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On Comparison of Spline Regularization with Exponential Sampling Method for Laplace Transform Inversion

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Abstract

In this paper we have converted the Laplace transform to an integral equation of the first kind of convolution type, which is an ill–posed problem and used the spline regularization method to solve it. The method is applied to several test examples taken from [1, 5, 22]. It gives a good approximation to the true solution and compares well with the exponential sampling method given in [3, 4, 5]. The results are shown in the table and respective diagrams.

AMS(MOS) Subject Classification: 65R20, 65R30

Key Words Inversion of Laplace transform, ill–posed problem, convolution equation, cross validation, spline regularization, filter function.

1. Introduction.

Some years ago Ostrowsky et al. [12] introduced the exponential sampling method for inversion of the Laplace transform in photon correlation spectroscopy. There are many problems whose solution may be found in terms of a Laplace transform which, however, is too complicated for inversion using different methods. However, no single method gives optimum results for all purposes and all occasions.

For a detailed bibliography, the reader should consult Piessens [16] and Piessens and Branders [17].

The problem of the recovery of a real function $f(t)$, $t \geq 0$, given its Laplace transform

$$\int_0^\infty e^{-st} f(t)dt = g(s) \tag{1.1}$$

for real values of s, is an ill–posed problem and, therefore, affected by numerical instability.

The ill–posedness of Laplace transform inversion in the case where $f \in L^2\ (R_+)$ and $g(s)$ is known for all real and positive values of s, can be investigated by means of the Mellin transform [3, 4, 11]. In practice, however, $g(s)$ is known only in a finite set of points. The case of an infinite set of equidistant points was investigated by Papoulis [14]. Several methods and a comparison is given in Davies [7] and Talbot [20].

The previous methods do not include regularization techniques. Regularization methods have been discussed by Varah [22] and Essah and Delves [8]. Regularization by means of truncated singular function expansion is investigated by Bertero in [2]; other methods are also available in the literature for the numerical evaluation of the Laplace transform inversion which have been described by Norden [12] and Salzer [18].

2. Description of the Method

In (1.1) given $g(s)$, $s \geq 0$ we wish to find $f(t)$, $t \geq 0$ and $f(t) = 0$ for $t < 0$, so that (1.1) holds.

Frequently, $g(s)$ is measured at certain points. We assume $g(s)$ is given analytically with known $f(t)$, so that we can measure the error in the numerical solution.

We shall convert the Laplace transform into the first kind integral equation of convolution type. We make the following substitution in equation (1.1).

$$s = a^x \quad \text{and } t = a^{-y} \quad \text{where } a > 1 \tag{2.1}$$

Then

$$g(a^x) = \int_{-\infty}^{\infty} \log a e^{-a^{x-y}} f(a^{-y}) a^{-y} dy \tag{2.2}$$

Multiplying both sides of (2.2) by a^x we obtain the convolution equation

$$\int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x), \quad -\infty \leq x \leq \infty \tag{2.3}$$

where

$$\left.\begin{array}{l} G(x) = a^x g(a^x) = sg(s) \\ K(x) = \log a a^x e^{-a^x} = \log a s e^{-s} \\ F(y) = f(a^{-y}) = f(t) \end{array}\right\} \tag{2.4}$$

In order that we can apply our deconvolution method to equation (2.3), it is necessary that $G(x)$ has essentially compact support, i.e., $G(x) \to 0$ as $x \to \pm\infty$ which is a property, we demand from our data function $G(x)$.

Let $B_j(H;x)$ be the n–th order cardinal B–spline (n even) with knots $(j - \frac{1}{2}n)H, \ldots, (j + \frac{1}{2}n)H$, i.e., $B_j(H;x) = Q_n(\frac{x}{H} - j + \frac{n}{2})$ where

$$Q_n(x) = \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} ((x-j)_+^{n-1} . \tag{2.5}$$

In addition let $MH = T$ where $M \leq N$ is an integral power of 2. We assume that $B_j(H;x)$ is periodically continued outside the interval $(0,T)$, with period T. Then $B_j(H;x)$ has a Fourier series

$$B_j(H;x) = \sum_{q=-\infty}^{\infty} \hat{B}_{jq} \exp(iw_q x) \tag{2.6}$$

where

$$\hat{B}_{jq} = \int_0^T B_j(H;x) \exp(-iw_q x) dx \tag{2.7}$$

and $w_q = \frac{2\pi q}{T}$.

Since $B_j(H;x)$ is simply a translation of $B_0(H;x)$ by an amount jH, we have

$$\hat{B}_{jq} = \hat{B}_{oq} \exp(-iw_q H)$$

where

$$\hat{B}_{oq} = H \left[\frac{\sin \frac{w_q H}{2}}{\frac{w_q H}{2}} \right]^4 \tag{2.8}$$

Tikhonov Regularization Using Cardinal Cubic B–splines

We shall approximate the convolution equation (2.3) by

$$\int_0^T K_N(x-y) F_M(y) dy = G_N(x) \tag{2.9}$$

where we assume that F, G and K have essentially finite support in $[0, T)$, F_M is a cubic spline ($n = 4$) of the form

$$F_M(x) = \sum_{j=1}^{M-1} \alpha_j B_j(H; x), \quad M \leq N \tag{2.10}$$

The real M dimensional vector

$$\underline{\alpha} = (\alpha_0, \ldots, \alpha_{M-1})^T$$

of unknown coefficients will be determined, the spline in equation (2.10) has the Fourier series

$$F_M(x) = \sum_{q=-\infty}^{\infty} \hat{F}_{M,q} \exp(i w_q x) \tag{2.11}$$

where

$$\begin{aligned} \hat{F}_{M,q} &= \sum_{j=0}^{M-1} \alpha_j \hat{B}_{jq} && \text{(2.11a)} \\ &= \hat{B}_{oq} \sum_{j=0}^{M-1} \alpha_j \exp\left(-\frac{2\pi i}{M} jq\right) \\ &= \sqrt{M} \hat{B}_{oq} \hat{\alpha}_s, \quad s \equiv q (\operatorname{mod} M), && \text{(2.12)} \end{aligned}$$

where

$$\underline{\hat{\alpha}} = \psi_M^H \underline{\alpha} \tag{2.13}$$

We find it advantageous to determine $\underline{\hat{\alpha}}$ rather than $\underline{\alpha}$, because of the simple properties available in discrete Fourier spaces. The vector $\underline{\alpha}$ in equation (2.10) may then be determined from the inverse M-dimensional FFT (Fast Fourier transform)

$$\underline{\alpha} = \psi_M \underline{\hat{\alpha}} \tag{2.14}$$

where ψ is the unitary matrix with elements

$$\psi_{rs} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} rs\right), \quad r, s = 0, 1, 2, \ldots, N-1$$

P–th Order Tikhonov Regularization [21].

Consider the smoothing functional

$$C(F_M;\lambda) = C(\underline{\alpha},\lambda) = \|K_N(x) * F_M(x) - G_N(x)\|_2^2 + \lambda\|F_M^{(P)}\|_2^2 \tag{2.15}$$

F_M^P is the p–th derivative .

Using Plancherel's theorem we have

$$\|K_N * F_M - G_N\|_2^2 = \frac{1}{N^2}\sum_{q=-\frac{1}{2}N}^{\frac{1}{2}N} |\hat{K}_{N,q}\hat{F}_{M,q} - \hat{G}_{N,q}|^2$$

Hence using equation (2.12)

$$\|K_N{*}F_M{-}G_N\|_2^2 = \frac{1}{N^2}\sum_{q=-\frac{1}{2}N}^{\frac{1}{2}N}\left[\left(\sqrt{M}\hat{B}_{oq}\hat{K}_{N,q}\hat{\alpha}_s - \hat{G}_{N,q}\right)\left(\sqrt{M}\,\hat{B}_{oq}\overline{\hat{K}}_{N,q}\overline{\hat{\alpha}}_s - \overline{\hat{G}}_{N,q}\right)\right] \tag{2.16}$$

where

$$s \equiv q (\operatorname{mod} M)$$

Also, Plancherel's theorem applied to the regularizing functional in equation (2.15) gives

$$\begin{aligned} \|F_M^{(P)}\|^2 &= \sum_{q=-\infty}^{\infty} w_q^{2p}|\hat{F}_{M,q}|^2 = 2\sum_{q=1}^{\infty} w_q^{2p}|\hat{F}_{M,q}|^2 \\ &= 2M\sum_{q=1}^{\infty} w_q^{2p}\hat{B}_{oq}^2|\hat{\alpha}_s|^2 \quad \text{where } s \equiv q(\operatorname{mod} M) \end{aligned} \tag{2.17}$$

The simplification of expression (2.17) requires the use of an attenuation factor τ_q. For cubic cardinal splines ($n = 4$) it is shown by Stoer [19] and Gautschi [9] that

$$\tau_q = \left[\frac{\sin\frac{\pi q}{M}}{\frac{\pi q}{M}}\right]^4 \frac{3}{1+2\cos^2(\frac{\pi q}{M})}. \tag{2.18}$$

In expression (2.17) we wish to arrange the summation over q to summation over s, where $s \equiv q(\bmod M)$. Define the matrix

$$W^{(1)} = \begin{bmatrix} \operatorname{diag} \sqrt{M}\hat{B}_{o,s}\hat{K}_{N,s} \\ \cdots\cdots\cdots \\ \operatorname{diag} \sqrt{M}\hat{B}_{0,M-s}\overline{\hat{K}}_{N,M-s} \end{bmatrix} \quad \begin{matrix} \text{order } N \times M \\ s = 0, 1, \ldots, M-1 \end{matrix} \tag{2.19}$$

From the property $\hat{K}_{N,q} = \overline{\hat{K}}_{N,N-q}$ of discrete FTs it then follows that expression (2.16) simplifies to

$$\|K_N * F_M - G_N\|_2^2 = \|W^{(1)}\underline{\hat{\alpha}} - \underline{\hat{G}}_N\|_2^2 \tag{2.20}$$

and (2.17) can be written as

$$\begin{aligned} \|F_M^{(P)}\|^2 &= 2M \sum_{s=1}^{M-1} \left\{ |\hat{\alpha}_s|^2 \sum_{n=0}^{\infty} w_{Mn+s}^{2p} \hat{B}_{0,Mn+s}^2 \right\} \\ &= 2M \sum_{s=1}^{M-1} \tau_s |\hat{\alpha}_s|^2 \end{aligned} \tag{2.21}$$

where

$$\tau_s = \sum_{n=0}^{\infty} w_{Mn+s}^{2p} \hat{B}_{0,Mn+s}^2 \tag{2.22}$$

$$= (2\pi)^{2p} \sum_{n=0}^{\infty} (Mn+s)^{2p} H^2 \left[\frac{\sin \frac{\pi(Mn+s)}{M}}{\frac{\pi(Mn+s)}{M}} \right]^8$$

$$\tau_s = (2\pi)^{2p} H^2 s^8 \left[\frac{\sin \frac{\pi s}{M}}{\frac{\pi s}{M}} \right]^8 \sum_{n=0}^{\infty} (Mn+s)^{2p-8}$$

$$= (2\pi)^{2p} s^8 \hat{B}_{0,s}^2 \sum_{n=0}^{\infty} (Mn+s)^{2p-8} \tag{2.23}$$

Since $\hat{\alpha}_s = \overline{\hat{\alpha}}_{M-s}$, equation (2.21) further simplifies to

$$\|F_M^{(P)}\|_2^2 = 2M \sum_{s=1}^{\frac{1}{2}M} (\tau_s + \tau_{M-s}) |\hat{\alpha}_s|^2 \tag{2.24}$$

In particular, when $p = 2$, from (2.23) it follows that

$$\tau_s = (2\pi)^4 s^4 \hat{B}_{0,s}^2 \sum_{n=0}^{\infty} \left(\frac{s}{Mn+s} \right)^4$$

while

$$\tau_{M-s} = (2\pi)^4 s^4 \hat{B}_{0,s}^2 \sum_{n=1}^{\infty} \left(\frac{s}{Mn-s}\right)^4$$

so that

$$\begin{aligned}
\tau_s + \tau_{M-s} &= (2\pi)^4 s^4 \hat{B}_{0,s}^2 \sum_{n=-\infty}^{\infty} \left(\frac{s}{Mn+s}\right)^4 \\
&= \frac{(2\pi)^4 s^4 \hat{B}_{0,s}^2 \left[1 + 2\cos^2\left(\frac{\pi s}{M}\right)\right]}{3\left[\frac{\sin\frac{\pi s}{M}}{\frac{\pi}{M}}\right]^4} \\
&\qquad \text{(see Pennisi [15])} \\
&= \frac{16}{3} M^2 \sin^4\left(\frac{\pi s}{M}\right)\left[1 + 2\cos^2\left(\frac{\pi s}{M}\right)\right] \qquad (2.25)
\end{aligned}$$

Defining the $M \times M$ matrix

$$W^{(2)} = \operatorname{diag}\left\{\left[M\left(\tau_s + \tau_{M-s}\right)\right]^{1/2}\right\} \qquad (2.26)$$

it follows from (2.24) that

$$\left\|F_M^{(P)}\right\|_2^2 = \left|W^{(2)}\hat{\underline{\alpha}}\right\|^2 \qquad (2.27)$$

Thus, from equations (2.20) and (2.27) we may express the smoothing functional (2.15) as

$$C(\underline{\alpha}, \lambda) = \left\|W^{(1)}\hat{\underline{\alpha}} - \hat{\underline{G}}_N\right\|_2^2 + \lambda\left\|W^{(2)}\hat{\underline{\alpha}}\right\|_2^2 \qquad (2.28)$$

The minimizer of (2.28) is clearly

$$\hat{\underline{\alpha}} = (W + \lambda V)^{-1} W^{(1)H} \hat{\underline{G}}_N \qquad (2.29)$$

where

$$\left.\begin{aligned} W &= W^{(1)H} W^{(1)} \\ V &= W^{(2)H} W^{(2)} \end{aligned}\right\} \qquad (2.30)$$

It is not necessary to invert the matrix $W + \lambda V$ directly because it is diagonal. From equations (2.19), (2.26), (2.29) and (2.30) it follows that

$$\hat{\alpha}_s = \frac{1}{\sqrt{M}} \frac{\hat{B}_{0,s}\overline{\hat{K}}_{N,s}\hat{G}_{N,s} + \hat{B}_{0,M-s}\hat{K}_{N,M-s}\overline{\hat{G}}_{N,M+s}}{\left[\hat{B}_{0,s}^2 \left[\left|\hat{K}_{N,s}\right|^2 + \hat{B}_{0,M-s}^2 \left|\hat{K}_{N,M-s}\right|^2\right] + N^2\lambda\left(\tau_s + \tau_{M-s}\right)\right]}$$

$$\hat{\alpha}_s = \frac{1}{\sqrt{M}} \frac{\hat{B}_{0,s}\left[\overline{\hat{K}}_{N,s}\hat{G}_{N,s} + \left(\frac{s}{M-s}\right)^4 \hat{K}_{N,M-s}\overline{\hat{G}}_{N,M+s}\right]}{\hat{B}_{0,s}^2 \left[\left|\hat{K}_{N,s}\right|^2 + \left(\frac{s}{M-s}\right)^8 \left|\hat{K}_{N,M-s}\right|^2\right] + N^2\lambda\left(\tau_s + \tau_{M-s}\right)} \qquad (2.31)$$

since

$$\hat{B}_{0,M-s} = \left(\frac{s}{M_n - s}\right)^4 \hat{B}_{0,s} \qquad (2.32)$$

We can easily verify that $\hat{\alpha}_s = \overline{\hat{\alpha}}_{M-s}$ so that the inverse FFT gives $\underline{\alpha} = \psi_M \underline{\hat{\alpha}}$ is a real vector as required.

The Filter for Cardinal B–Spline Regularization

The Fourier coefficients of the regularized (filtered) solution $F_M(x) \in B_M(0,T)$ clearly depends on λ through equations (2.11a), (2.13) and (2.31). In equation (2.31), we denote the dependence of $\hat{\alpha}_s$ on λ by writing $\hat{\alpha}_s = \hat{\alpha}_s(\lambda)$. Thus the Fourier coefficients of the filtered solution are

$$\hat{F}_{M,q}(\lambda) = \sqrt{M}\hat{B}_{0,q}\hat{\alpha}_s(\lambda), \quad s \equiv q \pmod{M}$$

whereas those of the unregularized (unfiltered) solution are

$$\hat{F}_{M,q}(0) = \sqrt{M}\hat{B}_{0,q}\hat{\alpha}_s(0).$$

Clearly the underlying filter $Z_{q;\lambda}$ must satisfy

$$\hat{F}_{M,q}(\lambda) = Z_{q;\lambda}\hat{F}_{M,q}(0)$$

so that we can deduce

$$Z_{q;\lambda} = \frac{\hat{\alpha}_s(\lambda)}{\hat{\alpha}_s(0)} \tag{2.33}$$

$$= \frac{\hat{B}^2_{0,s}[|\hat{K}_{N,s}|^2 + (\frac{s}{M_n - s})^8 |\hat{K}_{N,M-s}|^2]}{\hat{B}^2_{0,s}[|\hat{K}_{N,s}|^2 + (\frac{s}{M_n - s})^8 |\hat{K}_{N,M-s}|^2] + N^2\lambda(\tau_s + \tau_{M-s})} \tag{2.34}$$

The filter will of course apply to every Fourier coefficients $q = 0, \pm 1, \pm 2, \ldots$, but will have only M possible values depending on q modulo M. The regularization parameter λ is still to be determined.

Determination of Regularization Parameter λ

Let the filtered solution $F_M(x) \in B_M\ (0,T)$, which minimizes $\|K_N * F_M - G_N\|_2^2 + \lambda \|F''_M\|_2^2$ be given by (we have $p = 2$)

$$F_M(x) = \sum_{q=-\infty}^{\infty} \hat{F}_{M,q} \exp(i w_q x) \tag{2.35}$$

Consider

$$\hat{G}_{N,\lambda,q} = \hat{K}_{N,q}\hat{F}_{M,q} \quad , \quad q = 0, 1, \ldots, N-1 \tag{2.36}$$

$$= \begin{cases} \sqrt{M}\hat{B}_{0,q}\hat{K}_{N,q}\hat{\alpha}_s & , \quad s \equiv q(\bmod M) \\ 0 & , \quad \text{otherwise} \end{cases} \quad q = 0, 1, \ldots, N-1$$

We now introduce the $N \times N$ influence matrix

$$A(\lambda) = \psi_N \hat{A}(\lambda) \psi_N^H$$

where

$$\underline{\hat{G}}_{N,\lambda} = \hat{A}(\lambda)\underline{\hat{G}}_N \tag{2.37}$$

$\hat{A}(\lambda)$ is block diagonal with the following structure

$$\hat{A}(\lambda) = \left[\begin{array}{c|c} \text{diag } \underline{a}_1 & \text{diag } \underline{a}_2 \\ \hline \text{diag } \underline{a}_3 & \text{diag } \underline{a}_4 \end{array}\right] \tag{2.38}$$

where $\underline{a}_k \in C^M$, $\quad K = 1,2,3,4$ and

$$a_{1,s} = \begin{cases} \dfrac{\sqrt{M}(\hat{B}_{0,s})^2|\hat{K}_s|^2}{D_s} & s=0 \\ \frac{\sqrt{M}(\hat{B}_{0,s})^2|\hat{K}_s|^2}{2D_s} & 1 \le s \le M-1 \end{cases}$$

$$a_{2,s} = \begin{cases} 0 & s=0 \\ \dfrac{\sqrt{M}(\hat{B}_{0,s})^2(\frac{s}{M-s})^4\hat{K}_s\overline{\hat{K}}_{M+s}}{2D_s}, & 1 \le s \le M-1 \end{cases}$$

$$a_{3,s} = \begin{cases} \dfrac{\sqrt{M}\hat{K}_{M+s}\hat{B}_{0,M+s}\hat{B}_{0,s}\overline{\hat{K}}_s}{D_s}, & s=0 \\ \dfrac{\sqrt{M}\hat{K}_{M+s}\hat{B}_{0,M+s}\hat{B}_{0,s}\overline{\hat{K}}_s}{2D_s}, & 1 \le s \le M-1 \end{cases}$$

$$a_{4,s} = \begin{cases} 0 & s=0 \\ \dfrac{\sqrt{M}\hat{B}_{0,M+s}\hat{B}_{0,s}(\frac{s}{M+s})^4|\hat{K}_{M+s}|^2}{2D_s}, & 1 \le s \le M-1 \end{cases}$$

where

$$D_s = M\hat{B}^2_{0,s}[|\hat{K}_s|^2 + (\frac{s}{M-s})^8|\hat{K}_{M-s}|^2] + \lambda N^2(\tau_s + \tau_{M-s}).$$

For simplicity of notation we have written $\hat{K}_s$ for $\hat{K}_{N,s}$ in $a_{1,s}, a_{2,s}, a_{3,s}, a_{4,s}$ and D_s. The optimal λ as defined by GCV method may be found in Wahba [23]. Now minimizing the expression

$$V(\lambda) = \frac{\frac{1}{N}\|(I - \hat{A}(\lambda))\underline{\hat{G}}_N\|_2^2}{[\frac{1}{N}\text{ Trace }(I - \hat{A}(\lambda))]^2} \tag{2.39}$$

which from equation (2.38) simplifies to

$$V(\lambda) = \frac{\frac{1}{N}\{\sum_{s=0}^{M-1}|(1-a_{1,s})\hat{G}_s - a_{2,s}\overline{\hat{G}}_{M-s}|^2 + \sum_{s=0}^{M-1}|(1-a_{4,s})\overline{\hat{G}}_{M-s} - a_{3,s}\hat{G}_s|^2\}}{[1 - \frac{1}{N}\sum_{s=0}^{M-1}(a_{1,s}+a_{4,s})]^2} \tag{2.40}$$

In order to minimize $V(\lambda)$ in equation (2.40) we have used a subroutine which uses a quadratic interpolation technique to obtain a minimum.

3. Numerical Results

In this section we tabulate the results of the above method applied to the test examples taken from Brianzi [5], Varah [22] and Ang [1]. All data functions have the property $g(s) = 0(s^{-1})$ and no noise is added apart from machine rounding error; only optimal results have been quoted in the table and demonstrated in the diagrams. In each of the test examples, 64 sample points are used to calculate the discrete Fourier coefficients.

In our numerical calculations, we need to choose two numbers $x_{\min}$ and $x_{\max}$. We find $x_{\min}$ and $x_{\max}$ as the smallest and largest solutions of the non-linear equation $G(x) = \epsilon$. We may then pose the deconvolution problem (2.3) on the interval $[0, T]$, where $T = x_{\max} - x_{\min}$. Since the size of the essential support of $G(x)$ depends upon 'a', we have for a fixed number N of equidistant data points $\{x_n\}$, $h = T/N$, we have minimized (2.40) with respect to λ for values of $a \geq e$ and compared the L_∞ error of the resulting solution with the values of the true solutions. We find the minimum value of $V(\lambda)$ which yields the L_∞ error of the regularized solution as the least.

Example 1 (Brianzi [5])

$$\begin{aligned} g(s) &= \frac{1}{(s+1)^2} \\ f(t) &= t\,e^{-t} \end{aligned}$$

The optimal results are shown in Table 1 and diag 1.

Example 2 (Brianzi [5])

$$\begin{aligned} g(s) &= \frac{1}{s^2+2s+2} \\ f(t) &= e^{-t}\sin t \end{aligned}$$

The optimal results are shown in Table 1 and diag 2.

Example 3 (Varah [22])

$$\begin{aligned} g(s) &= \frac{2}{(s+\frac{1}{2})^3} \\ f(t) &= t^2\, e^{-t/2} \end{aligned}$$

The optimal results are shown in Table 1 and diags 3, 4.

Example 4 (Ang [1])

$$\begin{aligned} g(s) &= \frac{\Gamma(\beta)}{(s+\beta)^\alpha}, \qquad \alpha > 0 \\ f(t) &= t^{\alpha-1}\, e^{-\beta t} \end{aligned}$$

We have taken $\alpha = 2, \;\; \beta = 5/2$.

The optimal result is shown in Table 1 and diag 5.

Table

Example	a	T	h	λ	$V(\lambda)$	$\|f - f_\lambda\|_\infty$	diag.
1	2.8	6.8	0.10625	0.1099×10^{-5}	0.7147×10^{-4}	0.02	1
2	4.0	11.0	0.17188	0.1291×10^{-10}	0.6431×10^{-7}	0.01	2
3	5.0	11.40	0.17812	0.210×10^{-9}	0.1188×10^{-5}	0.02	3
4	2.70	9.20	0.14375	0.11×10^{-4}	0.1312×10^{-3}	0.044	5

Conclusion

Our method worked well over all the test examples. The results obtained are very good as shown in diags (1–3, 5). As compared with exponential sampling method and Varah's method [2, 3, 22], our results compare reasonably well and over a wider range of the values of t as shown in the respective diagrams.

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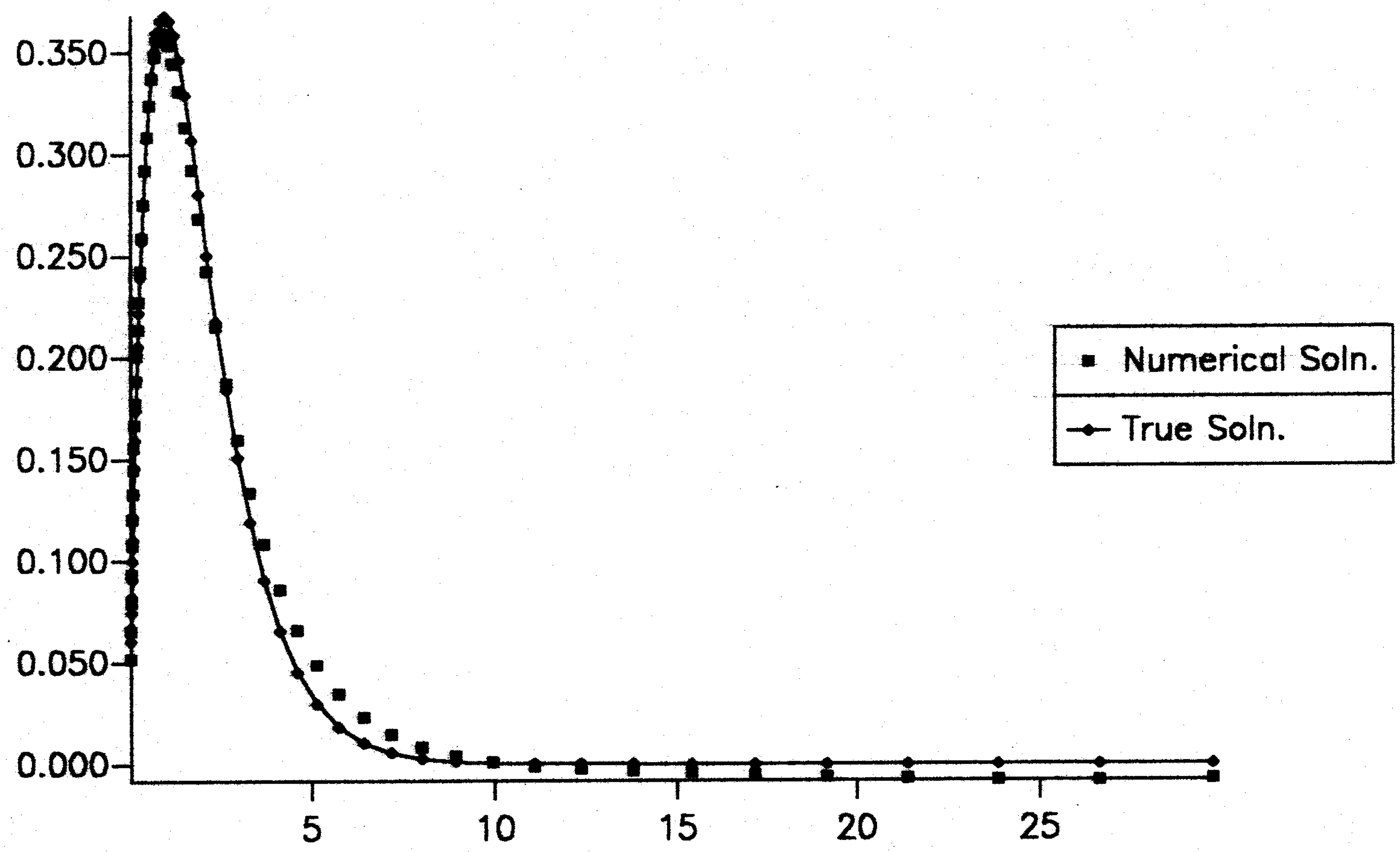
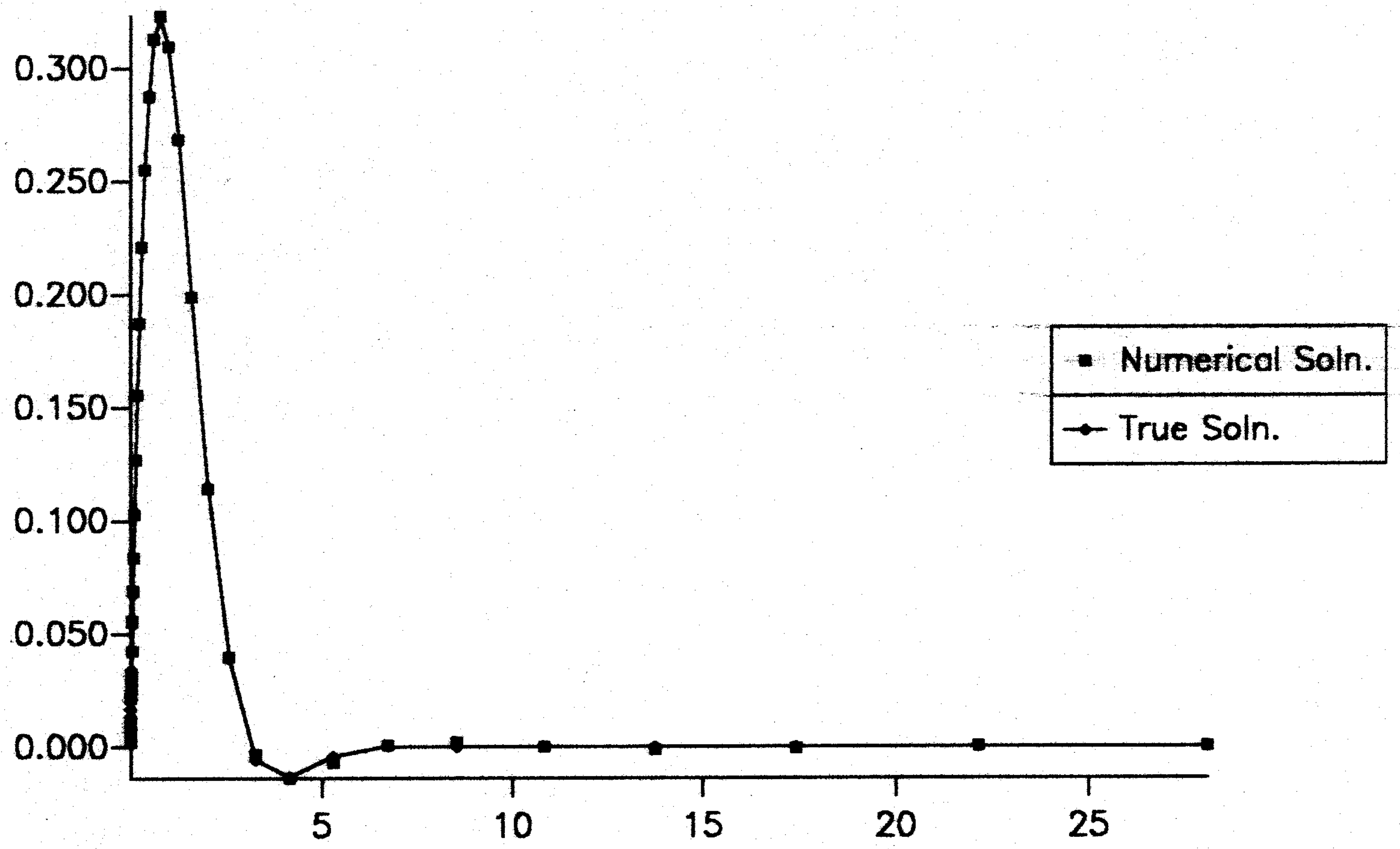
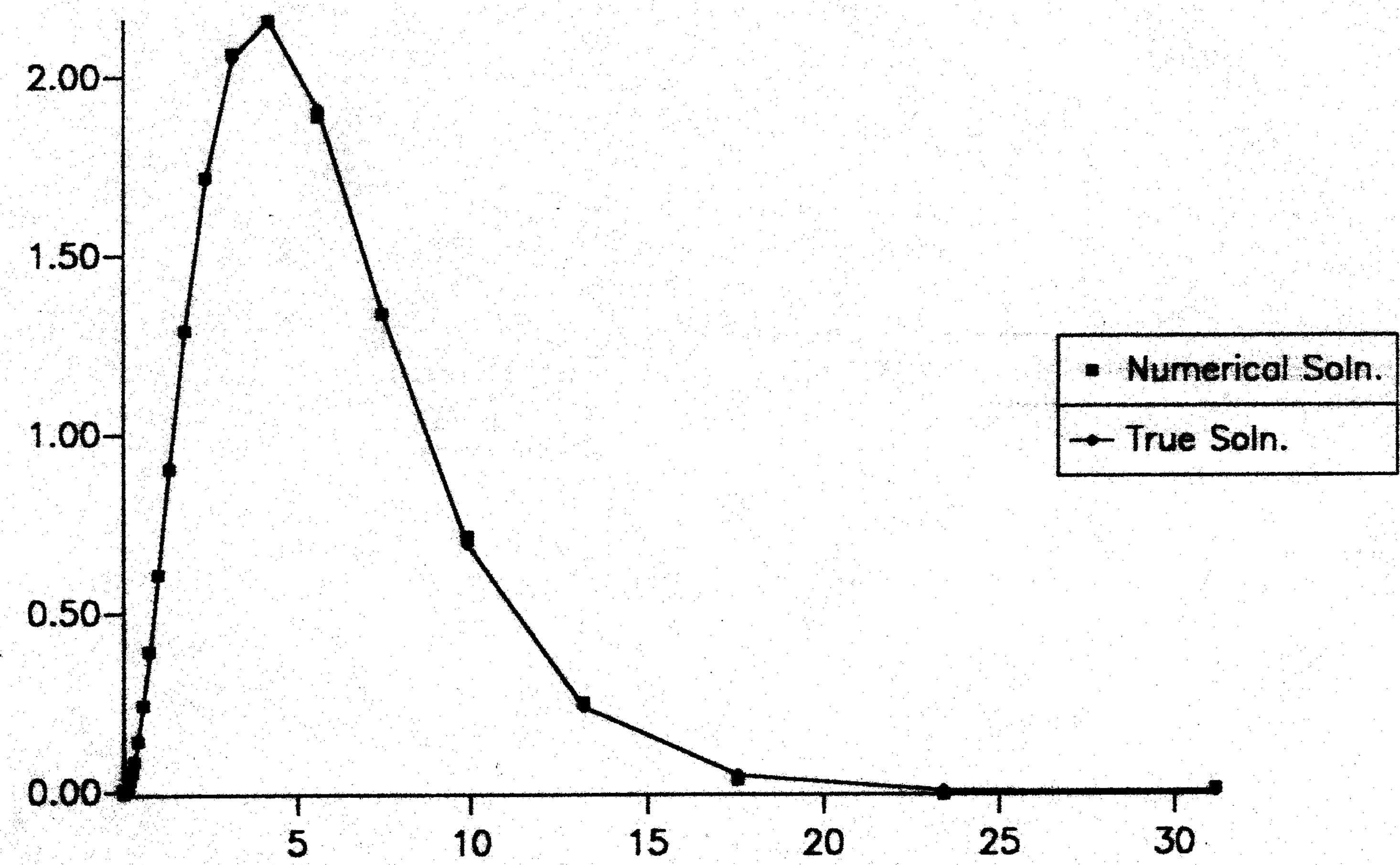
Diag(1) Spline Method.



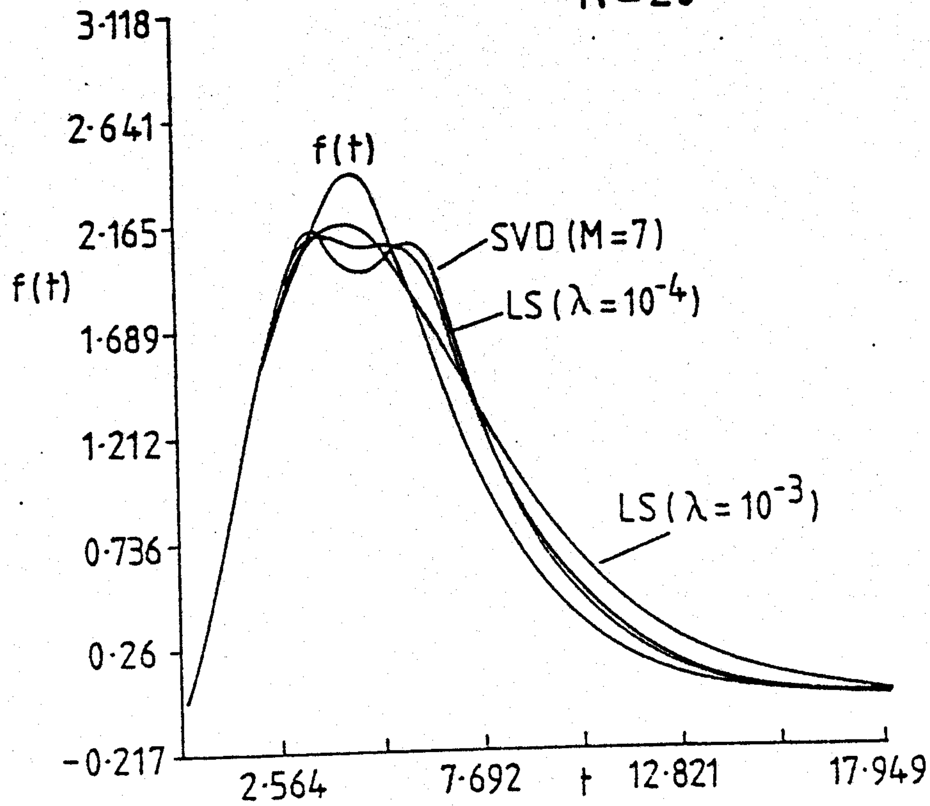
Diag(2) Spline Method.



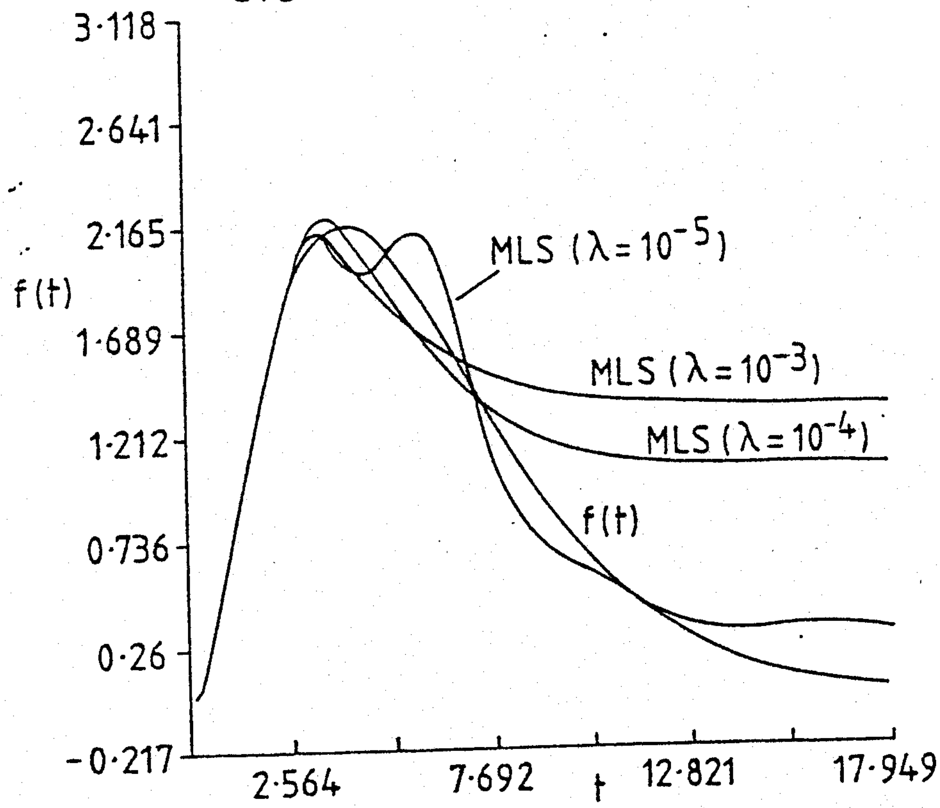
Diag(3) Spline Method.



DIAG (4) VARAH'S EXAMPLE 3
N = 20



SVD and LS solutions for Example 3.



MLS solutions for Example 3.

Diag(5) Spline Method.