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DEGREE OF POINTEDNESS OF A CONVEX FUNCTION

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Abstract

A convex function f is said to be pointed if its epigraph has a recession cone which is pointed. Partial pointedness of f refers to the case in which such a recession cone is only partially pointed. In this note we show that the degree of pointedness of f is related to the "thickness" of the effective domain of the conjugate function f^*.

Key words: Convex function, recession cone, pointedness, conjugate function, growth condition.

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1 Introduction.

Let K be a closed convex cone in some finite dimensional linear space X. It is easy to see that

$$\ell(K) := K \cap -K$$

is the largest subspace of X which is contained in K. The dimension of such a subspace can be used to measure the degree of pointedness of K.

Definition 1. The *degree of pointedness* of K is defined as the integer

$$p[K] := \dim X - \dim \ell(K).$$

If $p[K] = \dim X$, then one says that K is pointed. If $0 < p[K] < \dim X$, then one says that K is partially pointed.

Consider now a function $f : X \to R \cup \{+\infty\}$ whose effective domain

$$\text{dom} f := \{x \in X : f(x) < +\infty\}$$

is nonempty, and whose epigraph

$$\text{epi } f := \{(x, \lambda) \in X \times R : f(x) \leq \lambda\}$$

is convex and closed. Such class of functions is usually denoted by $\Gamma_0(X)$. The recession cone of the set epi f is defined by

$$(\text{epi} f)_\infty := \{(u, \alpha) \in X \times R : \ (u, \alpha) + \text{ epi } f \subseteq \text{ epi } f\}.$$

The recession function f_∞ of f is given by

$$f_\infty(u) := \sup\{f(x+u) - f(x) : x \in \text{dom} f\} \quad \text{for all} \quad u \in X.$$

Both notions are standard in the context of convex analysis and can be consulted, for instance, in the book by Rockafellar [3]. In this note we study the following new concept.

Definition 2. The *degree of pointedness* of the function $f \in \Gamma_0(X)$ is the integer

$$p[f] := \dim X - \dim \ell((\mathrm{epi} f)_\infty). \tag{1}$$

If $p[f] = \dim X$, then f is said to be pointed. If $0 < p[f] < \dim X$, then f is called partially pointed.

Remark. According to the above definition, f is pointed if and only if $(\mathrm{epi}\ f)_\infty$ is a pointed cone. This case has already been considered by Benoist and Hiriart–Urruty [1, Definition 2.3]. However, these authors do not address the question of the dimension of $\ell((\mathrm{epi} f)_\infty)$.

2 Pointedness and Conjugacy.

In this section we derive a simple formula for computing the degree of pointedness of a function $f \in \Gamma_0(X)$. Recall that the Legendre–Fenchel conjugate of f is the function $f^* \in \Gamma_0(X)$ defined by

$$f^*(y) := \sup_{x \in X}\{\langle y, x\rangle - f(x)\} \quad \text{for all } y \in X,$$

where $\langle \cdot, \cdot \rangle$ is a given inner product in the space X. The next theorem says that the degree of pointedness of f is equal to the dimension of the effective domain of f^*. The dimension of a nonempty convex set $A \subseteq X$ is defined as the dimension of the affine hull of A (cf. [3, p. 12]).

Theorem 1. *Let X be a finite dimensional linear space, and let $f \in \Gamma_0(X)$. Then,*

$$\dim(\mathrm{dom} f^*) + \dim \ell((\mathrm{epi} f)_\infty) = \dim X. \tag{2}$$

This theorem is obtained by combining the following two lemmata. The notation $\langle C \rangle$ refers to the linear space spanned by the set C, and $\langle C \rangle^\perp$ stands for the orthogonal complement of $\langle C \rangle$.

Lemma 1. *Let X and f be as in Theorem 1. Then,*

$$\ell((\mathrm{epi} f)_\infty) = \langle \mathrm{dom} f^* \times \{-1\} \rangle^\perp.$$

In particular, the space $X \times R$ can be decomposed as a direct sum as indicated below:

$$X \times R = \langle \mathrm{dom} f^* \times \{-1\} \rangle \oplus \ell((\mathrm{epi} f)_\infty).$$

Proof. By definition one has

$$(u, \alpha) \in \ell((\mathrm{epi} f)_\infty) \Leftrightarrow (u, \alpha) \in (\mathrm{epi} f)_\infty \cap -(\mathrm{epi} f)_\infty.$$

Since $(\mathrm{epi} f)_\infty = \mathrm{epi} f_\infty$, one can also write

$$\begin{aligned} (u, \alpha) \in \ell((\mathrm{epi} f)_\infty) \quad &\Leftrightarrow \quad f_\infty(u) \leq \alpha \quad \text{and} \quad f_\infty(-u) \leq -\alpha \\ &\Leftrightarrow \quad f_\infty(u) \leq \alpha \leq -f_\infty(-u). \end{aligned}$$

Now we use the fact that f_∞ is the support function of the set dom f^* (cf. [2]), i.e.

$$f_\infty(u) = \sup\{\langle y, u \rangle : y \in \mathrm{dom} f^*\}.$$

One has also

$$-f_\infty(-u) = \inf\{\langle y, u \rangle : \quad y \in \mathrm{dom} f^*\}.$$

Hence

$$(u,\alpha) \in \ell((\text{epi} f)_\infty) \;\Leftrightarrow\; \langle y, u\rangle = \alpha \quad \text{for all} \quad y \in \text{dom} f^*$$
$$\Leftrightarrow\; \langle (y,\beta),(u,\alpha)\rangle = 0 \quad \text{for all} \quad (y,\beta) \in \text{dom} f^* \times \{-1\}.$$

This completes the proof of the lemma. □.

Lemma 2. *Let A be a nonempty convex set in X. Then*

$$\dim A = \dim \langle A \times \{-1\}\rangle - 1. \tag{3}$$

Proof. The proof of this technical lemma is essentially an exercise in linear algebra. As shown in [3, Theorem 2.4], the dimension of A is the maximum of the dimensions of the various simplices included in A. If m denotes the dimension of A, then A contains $m+1$ points $a_0, a_1, \ldots, a_m$ for which the vectors

$$a_1 - a_0, a_2 - a_0, \ldots, a_m - a_0 \quad \text{are linearly independent.} \tag{4}$$

It can be shown that the above condition is equivalent to

$$(a_0,-1),(a_1,-1),\ldots,(a_m,-1) \quad \text{are linearly independent.} \tag{5}$$

This means that $A \times \{-1\}$ contains at least $(m+1)$ linearly independent vectors, and thus

$$\dim \langle A \times \{-1\}\rangle \geq \dim A + 1.$$

To prove the reverse inequality, let $m := \dim \langle A \times \{-1\}\rangle - 1$. In this case, $A \times \{-1\}$ contains $m+1$ linearly independent vectors, say $(a_0,-1),\ldots,(a_m,-1)$. Since (5) is equivalent to (4), we conclude that A contains an m–dimensional simplex. Hence,

$$\dim A \geq \dim \langle A \times \{-1\}\rangle - 1.$$

This completes the proof. □

Remark. By combining Theorem 1 and [3, Theorem 13.4], one sees that $\ell((\text{epi} f)_\infty)$ has the same dimension as the set $\{u \in X : f_\infty(u) = -f_\infty(-u)\}$. The latter set is known as the lineality space of f.

We end this section by illustrating Theorem 1 with the help of two examples.

Example 1. Consider the space $X = S_n$ of symmetric matrices of order $n \times n$ equipped with the usual inner product $\langle x, y\rangle := \text{trace}(xy)$. The variance of a matrix $x \in S_n$ is defined by

$$(\text{var})(x) := \frac{1}{n}\langle x, x\rangle - \left(\frac{\text{trace } x}{n}\right)^2.$$

The function var: $S_n \to R$ is convex, and its conjugate $(\text{var})^* : S_n \to R \cup \{+\infty\}$ is given by

$$(\text{var})^*(y) = \begin{cases} \frac{n}{4}\langle y, y\rangle & \text{if trace } y = 0, \\ +\infty & \text{otherwise} . \end{cases}$$

The dimension of $\text{dom}(\text{var})^* = \{y \in S_n : \text{ trace } y = 0\}$ is equal to $\dim S_n - 1$. Hence, the function var is partially pointed and its degree of pointedness is

$$p[\text{var}] = \dim S_n - 1 = (n^2 + n - 2)/2.$$

Example 2. Let $X = S_n$ be as in Example 1. Let $\lambda_{\max}(x)$ denote the largest eigenvalue of the matrix $x \in S_n$. It is known that $\lambda_{\max} : S_n \to R$ is equal to the support function of the set

$$A = \{x \in S_n : \ x \text{ is positive semidefinite, trace } x = 1\}.$$

Hence $\text{dom}(\lambda_{\max})^* = A$ is a set of dimension equal to $\dim S_n - 1$. Thus, $\lambda_{\max}$ is partially pointed and $p[\lambda_{\max}] = (n^2 + n - 2)/2$. In this example one can also compute $p[\lambda_{\max}]$

by using the very definition of this number. The set $\text{epi}(\lambda_{\max})$ is a closed convex cone, so it coincides with its recession cone. As a matter of calculus, one has

$$\begin{aligned}(x,\lambda) \in \ell(\text{epi}(\lambda_{\max})) \quad &\Leftrightarrow \quad \lambda_{\max}(x) \le \lambda \text{ and } \lambda_{\max}(-x) \le -\lambda \\ &\Leftrightarrow \quad \text{all the eigenvalues of } x \text{ are equal to } \lambda.\end{aligned}$$

If $i \in S_n$ denotes the identity matrix, then

$$\ell(\text{epi}(\lambda_{\max})) = \{(\lambda i, \lambda) : \quad \lambda \in R\} = R\{(i,1)\} \subset S_n \times R$$

is a space of dimension 1.

3 Pointedness and Growth Condition.

Recall that each $f \in \Gamma_0(X)$ can be minorized by some affine function, i.e. one can find $y \in X$ and $b \in R$ such that

$$f(x) \ge \langle y, x\rangle + b \quad \text{for all} \quad x \in X. \tag{6}$$

Is it possible to obtain a more precise information on the growth of f? In other words, is it possible to write a stronger growth condition for f? As already observed by Benoist and Hiriart–Urruty [1, Theorem 2.4], a growth condition of the type

$$f(x) \ge r\|x\| + \langle y, x\rangle + b \quad \text{for all} \quad x \in X, \tag{7}$$

with $r > 0$, characterizes the class of functions $f \in \Gamma_0(X)$ which are pointed. This observation leads us to think that a partially pointed function $f \in \Gamma_0(X)$ satisfies a growth condition which is intermediate between (6) and (7). The purpose of the next theorem is to display in a clear-cut manner the relationship between growth condition and degree of pointedness. To start with, observe that the conditions (6) and (7) can be written in a common format, namely:

$$f(x) \geq \psi_A^*(x) + b \quad \text{for all} \quad x \in X,$$

where ψ_A^* denotes the support function of $A \subseteq X$. In the former case A corresponds to the zero–dimensional set $\{y\}$, and in the latter case A is the full dimensional ball $B(y,r) := \{z \in X : \|z - y\| \leq r\}$.

Theorem 2. *The degree of pointedness of the function $f \in \Gamma_0(X)$ is given by*

$$p[f] = \max_{A \in C(X)} \{\dim A : f - \psi_A^* \text{ is minorized}\}, \tag{8}$$

where $C(X)$ denotes the class of nonempty convex sets in X. One can also write

$$p[f] = \max_{A \in C(X)} \{\dim A : f_\infty \geq \psi_A^*\}, \tag{9}$$

where the maximum in (9) is attained at $A = \mathrm{dom} f^*$ (or, more generally, at any set A which is contained in the closure of dom f^* and which has the same dimension as dom f^*).

Proof. The inequality $f_\infty \geq \psi_A^*$ is equivalent to the inclusion

$$\overline{A} \subseteq \overline{\mathrm{dom} f^*},$$

where the upper bar denotes the closure operation in X. Since the closure operation does not affect the dimension of a convex set (cf. [3, Theorem 6.2]), the maximum in (9) is attained at any $A \subseteq \overline{\mathrm{dom}\, f^*}$ which has the same dimension as dom f^*. Now we prove that $p[f] = m[f]$, where $m[f]$ denotes the term on the right–hand side of (8). It is fairly clear that

$$f - \psi_A^* \text{ is minorized } \Rightarrow f_\infty \geq \psi_A^*.$$

Thus (9) yields the inequality $p[f] \geq m[f]$. According to Toland–Singer duality theorem (cf. [4], [5]), for all $A \in C(X)$, one has:

$$\inf_{x \in X} \{f(x) - \psi_A^*(x)\} = -\sup_{y \in \overline{A}} f^*(y).$$

Hence,

$$m[f] = \max_{A \in C(X)} \{\dim A : f^* \text{ is majorized over } \overline{A}\}. \tag{10}$$

Take any $y \in ri$ (domf^*), where "ri" stands for the relative interior (cf. [3, p. 44]). Then, for some $r > 0$ sufficiently small, one has

$$A_r := B(y,r) \cap \overline{\text{dom} f^*} \subset ri \ (\text{dom } f^*).$$

According to Rockafellar [3, Theorem 10.1], the function f^* is continuous relative to ri (dom f^*). Hence, f^* is majorized over the compact set A_r, and

$$m[f] \geq \dim A_r.$$

But

$$\dim A_r = \dim(\overline{\text{dom} f^*}) = \dim(\text{dom } f^*) = p[f].$$

This proves the reverse inequality $m[f] \geq p[f]$, and completes the proof of the theorem. □

Remark. Since f and ψ_A^* may take the value $+\infty$ at the same time, we have implicitly adopted the rule $(+\infty) - (+\infty) = +\infty$. The maximum in (8) is attained at a set A of the form $A = B(y,r) \cap \overline{\text{dom} f^*}$. This means that the formula (8) remains true if one defines $C(X)$ as the class of nonempty convex compact sets in X.

4 Calculus Rules for the Degree of Pointedness.

Suppose $f \in \Gamma_0(X)$ is constructed from other functions, say $f_1, \ldots, f_N$, whose degrees of pointedness are known, or can be easily computed. In this case it is helpful to have a formula which relates $p[f]$ to the degrees $p[f_1], \dots, p[f_N]$ of the component functions. The calculus rules recorded in the first two propositions can be proved in a fairly simple way by using Theorem 1.

Proposition 1. *Let $f \in \Gamma_0(X)$. Then,*

(a) $p[f] \leq p[g]$ *for all* $g \in \Gamma_0(X)$ *such that* $f \leq g$;

(b) $p[f + \ell] = p[f]$ *for all affine functions* $\ell : X \to R$;

(c) $p[\lambda f] = p[f]$ *for all* $\lambda > 0$;

(d) $p[f(\cdot/\lambda)] = p[f]$ *for all* $\lambda > 0$;

(e) $p[f_c] = p[f]$, *where* $c \in X$ *and* $f_c(x) = f(x - c)$.

Proposition 2. *Let $X = \prod_{k=1}^{N} X_k$ be the Cartesian product of the finite dimensional spaces $X_1, \dots, X_N$. Suppose*

$$f(x) = f_1(x_1) + \cdots + f_N(x_N) \quad \text{for all } x \in X,$$

where $f_k \in \Gamma_0(X_k)$ for all $k = 1, \dots, N$. Then,

$$p[f] = \sum_{k=1}^{N} p[f_k]. \tag{11}$$

In particular, f is pointed if and only if all the f_k's are pointed.

In the next four propositions we consider some important functional operations arising in the context of convex analysis, namely, pointwise maximum, addition, closed convex hull, and infimal–convolution.

Proposition 3. *Let $f_1, \dots f_N \in \Gamma_0(X)$ be finite at some common point, and let $f = \max_{1 \leq k \leq N} f_k$. Then,*

$$p[f] \geq \max_{1 \leq k \leq N} p[f_k]. \tag{12}$$

In particular, f is pointed if at least one of the f_k's is pointed.

Proof. Formula (12) follows from Proposition 1(a). Indeed, since each $f_k \le f$, one has

$$p[f_k] \le p[f] \quad \text{for all} \quad k = 1, \dots, N.$$

An alternative proof of (12) is based on Theorem 2 and runs as follows. Let $A_1, \dots, A_N \in C(X)$ and $b_1, \dots, b_N \in R$ be such that dim $A_k = p[f_k]$, and

$$f_k(x) \ge \psi^*_{A_k}(x) + b_k \quad \text{for all} \quad x \in X.$$

Then,

$$f(x) \ge \psi^*_A(x) + b \quad \text{for all} \quad x \in X,$$

where $b = \min_{1 \le k \le N} b_k$ and $A = \text{co} \bigcup_{k=1}^{N} A_k$ is the convex hull of the sets $A_1, \dots, A_N$. Since $f - \psi^*_A$ is minorized, we have

$$p[f] \ge \dim A \ge \max_{1 \le k \le N} \dim A_k. \qquad \square$$

Proposition 4. *Let $f_1, \dots, f_N \in \Gamma_0(X)$ be finite at some common point, and let $f = \sum_{k=1}^{N} f_k$. Then*

$$p[f] \ge \mathit{dim}(\text{dom} f_1^* + \dots + \text{dom} f_N^*) \ge \max_{1 \le k \le N} p[f_k]. \tag{13}$$

In particular, f is pointed if at least one of the f_k's is pointed.

Proof. It is known that f^* is equal to the lower–semicontinuous hull of the function

$$y \in X \mapsto h(y) := \inf\{f_1^*(y_1) + \dots + f_N^*(y_N) : y_1 + \dots + y_N = y\}.$$

Thus,

$$\text{dom} f^* \supseteq \text{dom} h = \text{dom} f_1^* + \dots + \text{dom} f_N^*$$

and

$$\dim \mathrm{dom} f^* \geq \dim(\mathrm{dom} f_1^* + \cdots + \mathrm{dom} f_N^*) \geq \max_{1\leq k\leq N} \dim \mathrm{dom} f_k^*.$$

Theorem 1 completes the proof of (13). □

Remark. If $\bigcap_{k=1}^{N} \mathrm{ri}(\mathrm{dom} f_k)$ is nonempty, then h is lower–semicontinuous, and the first inequality in (13) becomes $p[f] = \dim(\mathrm{dom}\, f_1^* + \cdots + \mathrm{dom} f_N^*)$.

Proposition 5. *Let f be the closed convex hull of the functions $f_1, \ldots, f_N \in \Gamma_0(X)$. Suppose all the f_k's are minorized by a common affine function. Then*

$$p[f] = \dim \bigcap_{k=1}^{N} \mathrm{dom} f_k^* \quad \leq \min_{1\leq k\leq N} p[f_k]. \tag{14}$$

Proof. Since $f^* = \max\{f_k^* : 1 \leq k \leq N\}$, one has

$$\mathrm{dom} f^* = \bigcap_{k=1}^{N} \mathrm{dom} f_k^*.$$

Formula (14) follows by applying Theorem 1.

Proposition 6. *Let f be the infimal–convolution of the functions $f_1, \ldots, f_N \in \Gamma_0(X)$. Suppose the sets $ri(dom\ f_k)$, $k = 1, \ldots, N$ have a point in common. Then*

$$p[f] = \dim \bigcap_{k=1}^{N} \mathrm{dom} f_k^* \quad \leq \min_{1\leq k\leq N} p[f_k]. \tag{15}$$

Proof. The proof is the same as in Proposition 4. This time one starts with the equality

$$(f)^* = \sum_{k=1}^{N} f_k^*.$$ □

An important use of the infimal–convolution operation is the regularization of a given function $f \in \Gamma_0(X)$. The regularized version of f is defined by

$$x \in X \mapsto [f \Box \theta](x) := \inf_{u\in X}\{f(u) + \theta(x-u)\},$$

where the function $\theta \in \Gamma_0(X)$ is referred to as a "kernel".

Corollary 1. *Let* $\theta \in \Gamma_0(X)$ *be coercive in the sense that* $\theta(x)/\|x\| \to +\infty$ *as* $\|x\| \to \infty$. *Then*

$$p[f \square \theta] = p[f] \quad \textit{for all} \quad f \in \Gamma_0(X).$$

Proof. That $f \square \theta \in \Gamma_0(X)$ follows from the coercivity of θ. One also has dom $\theta^* = X$. Thus, $\text{dom}(f \square \theta)^* = \text{dom} f^* \cap \text{dom} \theta^*$ has the same dimension as dom f^*. □

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