0, 0 \leq i < d \} \), a homomorphism \( (n, r) \to \alpha_{(n, r)} \) of \( C \), the bicyclic semigroup, into the semigroup of endomorphisms of \( T \) such that

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(1) For each \( k \in I^0 \), there exists \( e_{kd} \in E(T_{kd}) \) such that \( g_{\alpha_{(k,k)}} = g_{e_{kd}} \)

(2) For each \( k, r, s \in I^0 \)

\[
T_{i+k d\alpha_{(r,s)}} \subseteq \begin{cases} 
T_{sd} & \text{if } r > k \\
T_{i+(k+s-r)d} & k \geq r
\end{cases}
\]

Furthermore, \( S \cong \{((n,k),g_{ki}) : g_{ki} \in T_{i+k d}, n, k \in I^0, 0 \leq i < d\} \) under the multiplication

3) \(((n,k),g_{ki})((r,s),g_{sj}) = ((n,k)(r,s),g_{ki}\alpha_{(r,s)}g_{sj})\) where juxtaposition denotes multiplication in \( C \) and \( T \). \( S \) has precisely \( d \) \( D \)-classes.

Conversely, let \( d \) be a positive integer and let \( T \) be an \( \omega \)-chain of right groups \( \{T_{i+k d} : k \in I^0, 0 \leq i < d\} \) and \((n,r) \rightarrow \alpha_{(n,r)}\) be a homomorphism of \( C \) into the semigroup of endomorphisms of \( T \) such that (1) and (2) are valid. Then \( \{((n,k),g_{ki}) : g_{ki} \in T_{i+k d}, n, k \in I^0, 0 \leq i < d\} \) under the multiplication (3) is a simple \( \omega-L\)-unipotent semigroup with \( d \) \( D \)-classes.

**Corollary 3.5.** Let \( S \) be a simple \( \omega-L\)-unipotent semigroup. Then there exists a positive integer \( d \), an \( \omega \)-chain \( T \) of right groups \( \{T_{i+k d} : k \in I^0, 0 \leq i < d\} \), endomorphisms \( \alpha \) and \( \beta \) of \( T \) and a sequence \( \{\rho_{e kd} : e_{kd} \in E(T_{kd}), k \in I^0\} \) of inner right translation of \( T \) such that

(a) \( T_{i+k d\alpha} \subseteq T_0 \) if \( k = 0 \), \( T_{i+k d\alpha} \subseteq T_{i+(k-1)d} \) if \( k > 0 \), \( T_{i+k d\beta} \subseteq T_{i+(k+1)d} \)

(b) \( \alpha^n \beta^m = \rho_{e_{kd}} \) for \( n \geq 1 \) and \( \beta \alpha = \rho_{e_0} \)

(c) \( \rho_{e_0} \alpha = \alpha = \alpha \rho_{e_0} \) \( \rho_{e_0} \beta = \beta = \beta \rho_{e_0} \)
Furthermore, \( S \cong \{(n, k), g_{ki} \in T_{i+k\delta}; n, k \in I^0, \ 0 \leq i < d\} \) under the multiplication

\[
(d) \ ((n, k), g_{ki})(r, s), h_{sj}) = \begin{cases} 
((n, k)(r, s), g_{ki}\alpha^{r-s}h_{sj}) & \text{if } r > s \\
((n, k)(r, s), g_{ki}\beta^{s-r}h_{sj}) & \text{if } s > r \\
((n, k)(r, s), g_{ki}h_{sj}) & \text{if } s = r
\end{cases}
\]

where juxtaposition denotes multiplication in \( C \) and \( T \cdot S \) has \( d \mathcal{D} \)-classes.

Conversely, let \( d \) be a positive integer and let \( T \) be an \( \omega \)-chain of right groups \( (T_{i+k\delta}: k \in I^0, 0 \leq i < d) \), let \( \alpha \) and \( \beta \) be endomorphisms of \( T \), and let \( \{\rho_{e_{nd}}: e_{nd} \in E(T_{nd})\} \) be a sequence of right translation of \( T \) such that (a) – (c) are valid. Then \( \{(n, k), g_{ki} \in T_{i+k\delta}, n, k \in I^0, 0 \leq i < d\} \) under the multiplication (d) is a simple \( \omega \mathcal{L} \)-unipotent semigroup with \( d \mathcal{D} \)-classes.

**Proof.** We first establish the direct part. In the notation of Theorem 3.4, let \( \alpha = \alpha_{(1, 0)} \) and \( \beta = \alpha_{(0, 1)} \). By (2) of Theorem 3.4, \( \alpha \) and \( \beta \) are endomorphisms of \( T \) satisfying (a). Let the sequence \( (\rho_{e_{kd}}: k \in I^0) \) be defined by (1) of Theorem 3.4. Then using (1) of Theorem 3.4 and the fact \((n, r) \to \alpha_{(n, r)} \) is a homomorphism, we obtain \( \rho_{e_{nd}} = \alpha_{(n, n)} = \alpha_{(n, 0)(0, n)} = \alpha_{(n, 0)}\alpha_{(0, n)} = \alpha^n\beta^n \) and \( \rho_{e_0} = \alpha_{(0, 0)} = \alpha_{(0, 1)}\alpha_{(1, 0)} = \beta \alpha \). Hence (b) is valid. Since \((0, 0)\) is the identity of the bicyclic semigroup and \((n, r) \to \alpha_{(n, r)} \) is a homomorphism, (c) is valid. Using (a), (b) and the fact \((n, r) \to \alpha_{(n, r)} \) is a homomorphism, we obtain (d). To establish the converse, let us define \( \alpha_{(n, r)} = \alpha^n\beta^r \) with \( \alpha^0 = \beta^0 = \rho_{e_0} \). By [3, Lemma 1.3], (b), (c), we obtain \( \alpha_{(n, r)}\alpha_{(p, q)} = (\alpha^n\beta^r)(\alpha^p\beta^q) = \alpha^{n+p-\min(r, p)}\beta^{r+q-\min(r, p)} = \alpha_{(n+p-\min(r, p), r+q-\min(r, p))} = \alpha_{(n, r)(p, q)} \) where the last juxtaposition of parenthesis denotes multiplication in \( C \).
Hence, $(n, r) \to \alpha_{(n, r)}$ is a homomorphism of $C$ into the semigroup of endomorphisms of $T$. Utilizing (a), we obtain (2) of Theorem 3.4. By (b) and the fact $\alpha^0 = \beta^0 = \rho_{e_0}$, (1) of Theorem 3.4 is valid. Utilizing (d), (b), and the fact $\alpha^0 = \beta^0 = \rho_{e_0}$, (3) of Theorem 3.4 is valid. Hence $((n, k), g_{ki}) : g_{ki} \in T_{i+k_d}, n, k \in I^0, 0 \leq i < d$ under the multiplication (d) is a simple $\omega$–$\mathcal{L}$–unipotent semigroup with $d$ $\mathcal{D}$–classes by Theorem 3.4.

Note 3.6. Let $(T, \alpha_{(r, s)})$ denote the structure given in Theorem 3.3. The following results which we use below are given in [18]. $E(T, \alpha_{(r, s)}) = ((k, k), g_{k\delta}) : k \in I^0, \delta \in Y, g_{k\delta} \in E(T_{(k, \delta)}))$. $E(T, \alpha_{(r, s)})$ is the $\omega Y$–semilattice $\Lambda$ of right zero semigroups $(E_{(k, \delta)} : k \in I^0, \delta \in Y)$ where $E_{(k, \delta)} = (((k, k), g_{k\delta}) : g_{k\delta} \in E(T_{(k, \delta)}))$. Let $((n, k), g_{n\delta}), ((r, s), h_{s\eta}) \in (T, \alpha_{(r, s)})$. Then $((n, k), g_{k\delta}) \mathcal{L}((r, s), h_{s\eta})$ iff $k = s, \delta = \eta$, and $(g_{k\delta}, h_{s\eta}) \in \mathcal{L}(eT); ((n, k), g_{k\delta}) \mathcal{R}((r, s), h_{s\eta})$ iff $n = r$ and $\delta = \eta; and ((n, k), g_{k\delta}) \mathcal{D}((r, s), h_{s\eta})$ if and only if $\delta = \eta$. We apply these results to the case where $Y$ is the chain $0 > 1 > \cdots > d - 1$ without explicit mention. If $k > 0$, let $E_{[k, i]} = \{(k, i, p) : p \in K\}$ and if $k = 0$, let $E_{[k, i]} = \{(k, i, p) : p \in P\}$. There exists an isomorphism $\varphi$ of $S$ onto $(T, \alpha_{(r, s)})$ for some $(T, \alpha_{(r, s)})$. Then $E_{[k, i]} \varphi = E_{(k, i)} = \{((k, k), g_{ki}) : g_{ki} \in E(T_{(k, i)})\}$. For fixed $k$ and $i$, let $(k, i, p) \varphi = ((k, k), \bar{p})$. Then $p \to \bar{p}$ defines an isomorphism of $K(P)$ onto $E(T_{(k, i)})$ regarded as right zero semigroups. Thus, for $k = 0, E(T_{(k, i)}) \cong P$ and, for $k \neq 0, E(T_{(k, i)}) \cong K$. Suppose $T_{(k, i)} = G \times A$ and $T_{(s, i)} = H \times B$, say $(G, H$ are groups and $A, B$ are right zero semigroups). Then $G \cong H_{e_{ki}}$ for any $e_{ki} \in E(T_{(k, i)})$ and $H \cong H_{e_{si}}$ for any
$e_{si} \in E(T_{(s,i)})$. Since $H_{((k,k),e_{bi})} \cong H_{e_{ki}}$ and $H_{((k,s),e_{ai})} \cong H_{e_{ai}}$, $G \cong H$. We write $T_{(k,i)} = G_i \times K \times (i + kd)$ for $k > 0$ and $T_{(k,i)} = G_i \times P \times (i + kd)$ for $k = 0$. Thus, $T \cong \bigcup((G_i \times P \times (i)) : 0 \leq i < d) \cup ((G_i \times K \times (i + kd)) : 0 \leq i < d, k \in N)$. We will use Note 3.6 without explicit mention.

**Lemma 3.7.** Let $T \cong \bigcup((G_i \times P \times (i)) : 0 \leq i < d) \cup ((G_i \times K \times (i + kd)) : 0 \leq i < d, k \in N)$. The multiplication on $T$ is given by

$$
(g, n, i + kd)(h, m, j + sd) = \begin{cases} 
(g(h\phi_{j+sd,i+kld}), n(h\gamma_{j+sd,i+kld}), i + kd) & \text{if } i + kd > j + sd \text{ and } k > s \\
(g(h\phi_{j+sd,i+kld}), m, i + kd) & \text{if } i + kd > j + sd \\
(g\phi_{i+kld,j+sd}, m, j + sd) & \text{if } i + kd \leq j + sd
\end{cases}
$$

where $\phi_{j+sd,i+kld}$ is a homomorphism of $G_j$ into $G_i$ for $j + sd < i + kd(s, k \in I^0, i, j \in \{0, 1, \ldots, d - 1\})$ and $\gamma_{j+sd,i+kld}$ is a homomorphism of $G_j$ into $G_K$, the full transformation group on $K$, where $j + sd < i + kd$ ($k \in N, s \in I^0, i, j \in \{0, 1, \ldots, d - 1\}$ such that if $a + id < b + jd < c + kd, \phi_{a+id,b+jd} = \phi_{a+id,c+kd}$ and $\phi_{a+id,b+jd}\gamma_{b+jd,c+kld} = \gamma_{a+id,c+kld}$ if $k > j > i$ or $k > j = i$ and $b > a$ and $\phi_{i+kld,i+kld} = \gamma_{G_i}$, the identity automorphism of $G_i$ ($0 \leq i < d, i \in I^0$). Furthermore $c > b$ and $k > i$ implies $\gamma_{a+id,c+kld} = \gamma_{a+id,b+kld}$.

**Proof.** We first assume $i + kd > j + sd$ and $k > s$. Thus, $T_{i+kld} = \{((g_i, p), i + kd) : p \in K, g_i \in G_i\}$ and $T_{j+sd} = \{((h_j, q), j + sd) : q \in K \text{ (if } s > 0) \in P \text{ (if } s = 0) \text{ and } h_j \in G_j\}$.

Using of Lemma of Clifford [4, Lemma 2.5], $((g_i, n), i + kd)((h_j, m), j + sd) =$
\[(g_l(h_j \phi_{j+sd,i+kd}), n((h_j, m), j + sd) \delta_{j+sd,i+kd}, i + kd) \] where \( \phi_{j+sd,i+kd} \) is a homomorphism of \( G_j \) into \( G_i \) and \( \delta_{j+sd,i+kd} \) is a homomorphism of \( T_{j+sd} \) into \( T_K \), the full transformation semigroup on \( K \). Let \( e_i \) denote the identity of \( G_i \). Since \((e_i, n, i + kd)(e_j, m, j + sd) = (e_i, n, i + kd), n((e_j, m, j + sd) \delta_{j+sd,i+kd}) = n \forall n \in K.\)

So, \((e_j, m, j + sd) \delta_{j+sd,i+kd} \) is the identity transformation on \( K \). So, \( T_{j+sd} \delta_{j+sd,i+kd} \) is a group. So, \( M = T_{j+sd} \delta_{j+sd,i+kd} \) is a subgroup of \( T_K \) which contains \( i_K \), the identity mapping on \( K \). If \( \alpha \in M \), there exists \( \beta \in M \) such that \( \alpha \beta = \beta \alpha = i_K \). So, \( x\alpha = y\alpha \) implies \( x = y \) and \( k \in K \) implies \( k = (k\beta)\alpha \). So, \( \alpha \in G_K \), the full transformation group on \( K \). Thus, \( \delta_{j+sd,i+kd} \) is a homomorphism of \( T_{j+sd} \) into \( G_K \). So, we may write \((h, m, j + sd) \delta_{j+sd,i+kd} = h \gamma_{j+sd,i+kd} \) where \( \gamma_{j+sd,i+kd} \) is a homomorphism of \( G_j \) into \( G_K \).

So, (a) is established. We next consider the case \( k = s \) and \( i > j \). Then,

\[(g, n, i + kd)(h, m, j + kd) = (g(h \phi_{j+kd,i+kd}), n((h, m, j + kd) \delta_{j+kd,i+kd}, i + kd)).\]

Since \((e_i, n, i + kd)(e_j, m, j + kd) = (e_i, m, i + kd), n((e_j, m, j + kd) \delta_{j+kd,i+kd}) = m \) for all \( n \in K \) (if \( k > 0 \) \( n \in P \) (if \( k = 0 \)). Recall \( \delta_{j+kd,i+kd} \) is a homomorphism of \( T_{j+kd} \) into \( T_{K(P)} \), the full transformation semigroup on \( K(P) \). Since \((e_j, m, j + kd) \mathcal{L}(h_j, m, j + kd) \) (in \( T_{j+kd} \)), \((e_j, m, j + kd) \delta_{j+kd,i+kd} \mathcal{L}(h_j, m, j + kd) \delta_{j+kd,i+kd} \). So, \( K(P)(h_j, m, j + kd) \delta_{j+kd,i+kd} = K(P)(e_j, m, j + kd) \delta_{j+kd,i+kd} = \{m\} \) by [3, Lemma 2.5, p. 52]. So, (b) is established.

Next, suppose \( i + kd \leq j + sd \). Then using [4, Lemma 2.5], \((g, n, i + kd)(h, m, j + sd) = (g \phi_{i+kd,j+sd}h, m, j + sd) \) where \( \phi_{i+kd,j+sd} \) is a homomorphism of \( G_i \) into \( G_j \).
and $\phi_{i+kd,i+kd}$ is the identity automorphism of $G_i$. Thus, (c) is established.

Suppose $a + id < b + jd < c + kd$ and $z_a \in G_a$. Then

\[
((e_c, n, c + kd)(e_b, r, b + jd)) \cdot (z_a, m, a + id) = \begin{cases} 
(z_a\phi_{a+id,c+kd}, n(z_a\gamma_{a+id,c+kd}), c + kd) & \text{if } k > j > i \\
(z_a\phi_{a+id,c+kd}, r(z_a\gamma_{a+id,c+kd}), c + kd) & \text{if } k = j > i, c > b \\
(z_a\phi_{a+id,c+kd}, n(z_a\gamma_{a+id,c+kd}), c + kd) & \text{if } k > j = i, b > a \\
(z_a\phi_{a+id,c+kd}, m, c + kd) & \text{if } k = j = i, c > b > a
\end{cases}
\]

while

\[
((e_c, n, c + kd)((e_b, r, b + jd)) \cdot (z_a, m, a + id)) = \begin{cases} 
(z_a\phi_{a+id,b+jd}\phi_{b+jd,c+kd}, n(z_a\phi_{a+id,b+jd}\gamma_{b+jd,c+kd}), c + kd) & \text{if } k > j > i \\
(z_a\phi_{a+id,b+jd}\phi_{b+jd,c+kd}, n(z_a\phi_{a+id,b+jd}\gamma_{b+jd,c+kd}), c + kd) & \text{if } k > j = i, b > a \\
(z_a\phi_{a+id,b+jd}\phi_{b+jd,c+kd}, r(z_a\gamma_{a+id,b+jd}), c + kd) & \text{if } k = j > i, c > b \\
(z_a\phi_{a+id,b+jd}\phi_{b+jd,c+kd}, m, c + kd) & \text{if } k = j = i, c > b > a
\end{cases}
\]

Comparing the above two expressions, we obtain that

\[
\phi_{a+id,b+jd}\phi_{b+jd,c+kd} = \phi_{a+id,c+kd} \text{ if } a + id < b + jd < c + kd
\]

and

\[
\phi_{a+id,b+jd}\gamma_{b+jd,c+kd} = \gamma_{a+id,c+kd} \text{ if } k > j > i \text{ or } b > a \quad \text{and} \quad k > j = i
\]

and

\[
\gamma_{a+id,c+kd} = \gamma_{a+id,b+kd} \text{ if } c > b \text{ and } k > i.
\]

**Theorem 3.8.** Let $S$ be an $\omega dPK$ regular simple semigroup. Then there exists a finite chain $\{0 > 1 > \cdots > d-1\}$ of groups $H$, a homomorphism $\theta$ of $H$ into the group
of units of $H$ ($\theta^0$ denotes the identity automorphism of $H$) and a homomorphism $\gamma$ of $H$ into $G_K$, the full transformation group on $K$ such that

$$S \cong ((I^0 \times \{0\}) \times (H \times P)) \cup ((I^0 \times N) \times (H \times K))$$

under the multiplication

$$((n, k), (g, p))((r, s), (h, q)) = \begin{cases} ((n + r - k, s), (g\theta^{r-k}h, q)) & \text{if } k \leq r \\ ((n, k + s - r), (g(h\theta^{k-r}), p(h\theta^{k-r-1}\gamma))) & \text{if } k > r \end{cases}$$

Thus $S \cong gBR(H, K, P, \theta, \gamma)$, the generalized Bruck-Reilly extension of $H$ determined by $K, P, \theta,$ and $\gamma$.

**Proof.** Let $S$ be an $\omega dPK$ regular simple semigroup. We first determine the endomorphisms $\alpha$ and $\beta$ of Corollary 3.5 for the $T$ given in Lemma 3.7. Using [3, Theorem 3.11], Corollary 3.5((a), (b), and (c)), and Lemma 3.7, we obtain

$$\begin{align*}
(1) \quad (g, n, i + kd)\alpha &= \begin{cases} (g\beta^{-1}_i, n_0, i + (k - 1)d) & \text{if } k = 0 \\
(g\beta^{-1}_{i+(k-1)d}, n\lambda^{-1}_{i+(k-1)d}, i + (k - 1)d) & \text{if } k > 0 \end{cases} \\
\end{align*}$$

where $\beta_{i+kd}$ $(0 \leq i < d, k \in I^0)$ is a collection of automorphisms of $G_i, \lambda_{i+kd}$ $(0 \leq i < d, k \in N)$ is a collection of automorphism of $K$, and $n_0 \in P$.

$$\begin{align*}
(2) \quad (g, n, i + kd)\beta &= \begin{cases} (g\beta_i, n_1, i + (k + 1)d) & \text{where } n_1 \in K \text{ if } k = 0 \\
(g\beta_{i+kd}, n\gamma_{i+kd}, i + (k + 1)d) & \text{if } k > 0 \end{cases} \\
\end{align*}$$

If $H$ is a group and $x, g \in H$, we write $gC_x = xgx^{-1}$. Utilizing the product $(e, n, i + (k + 1)d)(h, m, j + kd)$ (Lemma 3.7) and the fact $\beta$ is an endomorphism of $T$ (given by (2)), we obtain

$$\begin{align*}
(3) \quad \beta^{-1}_{j+kd}\phi_{j+kd,i+(k+1)d} = \phi_{j+(k+1)d,i+(k+2)d}\beta^{-1}_{i+(k+1)d} \\
\end{align*}$$
(4) \[ \gamma_{j+(k+1)d,i+(k+2)d} = \beta_{j+kd,i+kd,(k+1)d}^{-1} \beta_{j+k,i+kd}^{-1} \cdot \]

If \( s, k \in I^0, k > s \), and \( t_{i+s, 0, s} \), \( t_{i+(s-1)d, 0, s-1} \), \( \ldots \), \( t_{i+(k-1)d, 0, k-1} \) is a collection of endomorphisms of a semigroup \( U_i \), we write \( t_{i+s,k} = t_{i+s,0}t_{i+(s-1)d} \cdots t_{i+(k-1)d} \) and \( t_{i,0} \), the identity endomorphism of \( U_i \). Using (3) and mathematical induction, we obtain

(5) \[ (\phi_{i,d}\beta_{0}^{-1})(\phi_{0,d}\beta_{0}^{-1})^{s-k} = \phi_{i,(s-k)d}\beta_{0,0,s-k}^{-1} \quad \text{for } s > k. \]

Utilizing (1) and (5), we obtain

\[
(g, n, i + kd)\alpha^s = \begin{cases} 
(g\beta_{i-k,0,k}, n\lambda_{i-k,s-k}^{-1}, i + (k - s)d) & \text{if } k > s \text{ or } k = s > 1 \\
(g\beta_{i,0}, n_0, i + 0d) & \text{if } k = s = 1 \\
(g\beta_{i,k,0}, n_0, i + 0d) & \text{if } s > k.
\end{cases}
\]

Utilizing (2), we obtain

(7) \[ (g, n, i + kd)\beta^s = \begin{cases}
(g\beta_{i,k}, n_1\lambda_{i,k+1,k+s}, i + (k + s)d) & \text{if } k = 0 \\
(g\beta_{i,k}, n\lambda_{i,k,k+s}, i + (k + s)d) & \text{if } k > 0.
\end{cases} \]

We utilize the product of \((e_{i+1}, n, i + 1 + sd)\) and \((h, m, i + sd)\) (given by Lemma 3.7) and the fact \( \beta \) is an endomorphism of \( T \) (given by (2)) to obtain

8(a) \[ \beta_{i+sd}^{-1}\phi_{i+sd,i+1+sd} = \phi_{i+(s+1)d,i+1+(s+1)d}\beta_{i+1+sd}^{-1} \]

8(b) \[ \lambda_{i+sd} = \lambda_{i+1+sd} \quad \text{for } s > 0. \]

Let us define

\[ \theta(j,i) = \phi_{j,i+d}\beta_{i}^{-1} \]

\[ \delta_{i,j} = \phi_{i,j} \]

\[ \delta_{i,j} = \delta_{i}\delta_{i+1} \cdots \delta_{j-1} \quad (j > i) \]

\[ \delta_{i,i} = i_{G_i}, \text{ the identity automorphism on } G_i. \]
We next determine $\phi_{j+k,d,i+s,d}$ ((9), below) in terms of the above functions. For $s > k$, we use (3), Lemma 3.7, $(\phi_{a+id,b+jd,\phi_{b+jd,c+k}} = \phi_{a+id,c+k}$ for $a + id < b + jd < c + kd)$ and mathematical induction. For $s = k$, we use 8(a), Lemma 3.7 $(\phi_{a+id,b+jd,\phi_{b+jd,c+k}} = \phi_{a+id,c+k}$) and mathematical induction.

\begin{equation}
\phi_{j+k,d,i+s,d} = \begin{cases} 
\beta_{j,0,k}^{-1} \theta(j,i)\theta(i,i)\eta_{i}^{-k-1} \beta_{j,0,i} & \text{if } s > k \\
\beta_{j,0,i}^{-1} \gamma_{j,i} \beta_{j,0,i} & \text{if } s = k.
\end{cases}
\end{equation}

We next determine $\gamma_{j+k,d,i+s,d}$ ((10), below). For $s > k + 1$, use (3), (4), Lemma 3.7 ($\phi_{a+id,b+jd,\gamma_{b+jd,c+k}} = \gamma_{a+id,c+k}$) and mathematical induction. For $s = k + 1$, use (4).

\begin{equation}
\gamma_{j+k,d,i+s,d} = \begin{cases} 
\beta_{j,0,k}^{-1} \theta(j,i)\theta(i,i)\eta_{i}^{-k-2} \gamma_{i,i+1} \lambda_{i}^{-1} & \text{if } s > k + 1 \\
\beta_{j,0,i}^{-1} \gamma_{j,i} \beta_{j,0,i} \lambda_{i+1}^{-1} & \text{if } s = k + 1.
\end{cases}
\end{equation}

Using (3), Lemma 3.7 $(\phi_{a+id,b+jd,\phi_{b+jd,c+k}} = \phi_{a+id,c+k}$) and mathematical induction, we obtain

\begin{equation}
\beta_{0,0,k}^{-1} \phi_{0,j+s,d} = \phi_{kd,j+(k+s)d} \beta_{j,s,k+s}^{-1}.
\end{equation}

Next, use (6), (7), (9), (11) and Lemma 3.7 to obtain a multiplication for $S$ by means of Corollary 3.5 (d). Denote this multiplication by "o". Define $((n,k),(g,p,i+kd))z = ((n,k),(g,\beta_{i,0,k})p,i+kd))$. Then $z$ is a one-to-one mapping of $S$ onto itself. Define $((n,k),(g,p,i+kd)) * ((r,s),(h,q,j+sd)) = (((n,k),(g,p,i+kd))z^{-1} o ((r,s),(h,q,j+sd))z^{-1}))z$. Then $(S,*) \cong (S, o) \cong S$ (as given) and it is easily seen
that

\[
((n, k), (g, p, i + kd)) = \begin{cases}
(n, k + s - r), (g(h\theta(j, i)\theta(i, i)k^{r-1}), p\lambda^{-1}_{i, k-r+s}, i + (k - r + s)d) & \text{if } k > r > s \\
(n, k + s - r), (g(h\theta(j, i)\theta(i, i)k^{r-1}), p\lambda_{i, k-r+s}, i + (k - r + s)d) & \text{if } k > r \text{ and } s \geq r \\
(n + r - k, s), (g\theta(i, j)\theta(j, j)r^{-k-1}h, q, j + sd) & \text{if } r > k \\
(n, s), (g(h\delta_{j, i}), q, i + (k - r + s)d) & \text{if } r = k \text{ and } i > j \\
(n, s), ((g\delta_{i, j})h, q, j + sd) & \text{if } r = k \text{ and } i \leq j
\end{cases}
\]

(12)

Let \(((n, k), (g, p, i + kd))\psi = \begin{cases}
((n, k), (g, p\lambda^{-1}_{i, k}, i + kd)) & \text{if } k > 0 \\
((n, k), (g, p, i + kd)) & \text{if } k = 0
\end{cases}.

Define \(((n, k), (g, p, i + kd)) \cdot ((r, s), (h, q, j + sd)) = (((n, k), (g, p, i + kd))\psi^{-1} \cdot (r, s), (h, q, j + sd))\psi^{-1}\psi. Then using (12), (10), and 8(b), we obtain

\[
((n, k), (g, p, i + kd)). ((r, s), (h, q, j + sd)) = \begin{cases}
(n, k + s - r), (g(h\theta(j, i)\theta(i, i)k^{r-1}), p(h\theta(j, i)\theta(i, i)k^{r-2}\gamma_{i, i+d}, i + (k - r + s)d) & \text{if } k > r + 1 \\
(n, k + s - r), (g(h\theta(j, i)), p(h\gamma_{j, i+d}, i + (k - r + s)d)) & \text{if } k = r + 1 \\
(n + r - k, s), (g\theta(i, j)\theta(j, j)r^{-k-1}h, q, j + sd) & \text{if } r > k \\
(n, s), (g(h\delta_{j, i}), q, i + (k - r + s)d) & \text{if } r = k \text{ and } i > j \\
(n, s), ((g\delta_{i, j})h, q, j + sd) & \text{if } r = k \text{ and } i \leq j
\end{cases}
\]

(13)

Using (3) and mathematical induction, we obtain

\[
\theta(j, i)\theta(i, i)^t = \phi_{j, i+(i+1)d}\beta^{-1}_{i, 0, i+1}.
\]

(14)

We define

\[
\varphi_{j+rd, i+kd} = \phi_{j, i+(k-r)d}\beta^{-1}_{i, 0, k-r}
\]

(15)
(16) \[ \gamma_i = \gamma_{i,d}. \]

Using (14) and (15), we obtain

(17) \[ h \varphi_{j+r+d,i+k} = h \theta(j,i) \theta(i,i)^{k-r-1} \text{ for } k \geq r + 1. \]

Using (16) and Lemma 3.7 (\( \gamma_{a+i,c+k} = \gamma_{a+i,b+k} \) if \( c > b \) and \( k > i \)), we obtain

(18) \[ \gamma_j = \gamma_{j,i+d}. \]

Using (14), (15), and (18), we obtain

(19) \[ h \varphi_{j+(r+1)d,i+k} \gamma_i = h \theta(j,i) \theta(i,i)^{k-r-2} \gamma_{i,i+d} \text{ for } k > r + 1. \]

Using (15), Lemma 3.7 (\( \phi_{a+i,b+jd} \phi_{b+jd,c+kd} = \phi_{a+i,c+kd} \), if \( a+i < b+jd < c+kd \)) and the definition of \( \delta_{i,j} \) we obtain

(20) \[ \delta_{i,j} = \varphi_{i+kd,j+rd} \text{ if } r = k \text{ and } j \geq i. \]

Substitute (17), (18), (19), and (20) into (13) and call the resulting product \( \odot \). It is easily checked that \((n,k),(g_i,p,i+kd))\delta = ((n,k),(g_i,p))\) where \( g_i \in G_i \) defines a one-to-one mapping of \( S \) onto \( X = \bigcup_{i=0}^{d-1} (((I^0 \times \{0\}) \times (G_i \times P)) \cup ((I^0 \times N) \times (G_i \times K))). \) Define a product on \( X \) by \((n,k),(g_i,p)((r,s),(h_j,q)) = (((n,k),(g_i,p))\delta^{-1} \odot ((r,s),(h_j,q))\delta^{-1})\). Thus,

\[
((n,k),(g_i,p)(r,s),(h,q) =
\begin{cases} 
((n,k+s-r),(g_i(h_j \varphi_{j+r+d,i+k}),p(h_j \varphi_{j+(r+1)d,i+k} \gamma_i))) & \text{if } k > r + 1 \\
((n,k+s-r),(g_i(h_j \varphi_{j+r+d,i+k}),p(h_j \gamma_j))) & \text{if } k = r + 1 \\
((n+r-k,s),(g_i \varphi_{i+kd,j+rd}h_j,q)) & \text{if } r > k \\
((n,s),(g_i(h_j \varphi_{j+r+d,i+k}),q)) & \text{if } r = k \text{ and } i > j \\
((n,s),(g_i \varphi_{i+kd,j+rd}h_j,q)) & \text{if } r = k \text{ and } i \leq j
\end{cases}
\]

(21)
Clearly, \((S, \circ) \cong (X, \text{juxtaposition})\).

Using (3), we obtain
\[
\beta_{i,0}^{-1} \phi_{i,j+rd} = \phi_{i+sd,j+(s+r)d} \beta_{i,r}^{-1}.
\]

Using (15), (22), and Lemma 3.7 \((\phi_{a+id,b+jd} \phi_{b+jd,c+kd} = \phi_{a+id,c+kd}\) if \(a+id < b+jd < c+kd\)), we obtain
\[
\phi_{i+sd,j+rd} \phi_{j+rd,k+ud} = \phi_{i+sd,k+ud}\text{ if } i+sd < j+rd < k+ud.
\]

Define
\[
\varphi_m = \varphi_{m,m+1}.
\]

Using (23) and (24), we obtain
\[
\varphi_{m,n} = \varphi_m \varphi_{m+1} \cdots \varphi_{n-1} \text{ for } m < n.
\]

Using (15) and (24), we obtain
\[
\text{If } m \in I^0, \varphi_m = \varphi_s \text{ where } m = s + kd, 0 \leq s < d \text{ and } k \in I^b.
\]

\[H = G_0 \cup G_1 \cup \cdots \cup G_{d-1}\] under the multiplication \(xy = x \varphi_{i,\max(i,j)} y \varphi_{j,\max(i,j)}\).

Using (25) and [3, Theorem 4.11], \(H\) is a finite chain \((0 > 1 > 2 > \cdots > d-1)\) of the groups \(G_i\). It is easily seen that \(G_0\) is the group of units of \(H_0\). If \(x \in G_j\), define \(x\theta = x \varphi_{j,\varphi_{j+1}} \cdots \varphi_{d-1}\). By an easy calculation \(\theta\) is a homomorphism of \(H\) into \(G_0\), the group of units of \(H\).

Suppose \(k > r\). By (25) and the definition of the multiplication on \(H\),
\[g_i(h_j \theta^{k-r}) = g_i(h_j \theta^{k-r} \varphi_0 \varphi_1 \cdots \varphi_{i-1}).\]

Using (26), (25), and (15),
\[h_j \theta^{k-r} \varphi_0 \varphi_1 \cdots \varphi_{i-1} = h_j \varphi_j \varphi_{j+1}.\]
\[ g_i(h_j \theta^{k-r}) = g_i(h_j \varphi_{j+r}d_{i+k}) \text{ if } k > r \]

where the left multiplication is in \( H \) and the right multiplication is in \( T \).

Let us define \( x \gamma = x \gamma_j \) if \( x \in G_j \). We will show \( \gamma \) is a homomorphism of \( H \) into \( G_K \), the full transformation group on \( K \). Suppose \( x \in G_i \) and \( y \in G_j \) with \( i > j \).

Using the multiplication on \( H \), (15), (16), and Lemma 3.7 \( \varphi_{a+i}d_{b+j} \gamma_{b+j}d_{c+k} = \gamma_{a+i}d_{c+k} \) if \( k > j = i \) and \( b > a \), \( (xy)\gamma = (xy) \gamma_j = x \gamma_i(y \gamma_i) \gamma_i = (x \gamma_i)(y \gamma_i) \gamma_i, \gamma_i = x \gamma_i y \gamma_j = x \gamma j y \gamma \). The case \( j > i \) is similar, and the case \( i = j \) is obvious.

Using (25), (26), (16), Lemma 3.7 \( \varphi_{a+i}d_{b+j} \gamma_{b+j}d_{c+k} = \gamma_{a+i}d_{c+k} \) if \( k > j = i \) and \( b > a \), (15) and (23), \( h_j \theta^{k-r-1} \gamma = h_j \varphi_j \cdots \varphi_{d-1} \varphi_{d+1} \cdots \varphi_{(k-r-1)}d \cdots \varphi_{(d-1)+(k-r-2)}d \gamma_0 = h_j \varphi_{j,(k-r-1)}d \gamma_0d = h_j \varphi_{j,(k-r-1)}d \varphi_{0,i} \gamma_i, d = h_j \varphi_{j,(k-r-1)}d \varphi_{i} \gamma_i = h_j \varphi_{j,(k-r-1)}d \varphi_{i} \gamma_{i+(k-r-1)}d \gamma_i = \varphi_{j,i+(k-r-1)}d \gamma_i = \varphi_{j,i+(r+1)}d_{i+k} \gamma_i. \) Thus,

\[ h_j \theta^{k-r-1} \gamma = h_j \varphi_{j+(r+1)}d_{i+k} \gamma_i \text{ if } k > r + 1. \]

\( \theta^0 \) will denote the identity automorphism of \( H \). Thus,

\[ h_j \theta^{k-r-1} \gamma = h_j \gamma_i \text{ if } k = r + 1. \]

Using a proof analogous to the proof of (27), we obtain

\[ g_i \theta^{r-k} h_j = (g_i \varphi_{i+k}d_{j+r}) h_j \text{ if } r > k. \]
Using (15) and the definition of product in $H$,

\[(31) \quad g_i \vartheta^{r-k} h_j = g_i(h_j \varphi_{j+r, i+k}) \text{ if } r = k, i > j.\]

Similarly,

\[(32) \quad g_i \vartheta^{r-k} h_j = (g_i \varphi_{i+k, j+r}) h_j \text{ if } r = k, i \leq j\]

(note in (30) – (33), the left multiplication is in $H$ and the right multiplication is in $T$).

To complete the proof of Theorem 3.8, utilize the remarks above the proof of (27) ($H$ is a finite chain $(0 > 1 > \cdots > d - 1)$ of groups and $\theta$ is a homomorphism of $H$ into the group of units of $H$), the remark above the proof of (28) ($\gamma$ is a homomorphism of $H$ into $G_K$), and substitute (27), (28), (29), (30), (31), and (32) into (21).

Combining Theorems 2.3 and 3.8, we obtain

**Theorem 3.9.** A semigroup $S$ is isomorphic to a generalized Bruck-Reilly extension of a finite chain $(0 > 1 > \cdots > d - 1)$ ($d$ is a positive integer) of groups determined by $P$ and $K$ if and only if $S$ is an $\omega dPK$ regular simple semigroup.

**Corollary 3.10.** A semigroup $S$ is isomorphic to a Bruck-Reilly extension of a finite chain of groups if and only if $S$ is a simple $\omega$-inverse semigroup.

**Proof.** Let $S$ be a simple $\omega$-inverse semigroup. The $S$ has a finite number, $d$, of $D$-classes by Theorem 3.2. We may write $E(S)$ as the chain $(0, 0, p) > (0, 1, p) >$
\[ \cdots > (0, d - 1, p) > (1, 0, q) > \cdots > (1, d - 1, q) > (2, 0, q) > \cdots > \cdots. \] So \( S \) is a regular simple \( \omega dPK \) semigroup with \( p = \{ p \} \) and \( K = \{ q \} \). So, \( S \) is the required Bruck–Reilly extension by Theorem 3.9. The converse is a consequence of Corollary 2.4.

A bisimple semigroup \( S \) is termed \( E \)-bisimple if \( E(S) = \cup \{ E_k : k \in I^0 \} \), \( E_i \cap E_j = \emptyset \) if \( i \neq j \), each \( E_i \) is a right zero semigroup, and \( e \in E_i, f \in E_j \), and \( i > j \) imply \( e < f \) [16].

**Lemma 3.11.** \( S \) is an \( E \)-bisimple semigroup if and only if \( S \) is a regular simple \( \omega 1PK \) semigroup.

**Proof.** Suppose \( S \) is an \( E \)-bisimple semigroup. Then in the notation of Corollary 3.5, \( E_k = E(T_k) \). We note that \( \beta \) is a homomorphism of \( T_k \) into \( T_{k+1} \) and \( \alpha \) is a homomorphism of \( T_{k+1} \) into \( T_k \). Suppose \( k > 0 \). Then using Corollary 3.5(b), we have \( x\beta\alpha = xe_0 = x \) for \( x \in T_k \) and \( y\alpha\beta = ye_1 = y \) for \( y \in T_{k+1} \). So, \( \beta \) is an isomorphism of \( T_k \) onto \( T_{k+1} \). Hence, \( T_k \cong T_s \) if \( k, s > 0 \). So, \( E_k \cong E_s \) for \( k, s > 0 \). Hence, we may write \( E_0 = \{0\} \times \{0\} \times P \) and \( E_s = \{s\} \times 0 \times K \) for \( s > 0 \) where \( P \) and \( K \) are disjoint sets and \( (k, o, p)(r, o, q) = (k, o, p) \) if \( k > r \); \( (k, o, p)(r, o, q) = (r, o, q) \) if \( k < r \); and \( (k, o, p)(r, o, q) = (r, o, q) \) if \( k = r \). Hence, \( S \) is an \( \omega 1PK \) regular simple semigroup. Conversely, let \( S \) be a regular simple \( \omega 1PK \) semigroup. Let \( E_0 = \{(o, o, p) : p \in P \} \) and \( E_k = \{(k, o, q) : q \in K \} \) for \( k > 0 \). Then \( E(S) = U(\{E_j : j \in I^0 \}) \), \( E_i \cap E_j = \emptyset \) if \( i \neq j \), each \( E_i \) is a right zero semigroup, and \( e \in E_i, f \in E_j \),
and \( i > j \) imply \( e < f \). Thus \( S \) is an \( E \)-bisimple semigroup.

**Theorem 3.12.** A semigroup \( S \) is isomorphic to a generalized Bruck-Reilly extension of a group if and only if \( S \) is an \( E \)-bisimple semigroup.

**Proof.** Combine Theorem 3.9 and Lemma 3.11.

**Corollary 3.13.** A semigroup \( S \) is isomorphic to a Bruck-Reilly extension of a group if and only if \( S \) is a bisimple \( \omega \)-inverse semigroup.

**Proof.** Let \( S \) be a bisimple \( \omega \)-inverse semigroup. So, \( S \) is an \( E \)-bisimple semigroup. Hence, by Theorem 3.12, \( S \cong gBR(G, P, K, \theta, \gamma) \), say, where \( G \) is a group. Using Lemma 2.2(d), \( |P| = |K| = 1 \). So, \( S \cong BR(G, \theta) \). Conversely, if \( S = BR(G, \theta) \) where \( G \) is a group, \( S \cong gBR(G, P, K, \theta, \gamma) \) with \( |P| = |K| = 1 \). So, by Theorem 3.12 and Lemma 2.2(d) and (e), \( S \) is a bisimple \( \omega \)-inverse semigroup.

4. **Group Congruences of** \( gBR(T, P, K, \theta, \gamma) \).

In Theorem 4.6, we determine the group congruence of \( S = gBR(T, P, K, \theta, \nu) \) by means of admissible triples. In Theorem 4.11, we give the structure of \( S/\beta \) where \( \beta \) is the minimum group congruence on \( S \) and show that \( x\beta y \ (x, y \in S) \) if and only if \( xc = yc \) for some \( c \in E(S) \). In Theorems 4.12 and 4.13 respectively, we describe \( \vee \) and \( \wedge \) in \([\beta, S \times S]\), the lattice of group congruences of \( S \).

Let \( S = gBR(T, P, K, \theta, \gamma) \). Define \( ((n, k), (g, p)) \rho((r, s), (h, q)) \) if and only if \( n = r, k = s, \) and \( g = h \). Then \( \rho \) is a congruence relationship on \( S \) (Lemma 4.1)
and \( \rho \subseteq \sigma \) for any group congruence \( \sigma \) on \( S \) (Lemma 4.2), using these lemmas, \( \sigma \rightarrow \sigma/\rho = \{(x \rho, y \rho) \in S/\rho \times S/\rho : (x, y) \in \sigma \} \) defines an order-preserving bijection of the group congruences of \( S \) onto the group congruences \( S/\rho = BR(T, \theta) \) (Theorem 4.3). Using Theorem 4.3, Ault's determination of the group congruences of \( BR(U, \theta) \) by means of admissible triples (Theorem 4.4) and Piochi's correspondence between the group congruences of \( BR(T, \theta) \) and \( BR(U, \theta) \) (Theorem 4.5), we establishes Theorem 4.6.

**Lemma 4.1.** \( \rho \) is a congruence relation on \( S \) and \( S/\rho \cong BR(T, \theta) \).

**Proof.** By routine calculations, \( \rho \) is a congruence relation on \( S \) and \( (((n, k), (g, p)) \rho) \delta = ((n, k), (g)) \) defines an isomorphism of \( S \) onto \( BR(T, \theta) \).

**Lemma 4.2.** Let \( \sigma \) be a group congruence on \( S \). Then, \( \rho \subseteq \sigma \).

**Proof.** Suppose \( (((n, k), (g, p)) \rho)((r, s), (h, q)). \) Thus, \( n = r, k = s, \) and \( g = h. \) Let \( 1 \) denote the identity of \( T. \) Then, by Lemma 2(d), \( ((k, k), (1, p)), ((k, k), (1, q)) \in E(S). \) Thus, \( (((k, k), (1, p)) \sigma((k, k), (1, q)). \) Hence \( (((n, k), (g, p)) = (((n, k), (g, p))((k, k), (1, p)) \sigma((n, k), (g, p))((k, k), (1, q)) = ((n, k), (g, q)) = ((r, s), (h, q)). \)

**Theorem 4.3.** Let \( S = gBR(G, P, K, \theta, \gamma) \). If \( \sigma \) is a group congruence on \( S \), define

\[
\sigma/\rho = \{(x \rho, y \rho) \in S/\rho \times S/\rho : (x, y) \in \sigma \}.
\]

Then \( \sigma \rightarrow \sigma/\rho \) defines an order-preserving bijection of the group congruences of \( S \).
onto the group congruences of \( S/\rho = BR(T, \theta) \).

**Proof.** Let \( \sigma \) be a group congruence on \( S \). Since \( \sigma \subseteq \rho \) by Lemma 4.2, \( \sigma/\rho \) is a congruence relation on \( S/\rho \) by [2, Lemma 6.14]. Since \( (S/\rho)/(\sigma/\rho) \cong S/\sigma \) by [2, Theorem 6.15], \( \sigma/\rho \) is a group congruences on \( S/\rho \). Let \( \sigma_1, \sigma_2 \) be group congruences on \( S \). It is easily checked that \( \sigma_1 \leq \sigma_2 \) if and only if \( \sigma_1/\rho \leq \sigma_2/\rho \). Let \( \delta \) be any group congruence relation on \( S/\rho \). Define \( \sigma = \{ (a, b) \in S \times S : (a\rho, b\rho) \in \delta \} \). It is easy to show that \( \sigma \) is a congruence relation on \( S \) and \( \delta = \sigma/\rho \). Since \( S/\rho/\delta = (S/\rho)/(\sigma/\rho) \cong S/\sigma \), \( \sigma \) is a group congruence on \( S \). So, \( \sigma \rightarrow \sigma/\rho \) is a surjection.

The group congruences on \( BR(U, \theta) \) where \( U \) is a group were given by Ault [1]. We next give Petrich’s formulation [9, III, 5.19 Theorem] of Ault’s theorem. We will need the following definition.

Let \( S = BR(U, \theta)(U, \text{ a group}) \). Let \( H \) be a normal subgroup of \( U \), \( x \) be an element of \( U \), \( k \) be a non-negative integer, and assume the following conditions are satisfied

(i) \( g \in H \) if and only if \( g\theta \in H \)

(ii) \( (x\theta)H = xH \)

(iii) \( (g^{\theta^k}H = (x^{-1}gx)H \) for all \( g \in U \). \)

\( (H; x, k) \) is called an admissible triple. Define a relation \( \sigma = \sigma(H; x, k) \) on \( S \) by

(iv) \( ((m, n), g)\sigma((p, q), h) \) if and only if \( (n - m) - (q - p) = ak \) and \( (g^{\theta^m})(h^{-1}\theta^n) \in x^aH \) for some integer \( a \).
Theorem 4.4. (Petrich [9]; see also Ault [1]). Let $S = BR(U, \theta)$, where $U$ is a group. If $(H; x, k)$ is an admissible triple, then $\sigma(H; x, k)$ is a group congruence on $S$. Conversely, every group congruence on $S$ can be written in the form $\sigma(H; x, k)$ for some admissible triple $(H; x, k)$.

The group congruences on $BR(T, \theta)$ have been determined by Piochi [11]. We will need the following results.

Theorem 4.5 (Piochi, [11, Theorem 2.1]). Let $S = BR(T, \theta)$ and let $U$ be the group of units of $T$. Let $\Gamma$ be a group congruence on $S_0 = BR(U, \theta)$. Define the following relation on $S$: $((m, n), x)\Gamma^* ((p, q), y)$ if and only if $(m+1, x\alpha, n+1)\Gamma^* (p+1, y\alpha, q+1)$. The relation $\Gamma^*$ is the unique group congruence on $S$ whose restriction to $S_0$ is equal to $\Gamma$. Conversely, every group congruence on $S$ can be so constructed from its restriction to $S_0$.

In order to state Theorem 4.6, we will need the following definition. Let $S = gBR(T, P, K, \theta, \gamma)$ with $U$ the group of units of $T$. Let $H$ be a normal subgroup of $U$, let $x$ be an element of $U$, let $k$ be a non-negative integer and assume the following conditions are satisfied

(i) $g \in H$ if and only if $g\theta \in H$ ($g \in U$)

(ii) $(x\theta)H = xH$

(iii) $(g\theta^k)H = (x^{-1}gx)H$. 32
We call $[H; x, k]$ an admissible triple. Let $[H, x, k]$ be an admissible triple. Define a relation $\sigma = \sigma[H; x, k]$ on $S$ by $((m, n), (g, s))\sigma((p, q), (h, t))$ if and only if $(n - m) - (q - p) = ak$ and $g^{\theta_{n+2}(h^{\theta_{n+2}})}^{-1} \in x^a H$ for some integer $a$.

**Theorem 4.6.** Let $S = gBR(T, P, K, \theta, \gamma)$. If $[H; x, k]$ is an admissible triple, then $\sigma[H; x, k]$ is a group congruence on $S$. Conversely, every group congruence on $S$ can be written in the form $\sigma[H; x, k]$ for some admissible triple $[H; x, k]$.

**Proof.** Let $[H; x, k]$ be an admissible triple. Then $\Gamma = \sigma(H; x, k)$ is a group congruence relation on $BR(U, \theta)$ by Theorem 4.4. So $\Gamma^* = \{((m, n), g), ((p, q), h)) \in BR(T, \theta) \times BR((T, \theta) : ((m + 1, n + 1), g\theta)\Gamma((p + 1, q + 1), h\theta))\}$ is a group congruence on $BR(T, \theta)$ by Theorem 4.5. By Theorem 4.3, there exists a unique group congruence $\delta$ on $S$ such that $\delta/\rho = \Gamma^*$. Thus, $((m, n), (g, s))\delta((p, q), (h, t))$ iff $((m, n), g)\Gamma^*((p, q), h) \iff ((m + 1, n + 1), g\theta)\sigma(H; x, k)((p + 1, q + 1), h\theta) \iff (n - m) - (q - p) = ak$ and $g^{\theta_{n+1}(h^{\theta_{n+2}})}^{-1} \in x^a H$ for some integer $a$ iff $(n - m) - (q - p) = ak$ and $g^{\theta_{n+2}(h^{\theta_{n+2}})}^{-1} \in x^a H$. So, $\delta = \sigma[H; x, k]$ and, hence, $\sigma[H; x, k]$ is a group congruence on $S$.

Conversely, let $\delta$ be a group congruence on $S$. Thus, by Theorem 4.3, $\Gamma^* = \delta/\rho$ is a group congruence on $BR(T, \theta)$. Let $\Gamma$ be the restriction of $\Gamma^*$ to $BR(U, \theta)$. So, $\Gamma$ is a group congruence on $BR(U, \theta)$ Hence, $\Gamma = \sigma(H; x, k)$ for some admissible triple $(H; x, k)$ by Theorem 4.4. Thus, using the definition of $\delta/\rho$ and Theorem 4.5, $((m, n), (g, s))\delta((p, q), (h, t))$ iff $((m, n), g)\Gamma^*((p, q), h) \iff ((m + 1, n + 1), g\theta)\sigma(H; x, k)((p + 1, q + 1), h\theta) \iff (m - n) - (q - p) = ak$ and $g^{\theta_{n+2}(h^{\theta_{n+2}})}^{-1} \in x^a H$
for some integer $a$. So, $\delta = \sigma[H; x, k]$.

We next determine the minimum group congruence $\beta$ on $S = gBR(T, P, K, \theta, \gamma)$ and give the structure of $S/\beta$ (Theorem 4.11). Let $S_0 = BR(U, \theta)$ and let $\sigma$ denote the minimum group congruence on $S_0$. We first give the structure of the maximal group homomorphic image $V$ of $S_0$ (Theorem 4.7). So, $S_0/\sigma \cong V$. Let $S_1 = BR(T, \theta)$. We next show $\sigma^*$ (notation of Theorem 4.5) is the minimum group congruence on $S_1, S_1/\sigma^* \cong V$, and determine $\sigma^*$ (Theorem 4.8) (note Remark 4.9). We then show that $S$ has a minimum group congruence $\beta$ and that $S/\beta \cong S_1/\sigma^*$ (Lemma 4.10). Finally, we use Theorem 4.8, Lemma 4.10 and its proof, and Lemma 2(d) to prove Theorem 4.11.

**Theorem 4.7.** (Warne, [15, Theorem 3.4]). Let $S_0 = BR(U, \theta)$ where $U$ is a group and let $1$ denote the identity of $U$. If $N = \{g \in U | g^a = 1 \text{ for some } n \in \Gamma_0\}$, $N$ is a normal subgroup of $U$. If $(xN)\alpha = (x\theta)N$, $\alpha$ is an endomorphism of $U/N$. Let $g \rightarrow \bar{g}$ denote the natural homomorphism of $U$ onto $U/N$. Let us define a relation $\sigma$ on $(\Gamma_0)^2 \times U/N$ by the rule

$\delta = (((a, b), \bar{g}), ((c, d), \bar{h})) \in \delta$

if and only if there exists $x, y \in \Gamma_0$ such that $x + a = y + c, x + b = y + d$, and $\bar{g}x^\alpha = \bar{h}x^\alpha$. Then $\delta$ is an equivalence on $(\Gamma_0)^2 \times U/N$. Furthermore the rule

$((a, b), \bar{g})_\delta ((c, d), \bar{h})_\delta = (\bar{g}^{a+b} \bar{h}^b, a + c, b + d)_\delta$

defines a binary operation on $(\Gamma_0)^2 \times U/N/\delta = V$ whereby $V$ becomes a group which
is the maximal group homomorphic image of $S_0$. The canonical homomorphism of $S_0$ onto $V$ is given by $((a, b), g)\gamma = ((a, b), \overline{g})\delta$.

**Theorem 4.8.** Let $S_1 = BR(T, \theta)$. Then $S_1/\sigma^* \cong V$ where $\sigma^*$ denotes the minimum group congruence on $S_1$. Furthermore, if $x, y \in S_1, x\sigma^*y$ if and only if $xe = ye$ (or equivalently $ex = ey$) for some $e \in E(S_1)$.

**Proof.** Using Theorem 4.5, $\sigma^*$ (notation of theorem 4.5) is the minimum group congruence on $S_1$. By [11, Lemma 3.1], $S_1/\sigma^* \cong S_0/\sigma$. So, by Theorem 4.7, $S_1/\sigma^* \cong V$. Suppose $((m, n), g)\sigma^*((p, q), h)$. Thus, $((m, n), \overline{g})\delta = ((p, q), \overline{h})\delta$. Hence, there exists $x, y \in I^0$ such that $x + m = y + p, x + n = y + q$, and $\overline{g}\alpha = \overline{h}\alpha'$. So, there exists $k \in I^0$ such that $g\theta^{v+k} = h\theta^{u+k}$. Let $v = x + n + k = y + q + k$. So, $v \geq n, q$. Then $((m, n), g)((v, v), 1) = ((m + v - n, v), g\theta^{v+k}) = ((p + v - q, v), h\theta^{u+k}) = ((p, q), h)((v, v), 1)$. Let $e = ((v, v), 1)$. If $xe = ye, x\sigma^*y$ since $\sigma^*$ is a group congruence.

**Remark 4.9.** Theorem 4.8 has been proved with $V$ replaced by the construction of $S_0/\sigma^*$ given by W.D. Munn and N.R. Reilly [7, Theorem 3.4] by Piochi [11, Theorem 3.4]. Piochi uses the Munn–Reilly construction in his proof.

**Lemma 4.10.** Let $S = gBR(T, P, K, \theta, \gamma)$ and $S_1 = BR(T, \theta)$. Then $S$ has a minimum group congruence $\beta$. Furthermore, $((m, n), (g, p))\beta((r, s), (h, q))$ if and only if $((m, n), g)\sigma^*((r, s), h)$. So, $S/\beta \cong S_1/\sigma^*$.

**Proof.** We first show $(\rho \circ \sigma^*) \circ (\rho \circ \sigma^*)^{-1}$ is the minimum group congruence of
Clearly, \((\rho \circ \sigma^*) \circ (\rho \circ \sigma^*)^{-1}\) is a group congruence on \(S\). Suppose \((a, b) \in (\rho \circ \sigma^*) \circ (\rho \circ \sigma^*)^{-1}\). Then \((a \rho) \sigma^* = (b \rho) \sigma^*\). Hence, using Theorem 4.8, there exists \(e_1 \in E(S_1)\) such that \((a \rho)e_1 = (b \rho)e_1\). Using Lemma 2.2(d), \(e_1 = ((k, k), f)\) for some \(k \in I^0\) and \(f \in E(T)\). By Lemma 2.2(d), \(e = ((k, k), (f, t))(t \in P\) (if \(k = 0\), \(t \in K\) (if \(k \neq 0\)) \(\in E(S)\). So, \(e_1 = e \rho\). Hence, \((ae) \rho = (be) \rho\). So, \(ae = ((m, n), (g, p))\) and \(be = ((m, n), (g, q))\), say. Let \(g = ((n, n), (1, p))\) (1 is the identity of \(T\)). Thus \(aeg = beg\). Let \(\lambda\) be any group congruence relation on \(S\). Then \(a \lambda = a \lambda e \lambda f \lambda = b \lambda e \lambda f \lambda = b \lambda\). So, \((a, b) \in \lambda\) and \(\lambda \leq (\rho \circ \sigma^*) \circ (\rho \circ \sigma^*)^{-1}\). Thus, \((\rho \circ \sigma^*) \circ (\rho \circ \sigma^*)^{-1}\) is the minimum group congruence on \(S\) which we denote by \(\beta\). Hence \(((m, n), (g, p)) \beta (r, s), (h, q))\) if and only if \(((m, n), g) \sigma^*((r, s), h)\). So, \((a \beta) \gamma = (a \rho) \sigma^*\) defines an isomorphism of \(S/\beta\) onto \(S_1/\sigma^*\).

**Theorem 4.11.** Let \(S = gBR(T, P, K, \theta, \gamma)\). Then, \(S/\beta \cong V\) (notation of Theorem 4.7) where \(\beta\) denotes the minimum group congruence on \(S\). Furthermore, if \(x, y \in S, x \beta y\) if and only if \(xc = yc\) for some \(c \in E(S)\).

**Proof.** By Theorem 4.8 and Lemma 4.10, \(S/\beta \cong V\). Suppose \((a, b) \in \beta\). Using the proof of Lemma 4.10, \(aeg = beg\) where \(e = ((k, k), (f, t))\) and \(g = ((n, n), (1, p))\).

By a routine calculation \(ef \in E(S)\) (using Lemma 2.2(d)). Let \(ef = c\). If \(cx = cy\) with \(c \in E(S)\); \(x \beta y\) since \(\beta\) is a group congruence.

Petrich [10] has determined \(\sigma(H; x, k) \lor \sigma(K; y, \ell)(\sigma(H; x, k) \land \sigma(K; y, \ell))\) in \([\sigma, S_0 \times S_0]\). His determination will also give \(\sigma[H; x, k] \lor \sigma(K; y, \ell)(\sigma[H; x, k] \land\)
\( \sigma(K; y, \ell) \) in \([\beta, S \times S]\).

We need the following notation. A normal subgroup \( N \) of \( U \) is said to be \( \theta \)-admissible if \( N \theta \subseteq N \). \( \mathcal{T} \) will denote the set of all \( \theta \)-admissible subgroups of \( U \). If \( A \) is any nonempty subset of \( U \), \( \overline{A} \) denotes the intersection of all \( N \in \mathcal{T} \) containing \( A \). If \( x, y \in U, [x, y] = xyx^{-1}y^{-1} \) and \( \langle x \rangle \) is the cyclic group generated by \( x \). For brevity, we denote \( \sigma(H; x, k) \) by \([H; x, k]\) in Theorems 4.12 and 4.13 (below).

**Theorem 4.12.** Let \( S = gBR(T, P, K, \theta, \gamma) \) and \( \beta \) denote the minimum group congruence on \( S \). Let \([H; x, k], [K; y, \ell] \in [\beta, S \times S]\). Then,

\[
[H; x, k][K; y, \ell] = \begin{cases} 
\overline{P; x^{y^u}t} & \text{if } k \ell \neq 0, t = \text{gcd}(k, \ell) = ku + \ell v \\
\overline{P} &= \langle x, y \rangle < x^{t/(y^{k/t}H K)} > H K \\
\overline{(x)HK; y, \ell} & \text{if } k = 0, \ell \neq 0 \\
\overline{(x)H(y)K; 1, 0} & \text{if } k = \ell = 0
\end{cases}
\]

**Proof.** Using Theorem 4.5, \((L)T = L^*\) defines a lattice isomorphism of \([\sigma, S_0 \times S_0]\) onto \([\sigma^*, S_1 \times S_1]\). Using Theorem 4.3, \((\delta)V = \delta/\rho\) defines a lattice isomorphism of \([\beta, S \times S]\) onto \([\sigma^*, S_1 \times S_1]\). So \(T \circ V^{-1} = B\) defines a lattice isomorphism of \([\sigma, S_0 \times S_0]\) onto \([\beta, S \times S]\). Using the proof of Theorem 4.6, \((\sigma(H; x, k))B = \sigma[H; x, k]\). So, \(\sigma[H; x, k] \vee \sigma[K; y, \ell] = (\sigma(H; x, k) \vee \sigma(K; y, \ell))B\). Apply [10, Theorem 3.3].

**Theorem 4.13.** Let \( S, \beta, [H; x, k], [K; y, \ell] \) be as in the statement of Theorem 4.12.
Then
\[
[H \cap K; z, ts] \text{ if } k \ell \neq 0, t = \ell \text{cm}\{k, \ell\},
\]
\[
x \in x_1 H \cap y_2 K, \ s \text{ is the least such positive integer.}
\]
\[
[H \cap K; 1, 0] \text{ if } k \ell \neq 0, t = \ell \text{cm}\{k, \ell\},
\]
\[
x_1 H \cap y_2 K = \phi \text{ for all positive integers } r
\]
\[
[(x)H \cap K; 1, 0] \text{ if } k = 0, \ell \neq 0
\]
\[
[(x)H \cap (y)K; 1, 0] \text{ if } k = \ell = 0.
\]

Proof. By the proof of Theorem 4.12, \(\sigma[H; x, k] \land \sigma[K; y, \ell] = (\sigma(H; x, k) \land \sigma(K; y, \ell))B.\) Apply [10, Theorem 3.4].

References


