King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 170

March 1995

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ON A GENERALIZATION OF THE EULER GAMMA FUNCTION WITH APPLICATIONS

By

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Abstract

In this paper we have introduced a generalization of the Euler gamma function. Some properties of the function including recurrence relation, special cases, reflection formula, differential equation, and Laplace transforms are discussed. It is proved that the Fourier cosine and sine transforms of a class of functions can be simplified in terms of the generalized gamma function. A generalization of the classical psi (digamma) function is introduced as well. Some properties of the function are discussed. A tabular and graphical representation of the functions is also given.

1991 Mathematics Subject Classification: 33B20, 33D90, 33E99

Keywords and Phrases: Euler's gamma function, reflection formula, generalized incomplete gamma functions, Fourier sine and cosine transform, psi function, decomposition formula Laplace and Mellin transforms.

1. Introduction

Euler (1730) introduced the gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \quad \mathrm{Re}\alpha > 0, \tag{1.1}$$

in connection with the integral interpolation of the factorial function $\alpha!$ [5]. Several properties of the function (1.1) were discovered by Euler. Some of these properties include [5, 9]

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{1.2}$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \tag{1.3}$$

$$\Gamma(\alpha)\Gamma(1-\alpha) = \pi \csc \pi\alpha, \quad (\alpha \neq 0, \pm 1, \pm 2, \pm 3, \ldots), \tag{1.4}$$

$$\Gamma(-\alpha) = \frac{-\pi \csc \pi\alpha}{\alpha\Gamma(\alpha)}, \quad (\alpha \neq 0, \pm 1, \pm 2, \ldots). \tag{1.5}$$

The duplication formula [4, p. 44]

$$\Gamma(2\alpha) = \pi^{1/2}\, 2^{2\alpha-1}\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right), \tag{1.6}$$

was discovered by Legendre. Legendre (1811) considered the incomplete gamma functions [5, 9]

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1}e^{-t}dt, \quad \text{Re } \alpha > 0, \tag{1.7}$$

and

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1}e^{-t}dt, \tag{1.8}$$

for real values of x. The functions (1.2) – (1.3) have been found useful in the analytic representation of a considerable body of problems in physics, astronomy, communication and engineering [3, 5, 9, 13].

It can be seen that the functions (1.7) – (1.8) satisfy the decomposition formula

$$\gamma(\alpha, x) + \Gamma(\alpha, x) = \Gamma(\alpha). \tag{1.9}$$

Chaudhry and Zubair [2] introduced the generalized incomplete gamma functions

$$\gamma(\alpha, x; b) = \int_0^x t^{\alpha-1}e^{-t-bt^{-1}}dt, \quad (\text{ for } b = 0, \ \text{Re } \alpha > 0), \tag{1.10}$$

and

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1} e^{-t-bt^{-1}} dt, \tag{1.11}$$

in connection with the analytic study of a variety of transient heat conduction problems [3, 13]. The functions (1.10) – (1.11) satisfy the decomposition formula [2]

$$\gamma(\alpha, x; b) + \Gamma(\alpha, x; b) = 2b^{\alpha/2} K_\alpha(2\sqrt{b}), \qquad \text{Re } b > 0, \tag{1.12}$$

where K_α is the Macdonald function [9]. According to [12, p. 136]

$$\Gamma(\alpha) = \lim_{b \to 0} 2b^{\alpha/2} K_\alpha(2\sqrt{b}). \tag{1.13}$$

In view of (1.1) – (1.3) it seems natural to consider the integral function

$$\Gamma_b(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t-bt^{-1}} dt, \quad (\text{for } b = 0, \ \text{Re } \alpha > 0), \tag{1.14}$$

as a generalization of the Euler integral function (1.1). For $\text{Re}\, b > 0$, we have the closed form representation [11]

$$\Gamma_b(\alpha) = 2b^{\alpha/2} K_\alpha(2\sqrt{b}). \tag{1.15}$$

It should be noted that $\Gamma_b(\alpha)$ is defined for all values of α for $\text{Re}\, b > 0$. However, for $b = 0$ we must have $\text{Re}\, \alpha > 0$.

It is to be noted that Askey has used the notation $\Gamma_q(x)$ for the q–analogue of the Euler gamma function. For the definition and properties of $\Gamma_q(x)$ we refer to [1, 10]. We emphasize that besides its closed–form representation (1.15) for $\text{Re}\, b > 0$, the function $\Gamma_b(\alpha)$ has not been discussed independently in the literature. In this paper, we have proved some interesting properties of this function. Some of the classical results of the Euler gamma function are recovered as special cases. We have also introduced an extension

$$\psi_b(\alpha) = \frac{\partial}{\partial \alpha} \{\ln \Gamma_b(\alpha)\}, \tag{1.16}$$

of the classical psi (gamma) function [11]

$$\psi(\alpha) = \frac{d}{d\alpha}\{\ln \Gamma(\alpha)\}. \tag{1.17}$$

Some properties of the function $\psi_b(\alpha)$ are proved. For numerical and scientific purposes, tabular and graphical representations of the functions $\Gamma_b(\alpha)$ and $\psi_b(\alpha)$ are given as well.

2. Properties of $\Gamma_b(\alpha)$

In this section we state properties of the generalized gamma function. The proofs of these properties are straightforward. We have stated the properties as theorems.

Theorem (2.1) (Recurrence relation).

$$\Gamma_b(\alpha+1) = \alpha\Gamma_b(\alpha) + b\Gamma_b(\alpha-1). \tag{2.1}$$

Proof. Let M be the Mellin transform operator as defined by [7, p. 305]. Then $\Gamma_b(\alpha)$ can be expressed as the Mellin transform of $e^{-t-bt^{-1}}$ in α to give

$$\Gamma_b(\alpha) = M\left\{e^{-t-bt^{-1}};\alpha\right\}. \tag{2.2}$$

Using the property of [7, p. 307(9)] of the Mellin transform, we find that

$$-(\alpha-1)\Gamma_b(\alpha-1) = M\left\{(-1+bt^{-2})e^{-t-bt^{-1}};\alpha\right\},$$

that can be simplified to give

$$-(\alpha-1)\Gamma_b(\alpha-1) = -\Gamma_b(\alpha) + b\Gamma_b(\alpha-2),$$

which implies

$$\Gamma_b(\alpha) = (\alpha-1)\Gamma_b(\alpha-1) + b\Gamma_b(\alpha-2). \tag{2.3}$$

Replacing α by $\alpha+1$ in (2.3) yields the proof of (2.1). In particular, for $b=0$ in (2.1), we get the classical result (1.3) for the Euler gamma function. For $\alpha=0$ in (2.1) we get

$$\Gamma_b(1) = b\Gamma_b(-1), \quad b \neq 0. \tag{2.4}$$

Theorem (2.2) (Reflection formula)

$$\Gamma_b(-\alpha) = b^{-\alpha}\Gamma_b(\alpha), \quad b \neq 0. \tag{2.5}$$

Proof. The substitutions $t = b\tau^{-1}$ and $dt = -b\tau^{-2}\,d\tau$ in (1.14) yield the proof of (2.5). In particular, it follows from (2.5) that $\Gamma_1(\alpha)$ is a symmetric function of α.

Remark. According to [5] we do not have a simple differential equation satisfied by the Euler gamma function $\Gamma(\alpha)$. However, $\Gamma_b(\alpha)$ satisfies a simple differential equation in the parameter b.

Theorem (2.3) (Differential equation of $\Gamma_b(\alpha)$). $\Gamma_b(\alpha)$ is a solution to the differential equation

$$b\frac{\partial^2 u}{\partial b^2} + (1-\alpha)\frac{\partial u}{\partial b} - u = 0. \tag{2.6}$$

Proof. Firstly we note that for $n = 0, 1, 2, 3, \ldots$

$$\frac{\partial^n}{\partial b^n}\{\Gamma_b(\alpha)\} = (-1)^n\Gamma_b(\alpha-n). \tag{2.7}$$

From (2.3) and (2.7) we find that

$$\Gamma_b(\alpha) = -(\alpha-1)\frac{\partial}{\partial b}\{\Gamma_b(\alpha)\} + b\frac{\partial^2}{\partial b^2}\{\Gamma_b(\alpha)\}. \tag{2.8}$$

The rearrangement of the terms in (2.8) yields the proof of (2.6). In particular, we find that $\Gamma_x(1)$ is a solution to the ordinary differential equation

$$x\frac{d^2u}{dx^2} - u = 0. \tag{2.9}$$

Theorem (2.4) (Summation formula)

$$\Gamma_{a+b}(\alpha) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \Gamma_b(\alpha - n), \quad b > 0, \ a + b \geq 0. \tag{2.10}$$

Proof. According to the definition (1.14), we have

$$\Gamma_{a+b}(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t-(a+b)t^{-1}} dt. \tag{2.11}$$

Replacing $e^{-at^{-1}}$ in (2.11) by its series representation and interchanging the summation and integral signs we get (2.10).

Corollary. (see [9, p.100])

$$\left(\frac{a+b}{b}\right)^{\alpha/2} K_\alpha\left(2\sqrt{a+b}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} (-a/\sqrt{b})^n K_{\alpha-n}(2\sqrt{b}) \quad (\text{Re } b > 0, \text{Re}(a+b) > 0). \tag{2.12}$$

Proof. This follows from (1.15) and (2.10).

Theorem (2.5) (Special cases)

$$\Gamma_b\left(n + \frac{1}{2}\right) = \left(\frac{\pi}{\sqrt{b}}\right)^{1/2} b^{(2n+1)/4} e^{-2\sqrt{b}} \sum_{m=0}^{n} \frac{1}{m!} (4\sqrt{b})^{-m} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \quad (\text{Re } b > 0, n = 0, 1, 2, 3, \ldots) \tag{2.13}$$

Proof. According to [11, p. 969(8.432)(16)]

$$\Gamma_b\left(n + \frac{1}{2}\right) = 2b^{(2n+1)/4} K_{n+\frac{1}{2}}(2\sqrt{b}), \quad (\text{Re } b > 0, n = 0, 1, 2, 3, \ldots). \tag{2.14}$$

However [9, p. 10(40)]

$$K_{n+\frac{1}{2}}(2\sqrt{b}) = \left(\frac{\pi}{4\sqrt{b}}\right)^{1/2} e^{-2\sqrt{b}} \sum_{m=0}^{n} \frac{1}{m!} (4\sqrt{b})^{-m} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \quad (n = 0, 1, 2, 3, \ldots). \tag{2.15}$$

From (2.14) – (2.15) our proof of (2.13) is complete. In particular for $n = 0$ in (2.13),

$$\Gamma_b\left(\frac{1}{2}\right) = \sqrt{\pi} e^{-2\sqrt{b}}. \tag{2.16}$$

Theorem (2.6) (Partial differentiation at the integral values of α)

$$\frac{\partial}{\partial\alpha}\{\Gamma_b(\alpha)\}_{\alpha=n} = \frac{1}{2}\left[(\ln b)\Gamma_b(n) + n!\sum_{k=0}^{n-1}\frac{\Gamma_b(k)}{k!(n-k)}\right], \quad (\text{Re } b > 0, \;\; n = 0,1,2,3,\ldots). \tag{2.17}$$

Proof. Differentiating (1.15) with respect to α we get

$$\frac{\partial}{\partial\alpha}\{\Gamma_b(\alpha)\} = \frac{1}{2}\ln b\left\{2b^{\alpha/2}K_\alpha(2\sqrt{b})\right\} + 2b^{\alpha/2}\frac{\partial}{\partial\alpha}\left\{K_\alpha(2\sqrt{b})\right\}. \tag{2.18}$$

However, according to [11, p. 983(8.486(1)(9)]

$$\frac{\partial}{\partial\alpha}\left\{K_\alpha(2\sqrt{b})\right\}_{\alpha=n} = \frac{1}{2}n!b^{-n/2}\sum_{k=0}^{n-1}\frac{b^{k/2}K_k(2\sqrt{b})}{k!(n-k)}. \tag{2.19}$$

From (2.18) – (2.19) our proof of (2.17) is complete.

Theorem (2.7) (Laplace transform representation). Let L be the Laplace transform operator as defined by [7, p. 129]. Then

$$L\left\{t^{\alpha-1}e^{-1/t};p\right\} = p^{-\alpha}\Gamma_p(\alpha). \tag{2.20}$$

Proof. Straightforward. Moreover, using the reflection formula (2.5), we can write

$$L\left\{t^{\alpha-1}e^{-1/t};p\right\} = \Gamma_p(-\alpha). \tag{2.21}$$

In particular for $\alpha = -1/2$ in (2.21) we get [7, p. 146(28)]

$$L\left\{t^{-3/2}e^{-1/t};p\right\} = \Gamma_p\left(\frac{1}{2}\right) = \sqrt{\pi}\, e^{-2\sqrt{p}}, \quad p > 0. \tag{2.22}$$

Theorem (2.8) (Laplace transform of $\Gamma_b(\alpha)$).

$$L\left\{t^\nu\Gamma_t(\alpha);p\right\} = \Gamma(\nu+\alpha+1)\Gamma(\nu+1)p^{-\frac{1}{2}(2\nu+\alpha+1)}e^{1/2p}W_{-\nu-\frac{\alpha}{2}-\frac{1}{2},\frac{\alpha}{2}}(1/p)$$
$$(p > 0, \nu > -1, \nu+\alpha+1 > 0). \tag{2.23}$$

Proof. This follows from (1.14) when we use the fact [7, p. 199(37)]. In particular for $\nu = 0$ the Whittaker function in (2.23) simplifies to the incomplete gamma function [8, p. 432] to give [7, p. 199(36)]

$$L\{\Gamma_t(\alpha); p\} = \Gamma(\alpha+1)p^{-\alpha-1}e^{1/p}\Gamma(-\alpha, 1/p), \quad (\alpha > -1, p > 0). \tag{2.24}$$

Moreover, the substitution $\alpha = 0$ in (2.24) yields [7, p. 199(32)]

$$L\{\Gamma_t(0); p\} = -\frac{1}{p}\, e^{1/p}\mathrm{Ei}(-1/p), \quad p > 0. \tag{2.25}$$

Theorem (2.9). (Fourier Cosine and Sine transform representation) Let F_c and F_s be the Fourier cosine and sine transform operators as defined by [7, p. 7] and [7, p. 63] respectively. Then

$$F_c\left\{x^{\alpha-1}e^{-1/x}; \omega\right\} = \frac{\omega^{-\alpha}}{2}\left[e^{\frac{i\pi}{2}\alpha}\Gamma_{-i\omega}(\alpha) + e^{-\frac{i\pi}{2}\alpha}\Gamma_{i\omega}(\alpha)\right], \tag{2.26}$$

and

$$F_s\left\{x^{\alpha-1}e^{-1/x}; \omega\right\} = \frac{\omega^{-\alpha}}{2i}\left[e^{\frac{i\pi}{2}\alpha}\Gamma_{-i\omega}(\alpha) - e^{-\frac{i\pi}{2}\alpha}\Gamma_{i\omega}(\alpha)\right], \quad \omega > 0. \tag{2.27}$$

Proof. Substituting $b = i\omega$, $\omega > 0$ in (1.24) we get

$$\Gamma_{i\omega}(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\left\{\cos(\omega/t) - i\sin(\omega/t)\right\} dt. \tag{2.28}$$

The substitutions $t = 1/x$ and $dt = -\frac{dx}{x^2}$ in (2.28) lead to

$$\Gamma_{i\omega}(\alpha) = \int_0^\infty x^{-\alpha-1}e^{1/x}\{\cos(\omega x) - i\sin(\omega x)\}dx, \tag{2.29}$$

which implies

$$\Gamma_{i\omega}(\alpha) = F_c\left\{x^{-\alpha-1}e^{-1/x}; \omega\right\} - iF_s\left\{x^{-\alpha-1}\, e^{-1/x}; \omega\right\}. \tag{2.30}$$

Separating the real and imaginary parts in (2.30) yields

$$F_c\left\{x^{-\alpha-1}e^{-1/x}; \omega\right\} = \frac{1}{2}\left[\Gamma_{-i\omega}(\alpha) + \Gamma_{i\omega}(\alpha)\right], \tag{2.31}$$

and

$$F_s\left\{x^{-\alpha-1}e^{-1/x};\omega\right\} = \frac{1}{2i}\left[\Gamma_{-i\omega}(\alpha) - \Gamma_{i\omega}(\alpha)\right], \tag{2.32}$$

Replacing α by $-\alpha$ in (2.31) – (2.32) and using the reflection formula (2.5) for the generalized gamma function, our proof of the theorem is complete.

In particular, the substitution $\alpha = 0$ in (2.31) – (2.32) yields

$$F_c\left\{\frac{1}{x}e^{-1/x};\omega\right\} = \frac{1}{2}\left[\Gamma_{-i\omega}(0) + \Gamma_{i\omega}(0)\right],$$

and

$$F_s\left\{\frac{1}{x}e^{-1/x};\omega\right\} = \frac{1}{2i}\left[\Gamma_{-i\omega}(0) - \Gamma_{i\omega}(0)\right].$$

Remark. Besides the fact that $\Gamma_b(\alpha)$ is a generalization of the Euler gamma function $\Gamma(\alpha)$, it is important to note that these functions are related via the integral representation

$$\Gamma(\alpha) = \int_0^\infty \Gamma_b(\alpha - 1)db, \quad (\alpha \neq 0, -1, -2, -3, \ldots), \tag{2.33}$$

which follows from (1.14) when we replace α by $\alpha - 1$ and integrate both sides with respect to b from $b = 0$ to $b = \infty$. The relation (2.33) could be exploited to evaluate certain integrals involving the generalized gamma function.

3. Generalization of Psi (digamma) Function $\psi(\alpha)$

The psi function $\psi(\alpha)$ is defined as the logarithmic derivative of the Euler gamma function [11, p. 952]

$$\psi(\alpha) = \frac{d}{d\alpha}\{\ln \Gamma(\alpha)\} = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \tag{3.1}$$

It seems natural to define

$$\psi_b(\alpha) = \frac{\partial}{\partial\alpha}\{\ln \Gamma_b(\alpha)\} = \frac{1}{\Gamma_b(\alpha)}\frac{\partial}{\partial\alpha}\{\Gamma_b(\alpha)\}, \tag{3.2}$$

as the generalization of the psi function. From (1.14) and (3.2), we find that

$$\psi_b(\alpha) = \frac{1}{\Gamma_b(\alpha)} \int_0^\infty t^{\alpha-1}(\ln t)e^{-t-bt^{-1}}\,dt. \tag{3.3}$$

In this section some of the properties of this function including recurrence relation and special cases have been presented.

Theorem (3.1) (Recurrence relation)

$$\left(\frac{\Gamma_b(\alpha+1)}{\alpha\Gamma_b(\alpha)}\right)\psi_b(\alpha+1) - \left(\frac{b\Gamma_b(\alpha-1)}{\alpha\Gamma_b(\alpha)}\right)\psi_b(\alpha-1) = \frac{1}{\alpha} + \psi_b(\alpha) \quad (\alpha \neq 0). \tag{3.4}$$

Proof. According to Chaudhry and Ahmad [4] (see also [11, p. 607(11)])

$$\int_0^\infty \ln t(t^\alpha - \alpha t^{\alpha-1} - bt^{\alpha-2})e^{-t-bt^{-1}}\,dt = 2b^{\alpha/2}K_\alpha(2\sqrt{b}) \quad (\mathrm{Re}\, b > 0), \tag{3.5}$$

which can be written in terms of the generalized gamma and psi functions to give

$$\Gamma_b(\alpha+1)\psi_b(\alpha+1) - \alpha\Gamma_b(\alpha)\psi_b(\alpha) - b\Gamma_b(\alpha-1)\psi_b(\alpha-1) = \Gamma_b(\alpha). \tag{3.6}$$

Dividing (3.6) by $\alpha\Gamma_b(\alpha)$ and rearranging the terms complete the proof of (3.4). In particular letting $b \to 0^+$ in (3.4), the classical recurrence formula [11, p. 952]

$$\psi(\alpha+1) = \frac{1}{\alpha} + \psi(\alpha), \tag{3.7}$$

for the psi function is recovered.

Theorem (3.2) (Reflection formula)

$$\psi_b(-\alpha) = \ln b - \psi_b(\alpha), \quad \mathrm{Re}\, b > 0. \tag{3.8}$$

Proof. Replacing α by $-\alpha$ in (3.3) we get

$$\Gamma_b(-\alpha)\psi_b(-\alpha) = \int_0^\infty t^{-\alpha-1} \ln t e^{-t-bt^{-1}}\,dt, \qquad \mathrm{Re}\, b > 0. \tag{3.9}$$

The substitutions $t = bx^{-1}$ and $dt = -bx^{-2}dx$ in (3.9) yield

$$\Gamma_b(-\alpha)\psi_b(-\alpha) = b^{-\alpha}\int_0^\infty x^{\alpha-1}(\ln b - \ln x)e^{-x-bx^{-1}}\,dx, \qquad (3.10)$$

which implies

$$\psi_b(-\alpha) = \frac{b^{-\alpha}}{\Gamma_b(-\alpha)} \quad (\ln b\Gamma_b(\alpha) - \Gamma_b(\alpha)\psi_b(\alpha)). \qquad (3.11)$$

However, according to the reflection formula (2.5)

$$\frac{b^{-\alpha}}{\Gamma_b(-\alpha)} = \frac{1}{\Gamma_b(\alpha)}. \qquad (3.12)$$

From (3.11) – (3.12), our proof of (3.8) is complete.

Corollary

$$\int_0^\infty (\ln t)(t^\alpha + (b/t)^\alpha)\,e^{-t-bt^{-1}}\,\frac{dt}{t} = 2(\ln b)K_\alpha\,(2\sqrt{b}), \quad (b>0,\ -\infty<\alpha<\infty). \qquad (3.13)$$

Proof. This follows from (3.8) and (3.3). In particular for $\alpha = 0$ in (3.13), the result [11, p. 605, 4.356(1)]

$$\int_0^\infty (\ln t)e^{-t-bt^{-1}}\,\frac{dt}{t} = (\ln b)K_0\,(2\sqrt{b}), \quad b>0, \qquad (3.14)$$

is recovered.

Theorem (3.3) (Special cases when $\alpha = n$)

$$\psi_b(n) = \frac{1}{2}\left[\ln b + \frac{n!}{\Gamma_b(n)}\sum_{m=0}^{n-1}\frac{\Gamma_b(m)}{m!(n-m)}\right], \quad (b>0,\ n=0,1,2,3,\ldots). \qquad (3.15)$$

Proof. This follows from (2.17) and (3.2).

Remark. It follows from (3.3) that

$$\psi_b(\alpha-1)\Gamma_b(\alpha-1) = \int_0^\infty t^{\alpha-2}(\ln t)e^{-t-bt^{-1}}\,dt, \quad b>0. \qquad (3.16)$$

Integrating both sides of (3.16) with respect to b from $b=0$ to $b=\infty$ we get

$$\int_0^\infty \psi_b(\alpha-1)\Gamma_b(\alpha-1)db = \psi(\alpha)\Gamma(\alpha), \quad (b>0, \alpha\neq 0,-1,-2,\ldots). \qquad (3.17)$$

In particular, the substitution $\alpha = 1$ in (3.17) yields [11, p. 955(8.367)(1)]

$$\int_0^\infty \psi_b(0)\Gamma_b(0)db = -\gamma = -0.57721566490\ldots, \tag{3.18}$$

where γ is the Euler constant.

4. Tabular and Graphical Representations of the Functions $\Gamma_b(\alpha)$ and $\psi_b(\alpha)$

For numerical and scientific computations, the functions $\Gamma_b(\alpha)$ and $\psi_b(\alpha)$ can easily be tabulated by using IMSL FORTRAN subroutines for mathematical applications [14]. In this regard, the values of the functions are calculated by using the numerical integration subroutine QDAGI. The subroutine uses a globally adaptive scheme in an attempt to reduce the absolute error, which is fully documented in Ref. [15].

The normalized function $\Gamma_b^*(\alpha) = \Gamma_b(\alpha)/\Gamma(\alpha)$ and the generalized psi function $\psi_b(\alpha)$, for different values of the parameters α and b are given in both Tables (1) – (2) and Figs. (1) – (2), respectively. It should be noted that the first column of Table (1) represents the values of the Euler gamma function $\Gamma(\alpha)$ and the first column of Table (2) represents the values of the classical psi function $\psi(\alpha)$; so an easy comparison with existing tables (or approximations) can be made. We emphasize that the generalized psi function (cf. Eq. (3.3)) is defined somewhat in a normalized form. We therefore do not find it necessary to further normalize the function for tabular and graphical representations.

Acknowledgment.

The authors appreciate Prof. Asghar Qadir for reviewing the manuscript. The support provided by the King Fahd University of Petroleum and Minerals through the research project MS/GAMMA/171 is acknowledged.

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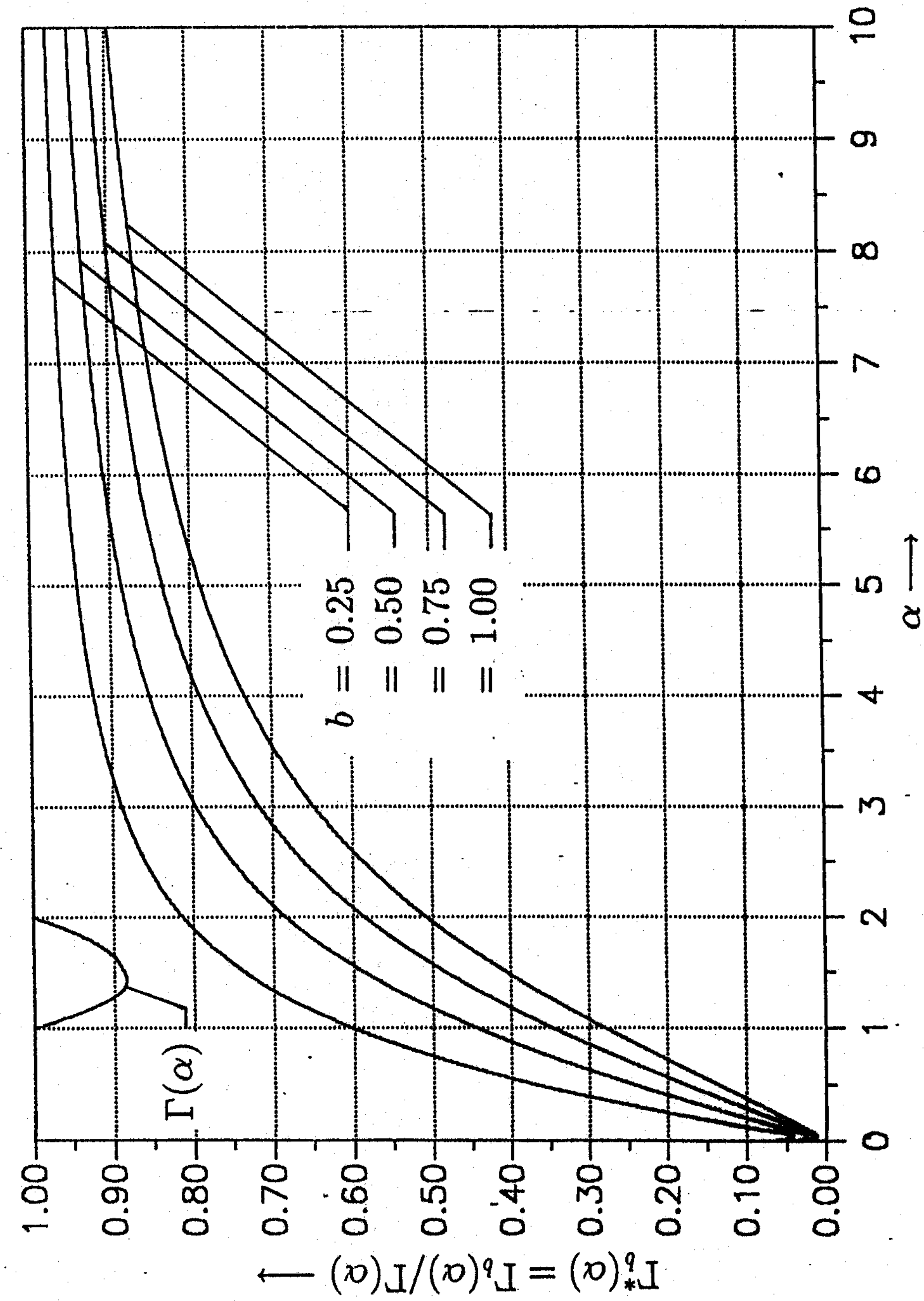
Table 1. Normalized representation of the generalized gamma function, $\Gamma_b(\alpha)$

α	$\Gamma(\alpha)$	$\Gamma^*_{0.25}(\alpha)$	$\Gamma^*_{0.50}(\alpha)$	$\Gamma^*_{0.75}(\alpha)$	$\Gamma^*_{1.00}(\alpha)$
0.1000	9.5106	0.0829	0.0487	0.0330	0.0240
0.1500	6.2193	0.1230	0.0735	0.0503	0.0368
0.2000	4.5901	0.1621	0.0983	0.0679	0.0500
0.2500	3.6254	0.1998	0.1231	0.0858	0.0636
0.3000	2.9914	0.2363	0.1477	0.1039	0.0776
0.3500	2.5461	0.2713	0.1721	0.1222	0.0918
0.4000	2.2181	0.3050	0.1962	0.1404	0.1062
0.4500	1.9681	0.3371	0.2199	0.1587	0.1207
0.5000	1.7724	0.3679	0.2431	0.1769	0.1353
0.5500	1.6161	0.3972	0.2659	0.1950	0.1500
0.6000	1.4892	0.4251	0.2881	0.2129	0.1648
0.6500	1.3848	0.4515	0.3097	0.2307	0.1795
0.7000	1.2980	0.4767	0.3308	0.2482	0.1942
0.7500	1.2254	0.5005	0.3513	0.2654	0.2088
0.8000	1.1642	0.5231	0.3711	0.2824	0.2232
0.8500	1.1124	0.5445	0.3904	0.2990	0.2376
0.9000	1.0686	0.5647	0.4090	0.3154	0.2518
0.9500	1.0314	0.5838	0.4270	0.3314	0.2659
1.0000	1.0000	0.6019	0.4443	0.3470	0.2797
1.2500	0.9064	0.6783	0.5224	0.4198	0.3459
1.5000	0.8862	0.7358	0.5869	0.4834	0.4060
1.7500	0.9191	0.7792	0.6400	0.5383	0.4598
2.0000	1.0000	0.8124	0.6835	0.5854	0.5075
2.2500	1.1330	0.8381	0.7193	0.6257	0.5495
2.5000	1.3293	0.8584	0.7490	0.6603	0.5865
2.7500	1.6084	0.8745	0.7738	0.6899	0.6189
3.0000	2.0000	0.8877	0.7946	0.7155	0.6474
3.2500	2.5493	0.8984	0.8122	0.7377	0.6725
3.5000	3.3233	0.9074	0.8273	0.7570	0.6947
3.7500	4.4230	0.9150	0.8403	0.7738	0.7144
4.0000	6.0000	0.9215	0.8515	0.7887	0.7320
4.2500	8.2851	0.9271	0.8614	0.8018	0.7477
4.5000	11.6317	0.9320	0.8701	0.8135	0.7617
4.7500	16.5862	0.9362	0.8778	0.8240	0.7744
5.0000	23.9999	0.9400	0.8846	0.8334	0.7859
5.5000	52.3426	0.9464	0.8963	0.8496	0.8059
6.0000	119.9996	0.9515	0.9059	0.8630	0.8225
6.5000	287.8843	0.9558	0.9139	0.8742	0.8366
7.0000	719.9971	0.9594	0.9207	0.8838	0.8487
7.5000	1871.2488	0.9624	0.9264	0.8921	0.8592
8.0000	5039.9805	0.9650	0.9315	0.8992	0.8683
8.5000	14034.3633	0.9673	0.9358	0.9055	0.8763
9.0000	40319.8828	0.9693	0.9397	0.9111	0.8835
9.5000	119292.0620	0.9711	0.9431	0.9160	0.8898
10.0000	362879.0620	0.9727	0.9461	0.9204	0.8955

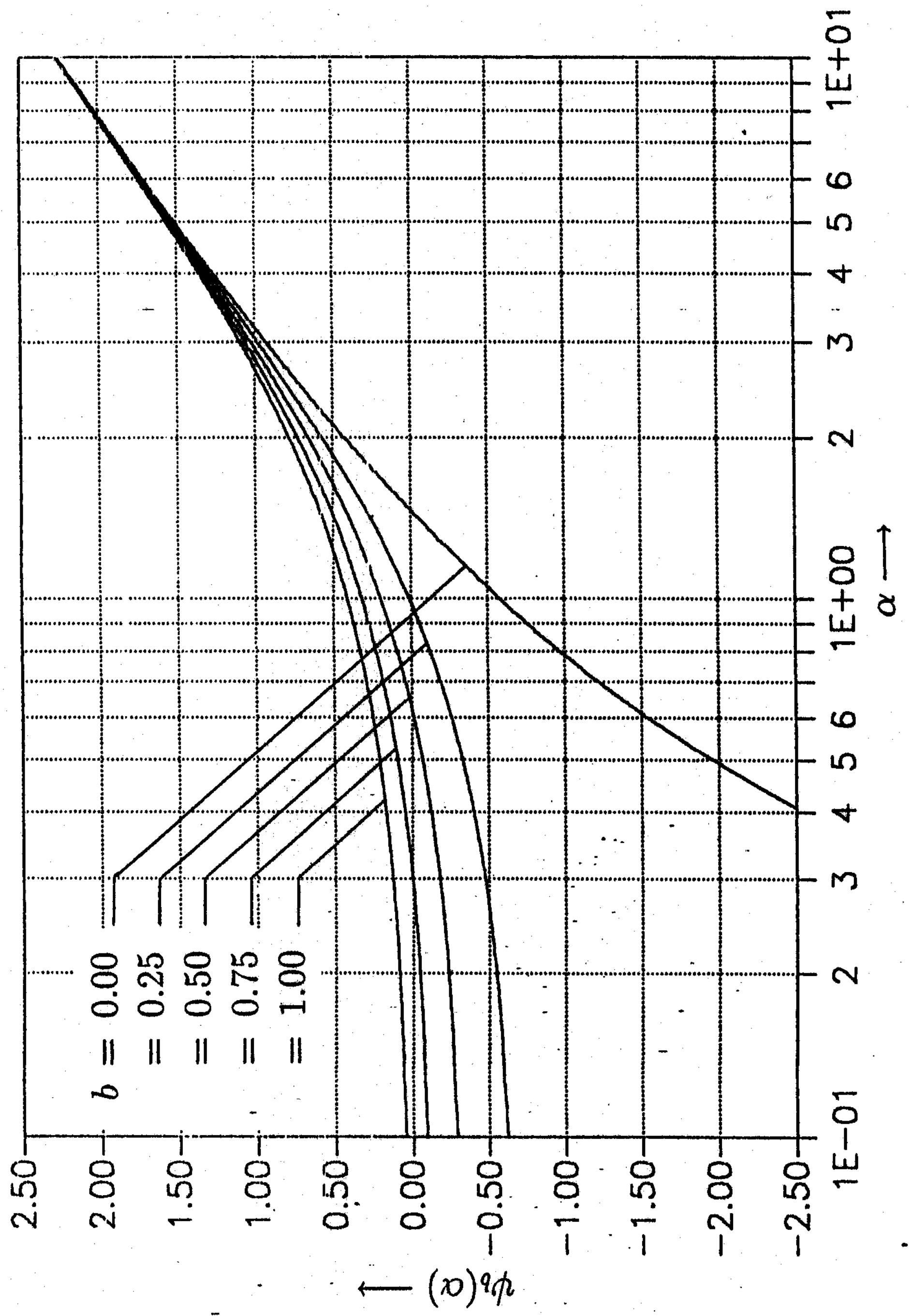
$$\Gamma^*_b(\alpha) = \Gamma_b(\alpha)/\Gamma(\alpha)$$

Table 2. Representation of the generalized psi function, $\psi_b(\alpha)$

α	$\psi(\alpha)$	$\psi_{0.25}(\alpha)$	$\psi_{0.50}(\alpha)$	$\psi_{0.75}(\alpha)$	$\psi_{1.00}(\alpha)$
0.1000	-4.1971	-0.6201	-0.2912	-0.0970	0.0415
0.1500	-4.3147	-0.5836	-0.2635	-0.0736	0.0622
0.2000	-4.0426	-0.5472	-0.2359	-0.0503	0.0829
0.2500	-3.6401	-0.5109	-0.2083	-0.0269	0.1035
0.3000	-3.2235	-0.4747	-0.1808	-0.0037	0.1242
0.3500	-2.9736	-0.4387	-0.1534	0.0196	0.1448
0.4000	-2.5608	-0.4029	-0.1261	0.0428	0.1654
0.4500	-2.2340	-0.3672	-0.0988	0.0659	0.1859
0.5000	-1.9634	-0.3318	-0.0716	0.0890	0.2063
0.5500	-1.7361	-0.2966	-0.0446	0.1119	0.2268
0.6000	-1.5405	-0.2617	-0.0177	0.1348	0.2471
0.6500	-1.3702	-0.2271	0.0090	0.1576	0.2674
0.7000	-1.2201	-0.1927	0.0356	0.1803	0.2876
0.7500	-1.0859	-0.1587	0.0621	0.2029	0.3078
0.8000	-0.9650	-0.1250	0.0884	0.2254	0.3278
0.8500	-0.8553	-0.0916	0.1145	0.2478	0.3478
0.9000	-0.7550	-0.0586	0.1404	0.2701	0.3677
0.9500	-0.6626	-0.0259	0.1661	0.2922	0.3875
1.0000	-0.5772	0.0063	0.1916	0.3142	0.4072
1.2500	-0.2275	0.1618	0.3160	0.4220	0.5040
1.5000	0.0365	0.3069	0.4347	0.5258	0.5979
1.7500	0.2475	0.4415	0.5472	0.6254	0.6885
2.0000	0.4228	0.5660	0.6534	0.7204	0.7756
2.2500	0.5725	0.6809	0.7535	0.8109	0.8591
2.5000	0.7032	0.7870	0.8475	0.8968	0.9389
2.7500	0.8189	0.8852	0.9359	0.9783	1.0152
3.0000	0.9228	0.9762	1.0190	1.0556	1.0880
3.2500	1.0170	1.0607	1.0971	1.1289	1.1573
3.5000	1.1032	1.1395	1.1706	1.1983	1.2235
3.7500	1.1825	1.2131	1.2400	1.2643	1.2865
4.0000	1.2561	1.2822	1.3055	1.3269	1.3467
4.2500	1.3247	1.3471	1.3676	1.3865	1.4041
4.5000	1.3889	1.4084	1.4264	1.4432	1.4590
4.7500	1.4492	1.4663	1.4823	1.4973	1.5115
5.0000	1.5061	1.5212	1.5355	1.5489	1.5618
5.5000	1.6111	1.6231	1.6346	1.6456	1.6562
6.0000	1.7061	1.7159	1.7254	1.7345	1.7433
6.5000	1.7929	1.8010	1.8089	1.8166	1.8240
7.0000	1.8728	1.8796	1.8863	1.8928	1.8992
7.5000	1.9468	1.9526	1.9583	1.9639	1.9694
8.0000	2.0156	2.0207	2.0257	2.0305	2.0353
8.5000	2.0801	2.0845	2.0888	2.0931	2.0973
9.0000	2.1406	2.1445	2.1483	2.1521	2.1558
9.5000	2.1977	2.2012	2.2046	2.2079	2.2113
10.0000	2.2517	2.2548	2.2579	2.2609	2.2638



1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0
1
2
3
4
5
6
7
8
9
10
Γ(α)
b = 0.25
= 0.50
= 0.75
= 1.00
α ⟶
Γ*_b(α) = Γ_b(α)/Γ(α) ⟶



b = 0.00
= 0.25
= 0.50
= 0.75
= 1.00
2.50
2.00
1.50
1.00
0.50
0.00
−0.50
−1.00
−1.50
−2.00
−2.50
$\psi_b(\alpha)$ ⟶
1E−01
2
3
4
5
6
1E+00
2
3
4
5
6
1E+01
α ⟶