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OPEN MAPPING PRINCIPLE FOR CONVEX MULTIFUNCTIONS UNDER PSEUDO–REGULARITY ASSUMPTION

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Abstract

We derive an open mapping principle for a closed convex multifunction $F : X \to Y$ between Banach spaces. We generalize the celebrated result of Robinson by relying only on a pseudo–regularity assumption and on the closedness of the horizon of F. These two hypotheses are shown to be minimal. Some applications are discussed.

Key words: Convex multifunction, open mapping principle, pseudo–interior, pseudo–regularity, horizon of sets and multifunctions.

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1 Introduction.

The well–known open mapping theorem of functional analysis can be extended to the case of a closed convex multifunction. As shown in a remarkable paper by Robinson [Ro], such an extension is possible under a suitable regularity assumption made on the range of the multifunction. The fundamental result of Robinson [Ro, Theorem 1] is as follows:

Theorem 1.1. *Let X and Y be Banach spaces, and let $F : X \rightarrow Y$ be a closed convex multifunction, i.e.*

$$Gr\ F := \{(x,y) \in X \times Y : \quad y \in F(x)\}$$

is a closed convex set in the product space $X \times Y$. Let y be a regular value of F, i.e. y is an interior point of

$$R(F) := \cup\{F(x) : x \in X\}.$$

Then,

$$\left\{ \begin{array}{l} \text{for each } x \in F^{-1}(y) \text{ there is a positive } \eta \text{ such that} \\ \quad y + \lambda\eta B_Y \subset F(x + \lambda B_X\} \quad \text{for all } \lambda \in [0,1], \end{array} \right. \tag{1.1}$$

where F^{-1} denotes the inverse of F, and B_X and B_Y are the closed unit balls in the spaces X and Y, respectively.

It is not difficult to show that the conclusion (1.1) amounts to saying that F is *open* at y, that is to say,

$$\left\{ \begin{array}{c} \text{for every } x \in F^{-1}(y) \text{ and } r > 0, \text{ one has} \\ y \in \text{int } F(x + rB_X). \end{array} \right. \tag{1.2}$$

The above definition and a new proof of Theorem 1.1 appear in Borwein [Bo, Theorem 6.2]. Several variants of the concept of openness can be found in the literature; see [Bo], [BZ], [Mo], [MS], [Pe] and the references therein. A discussion on these variants would lead us however too far away from the main concern of this paper. The attention of our work is focussed toward Robinson's regularity assumption

$$y \in \text{int } R(F). \tag{1.3}$$

To which extent is it possible to remove this interiority condition without destroying completely the conclusion (1.1)? It is interesting to note that Borwein [Bo, Theorem 6.2] replaces (1.3) by

$$y \in \text{core } R(F). \tag{1.4}$$

In this paper we show that it is possible to work with a much weaker regularity assumption, and still obtain a certain open behavior of F.

2 Pseudo–Interior and Horizon of a Convex Set.

There have been many attempts to generalize the definition of the interior of a convex set. Goodrich and Limber [GL] compile and compare some of those definitions. The reader may consult also the original sources: Borwein and Lewis [BL], Gowda and Teboulle [GT], among others. The following concept of interiority will play an important role in the sequel.

Definition 2.1. Let Q be a nonempty convex set in a linear space Y. The *pseudo-interior* of Q, denoted by $\text{pi}(Q)$, is the set of those $y \in Q$ for which

$$R_+(Q-y) := \bigcup_{\lambda \geq 0} \lambda(Q-y)$$

is a subspace.

The next proposition shows that $\text{pi}(Q)$ is the set of those $y \in Q$ for which $R_+(Q-y)$ is "maximized" with respect to the set–inclusion ordering.

Proposition 2.1. *Let $Q \subset Y$ be a convex set with nonempty pseudo–interior. Then the following statements are equivalent:*

(a) $\overline{y}$ is a pseudo–interior point of Q;

(b) $\overline{y} \in Q$ and $R_+(Q - y) \subset R_+(Q - \overline{y})$ for all $y \in Q$.

In particular,

$$R_+(Q - y_1) = R_+(Q - y_2) \quad \textit{for all } y_1, y_2 \in pi(Q).$$

Proof. Consider first the implication (a) $\Rightarrow$ (b). Assume that $\overline{y} \in \mathrm{pi}\ (Q)$. Let $y \in Q$ and $p \in R_+(Q - y)$. Then, there exist $\lambda \geq 0$ and $q \in Q$ such that $p = \lambda(q - y)$. Thus, p can be written as difference

$$p = \lambda(q - \overline{y}) - \lambda(y - \overline{y})$$

of two elements in the linear space $R_+(Q - \overline{y})$. This shows that $p \in R_+(Q - \overline{y})$. Consider now the reverse implication (b) $\Rightarrow$ (a). Pick up any $\tilde{y} \in \mathrm{pi}(Q)$. From the first part of the proof, it follows that

$$R_+(Q - \overline{y}) \subset R_+(Q - \tilde{y}).$$

But, under the assumption (b), one has also

$$R_+(Q - \tilde{y}) \subset R_+(Q - \overline{y}).$$

Thus, the set $R_+(Q - \overline{y}) = R_+(Q - \tilde{y})$ is a subspace. □

According to Proposition 2.1, for each convex set $Q \subset Y$ there is a subspace $H \subset Y$ such that

$$R_+(Q - y) = H \quad \text{for all } y \in \mathrm{pi}(Q).$$

In fact, the subspace H can be characterized in a very simple way.

Definition 2.2. The *horizon* $H(Q)$ of the convex set $Q \subset Y$ is the smallest subspace in Y which contains the Minkowski difference $Q - Q$.

Proposition 2.2. *Suppose the convex set Q has at least one pseudo–interior point. Then,*

$$R_+(Q - y) = H(Q) \quad \textit{for all } y \in pi(Q).$$

Proof. Let $y \in \text{pi}(Q)$. Then,

$$Q - q \subset R_+(Q - q) \subset R_+(Q - y) \quad \text{for all } q \in Q.$$

This proves that the subspace $R_+(Q - y)$ contains $Q - Q$, and therefore

$$H(Q) \subset R_+(Q - y).$$

To prove the reverse inclusion, take any $p \in R_+(Q - y)$, i.e.

$$p = \lambda(q - y) \quad \text{with } \lambda \geq 0 \text{ and } q \in Q.$$

If $\lambda = 0$, then $p = 0 \in H(Q)$. If $\lambda > 0$, then

$$\frac{p}{\lambda} = q - y \in Q - Q \subset H(Q).$$

Again, $p \in \lambda\, H(Q) = H(Q)$. □.

3 Open Mapping Principle under Pseudo–Regularity.

The aim of this section is to extend Robinson's result to the case in which y fails to be a regular value of F in the usual sense. As shown in the next theorem, it is still possible to establish an open mapping principle under a weaker regularity assumption.

Definition 3.1. Let $F : X \to Y$ be a closed convex multifunction between Banach spaces. Then,

(a) $y \in Y$ is called a pseudo–regular value of F if y is a pseudo–interior point of $R(F)$.

(b) the horizon of F is defined as the subspace $H(R(F)) \subset Y$.

Theorem 3.1. *Let $F : X \to Y$ be a closed convex multifunction between Banach spaces. Assume the following two hypotheses*

(H1) the horizon $W := H(R(F))$ of F is closed;

(H2) $y \in Y$ is a pseudo–regular value of F.

Then,

$$\begin{cases} \text{for each } x \in F^{-1}(y) \text{ there is a positive } \eta \text{ such that} \\ y + \lambda\eta B_W \subset F(x + \lambda B_X) \quad \text{for all } \lambda \in [0,1]. \end{cases} \tag{3.1}$$

Proof. As in Robinson's setting [Ro, Theorem 1], it is enough to carry out the proof only for $\lambda = 1$. Indeed, if the result holds for $\lambda = 1$, then for any $\lambda \in [0,1]$ one has

$$\begin{aligned} F(x + \lambda B_X) &= F(\lambda(x + B_X) + (1-\lambda)x) \\ &\supset \lambda F(x + B_X) + (1-\lambda)F(x) \\ &\supset \lambda(y + \eta B_W) + (1-\lambda)y \\ &= y + \lambda\eta B_W. \end{aligned}$$

For notational convenience, we consider first the case $x = 0$ and $y = 0$. Robinson [Ro, p. 132] observes that $Z := \overline{F(B_X)} \subset Y$ is a closed convex set containing the origin $0 \in Y$. Here the upper bar denotes the closure operation in Y. Without Robinson's regularity assumption $0 \in \text{int } R(F)$, the set Z may fail to absorb the whole space Y.

However, with the pseudo–regularity condition $0 \in \text{pi}[R(F)]$, it is possible to show that Z absorbs the space W. First, observe that

$$F(B_X) \subset R(F) \subset R_+(R(F)) = H(R(F)), \tag{3.2}$$

the last equality being a consequence of Proposition 2.2. Thus,

$$Z = \overline{F(B_X)} \subset \overline{H(R(F))} = H(R(F)) = W.$$

To prove the absorbing property of Z, take any nonzero vector $w \in W$. The last equality in (3.2) implies that $w \in \lambda R(F)$ for some $\lambda \geq 0$. In fact, one can take $\lambda > 0$ and write

$$\lambda^{-1} w \in F(u) \quad \text{for some } u \in X.$$

If $u \in B_X$, then $\lambda^{-1} w \in F(B_X)$, and $w \in \lambda Z$. If $u \notin B_X$, then take $\alpha = \|u\|^{-1} \in]0,1[$. As in Robinson's proof, we see that

$$\alpha\lambda^{-1} w = \alpha\lambda^{-1} w + (1-\alpha)0 \in \alpha F(u) + (1-\alpha)F(0) \subset F(\alpha u).$$

So

$$\alpha\lambda^{-1} w \in F(B_X), \quad \textit{and} \quad w \in \alpha^{-1}\lambda Z.$$

This proves that any nonzero vector in W is absorbed by Z. From now on we regard F as a multifunction from X into the Banach space W. Since $F(B_X) \subset W$, one has

$$Z = \overline{F(B_X)} = \text{cl}_W F(B_X),$$

where "cl_W" denotes the closure operation in the space W. Also

$$\eta' B_W \subset \text{cl}_W F(B_X),$$

for some $\eta' > 0$. If η is any positive number less than η', then

$$\eta B_W \subset \text{int}_W \text{cl}_W F(B_X),$$

where the interior operation is taken with respect to the space W. Observe that $F(B_X)$ is the projection of the closed convex set

$$\text{Gr } F \cap (B_X \times W) \subset X \times W$$

into W. By applying [Ro, Lemma 1] one gets

$$\eta B_W \subset \text{int}_W F(B_X) \subset F(B_X).$$

Finally, consider the case in which $(x, y) \neq (0, 0)$. Let $F_{x,y} : X \to Y$ be the closed convex multifunction whose graph is given by

$$\text{Gr } F_{x,y} = \text{Gr } F - (x, y).$$

More explicitly,

$$F_{x,y}(u) = F(x + u) - y \quad \text{for all } u \in X.$$

Since $R(F_{x,y}) = R(F) - y$, one sees that the horizon of $F_{x,y}$ is the closed subspace

$$H(R(F_{x,y})) = H(R(F) - y) = H(R(F)) = W.$$

Also, one observes that 0 is a pseudo–regular value of $F_{x,y}$. Since $0 \in F_{x,y}^{-1}(0)$, we know that there is a positive η such that

$$\eta B_W \subset F_{x,y}(B_X) = F(x + B_X) - y.$$

This completes the proof of the theorem. □

By taking into account Proposition 2.2, it is clear that the combination of (H1) and (H2) is equivalent to the single assumption

(H) $\quad \bigcup_{\lambda \geq 0} \lambda(R(F) - y)$ is a closed subspace.

However, it is better to separate (H1) and (H2) in order to understand the role played by each one of these hypotheses. The next two examples show that neither (H1) nor (H2) can be removed from the statement of Theorem 3.1.

Example 3.1. Let $X = Y = \ell^2(R)$ be the Hilbert space of square–summable sequences in R, and let $F : X \to Y$ be the linear continuous operator defined by

$$F(x) = (2^{-n} x_n)_{n \in N} \quad \text{for all } x = (x_n)_{n \in N} \in X.$$

This operator is considered in a different context by Bauschke and Borwein [BB, p. 435]. In this example the hypothesis (H1) does not hold. Indeed, the horizon of F coincides with $R(F)$, and the latter set is not closed. One can check that

$$v = (2^{-n})_{n \in N} \in \overline{R(F)} \backslash R(F).$$

Is it possible then to drop the hypothesis (H1) and state the conclusion (3.1) with $W = \overline{H(R(F))}$? The answer is no. To see this, take $y = 0$ and $x = 0$. Hypothesis (H2) holds because $R_+(R(F) - y) = R(F)$ is a subspace. However, there is no positive η such that $\eta B_W \subset F(B_X)$. This is because $\|v\|^{-1}\, v \in B_W$ but $\|v\|^{-1}\, v \notin R(F) \supset \eta^{-1} F(B_X)$.

Example 3.2. Let $F : R \to R^2$ be given by $F(x) = R_+ \times \{0\}$ for all $x \in R$. The horizon of this closed convex multifunction is the closed subspace $W = R \times \{0\}$, so the assumption (H1) holds. The point $y = (0,0)$ belongs to $R(F) = R_+ \times \{0\}$ but it is not a pseudo–regular value of F. The conclusion (3.1) fails in this example. Indeed, the ball $B_W = [-1, 1] \times \{0\}$ is not contained in a set of the form $\eta^{-1} F(x + B_R) = R_+ \times \{0\}$.

Closedness of the horizon of F and pseudo–regularity of y turn out to be conditions that are necessary and sufficient for obtaining the result of Theorem 3.1. The necessity of these conditions is the matter of the theorem stated below.

Theorem 3.2. *Suppose $F : X \to Y$ is a closed convex multifunction between normed linear spaces. Let W denote the closure of the horizon of F, and let $y \in R(F)$. If there are $x \in F^{-1}(y)$ and $\eta > 0$ such that*

$$y + \eta B_W \subset F(x + B_X),$$

then the horizon of F is closed and y is a pseudo–regular value of F.

Proof. Let us prove first that y is a pseudo–regular value of F. Since $F(x + B_X) \subset R(F)$, one can write

$$y + \eta B_W \subset R(F) \quad \text{for some } \eta > 0.$$

Hence

$$B_W \subset \frac{1}{\eta}(R(F) - y) \subset R_+(R(F) - y). \tag{3.3}$$

We need to prove that the cone $M := R_+(R(F) - y)$ is a subspace, i.e. $M \subset -M$. Take any $v \in M$. If $v \in B_W$, then $-v \in B_W \subset M$, and consequently $v \in -M$. If $v \notin B_W$, then we take $\epsilon > 0$ small enough so that $\epsilon v \in B_W$. In this case $-\epsilon v \in B_W$, and

$$-\epsilon v \in \frac{1}{\eta}(R(F) - y).$$

Thus

$$-v \in \frac{1}{\epsilon\eta}(R(F) - y) \subset M.$$

Let us prove now that F has a closed horizon. Suppose to the contrary that there is some vector v in $W \backslash H(R(F))$. In this case we can find a small $\epsilon > 0$ such that

$$\epsilon v \in B_W := W \cap B_Y \quad \text{and} \quad \epsilon v \notin H(R(F)).$$

But, from (3.3) and Proposition 2.2 it follows that

$$\epsilon v \in R_+(R(F) - y) = H(R(F)),$$

which is clearly a contradiction. This completes the proof of the theorem. □

The following definition emerges as a natural substitute of the concept of openness when Robinson's regularity assumption fails.

Definition 3.2. Let $F : X \to Y$ be a closed convex multifunction between Banach spaces. F is said to be *pseudo–open* at $y \in R(F)$ if

$$\left\{ \begin{array}{c} \text{for every } x \in F^{-1}(y) \text{ and } r > 0, \text{ one has} \\ y \in \mathrm{int}_W F(x + rB_X), \end{array} \right. \tag{3.4}$$

where W stands for the closure of the horizon of F.

The next result summarizes our discussion on Theorems 3.1 and 3.2.

Corollary 3.1. *Let $F : X \to Y$ be a closed convex multifunction between Banach spaces. Let $y \in R(T)$. Then the following statements are equivalent:*

(a) F is pseudo–open at y;

(b) F has closed horizon and (3.4) holds with $W = H(R(F))$;

(c) F has closed horizon and (3.1) holds with $W = H(R(F))$;

(d) F has closed horizon and y is a pseudo–regular value of F.

By applying Theorems 3.1 and 3.2 to the case of a constant multifunction, one gets straightforwardly:

Corollary 3.2. *Let Q be a nonempty closed convex set in a Banach space Y. Suppose the horizon $W := H(Q)$ of Q is closed. Then*

$$pi(Q) = int_W(Q).$$

Proof. Let X be any Banach space, and let $F : X \to Y$ be given by

$$F(x) = Q \quad \text{for all } x \in X.$$

The inclusion $\mathrm{pi}(Q) \subset \mathrm{int}_W(Q)$ is obtained by applying Theorem 3.1 to the constant multifunction F. The reverse inclusion follows from Theorem 3.2. □.

Observe that Theorem 3.1 is an extension of Robinson's result. Indeed, if $y \in Y$ is a regular value of F, then y is a pseudo–regular value of F, and the horizon $W = H(R(F))$ of F is the whole space Y. Theorem 3.1 extends also the classical open mapping theorem of functional analysis.

Corollary 3.3. *Let X and Y be two Banach spaces. Let $A : X \to Y$ be a linear continuous operator whose range $W = R(A)$ is closed. Then $0 \in \mathrm{int}_W A(B_X)$.*

4 Applications.

4.1 Distance Estimate for the Inverse Multifunction.

For each $y \in R(F)$, consider the abstract inverse problem

$$\text{Find } x \in X \text{ such that } y \in F(x). \tag{4.1}$$

In this case $F^{-1}(y)$ corresponds to the set of solutions to (4.1). How does the solution set $F^{-1}(y)$ behave with respect to changes in the "parameter" y? This question is at the origin of numerous papers dealing with the stability of variational problem under perturbations. Without further ado, we state the following result in which we use the notation:

$$D(F) := \{x \in X : F(x) \neq \phi\} \quad , \quad \text{domain of } F;$$

$$d[\cdot, A] = \inf_{a \in A} \| \cdot - a\| \quad , \quad \text{distance to the set } A \,.$$

Proposition 4.1. *Let $F : X \to Y$ be a closed convex multifunction between Banach spaces. Suppose the horizon $W = H(R(F))$ of F is closed. Let $y_0 \in pi(R(T))$ and $x_0 \in F^{-1}(y_0)$ be chosen. Then there is a positive η such that*

$$d[x, F^{-1}(y)] \leq \frac{1 + \|x - x_0\|}{\eta} d[y, F(x)] \quad \textit{for all } x \in D(F) \textit{ and } y \in y_0 + \eta B_W.$$

In particular,

$$d[x_0, F^{-1}(y)] \leq \eta^{-1}\, d[y, F(x_0)] \quad \textit{for all } y \in y_0 + \eta B_W.$$

Proof. The proof of this proposition is based on Theorem 3.1. It follows basically the same steps as in [Ro, Theorem 2]. □

Proposition 4.1 is an extension of a celebrated result due to Robinson and Ursescu (see [AC, p. 54] or [AE, p. 132]). It is important to observe that by dropping the classical regularity condition $y_0 \in \text{int}(R(T))$, we have lost the information on the behavior of F^{-1} outside of W. However, we still know how does F^{-1} behave on a W–neighborhood of y_0, i.e. on a neighborhood of y_0 contained in the horizon of F.

4.2 Subdifferentiability of Optimal–Value Functions.

A general convex program can be written in the abstract form

$$\psi(y) := \inf_{x \in X} \{f(x) : \quad y \in F(x)\}. \tag{4.2}$$

The variable y in the feasible set

$$F^{-1}(y) = \{x \in X : \quad y \in F(x)\}$$

is regarded as a perturbation parameter. The differential properties of the optimal-value function ψ are reflected by the subdifferential mapping

$$y \in Y \mapsto \partial\psi(y) := \{y^* \in Y^* : \psi(y') \geq \psi(y) + \langle y^*, y' - y\rangle \quad \forall\, y' \in Y\},$$

where Y^* is the topological dual of Y, and $\langle\cdot,\cdot\rangle : \; Y^* \times Y \to R$ is the usual duality mapping between the spaces Y and Y^*.

Proposition 4.2. *Let X and Y be Banach spaces. Let $f : X \to R$ be a convex function, and $F : X \to Y$ be a closed convex multifunction with closed horizon. Then*

$$\partial\psi(y) \neq \phi \quad \textit{for all } y \in pi[R(F)].$$

Proof. Observe that the effective domain

$$\text{dom}\,\psi := \{y \in Y : \quad \psi(y) < +\infty\}$$

of ψ coincides with the set

$$D(F^{-1}) := \{y \in Y : \; F^{-1}(y) \neq \phi\} = R(F).$$

Let W be the horizon of F. Pick up any $y \in \text{pi}[R(F)]$. By applying Theorem 3.1 one gets

$$y + \eta B_W \subset R(F) \quad \text{for some } \eta > 0.$$

Since ψ is finite over $y + \eta B_W$, we see that

$$w \in W \mapsto \psi_y(w) := \psi(y + w) - \psi(y)$$

is finite over the ball ηB_W. The function ψ_y is also convex. Hence, $\partial\psi_y(0)$ is nonempty, i.e. there is some $w^* \in W^*$ such that

$$\psi_y(w) \geq \psi_y(0) + \langle w^*, w - 0\rangle \quad \text{for all } w \in W.$$

In other words, ψ_y is minorized over the space W by the linear continuous function $\langle w^*, \cdot \rangle$. To complete the proof it suffices to apply the Hahn–Banach extension theorem, and observe that $\partial\psi(y) = \partial\psi_y(0)$. □

Proposition 4.2 is related to a general subdifferentiability result due to Volle:

Corollary 4.1. [Vo, Corollary 1]. *Let ψ be any proper convex lower–semicontinuous function over the Banach space Y. If $R_+(dom\ \psi - y)$ is a closed subspace, then $\partial\psi(y)$ is nonempty.*

Proof. Apply Proposition 4.2 to the case in which $X = R$, $f(x) = x$, and

$$F(x) = \{y \in Y : \psi(y) \leq x\}.$$ □

Remark. Volle's proof is based on the use of Attouch–Brézis constraint qualification condition as a way to guarantee the attainment of an infimal–convolution (cf. [AB, Theorem 1.1]).

4.3 Lipschitz Behavior of Optimal–Value Functions.

The following result deals with the Lipschitz behavior of the optimal–value function ψ. The proof is inspired from Robinson [Ro, Corollary 1], but we rely on the open mapping principle as stated in Theorem 3.1.

Proposition 4.3. *Let X, Y, f and F be as in Proposition 4.2. Let W be the horizon of F, and ψ be the optimal–value function defined by (4.2). Assume that $y \in pi[R(F)]$ and that f is bounded from below over $F^{-1}(y)$. Then, there are positive constants δ and L such that*

$$|\psi(y_1) - \psi(y_2)| \leq L\|y_1 - y_2\| \quad \forall\, y_1, y_2 \in y + \delta B_W.$$

Proof. Consider the multifunction $G : X \times R \to Y$ given by

$$G(x,r) = \begin{cases} F(x) & \text{if } f(x) \leq r, \\ \phi & \text{otherwise}. \end{cases}$$

A simple calculus shows that G has closed convex graph, and $R(G) = R(F)$. Thus $H(G) = H(F) = W$ and $y \in \text{pi}[R(G)]$. Now, pick up any $x_0 \in F^{-1}(y)$. According to Theorem 3.1, there is a positive η such that

$$y + \eta B_W \subset G((x_0, f(x_0)) + B_{X \times R}),$$

where

$$B_{X \times R} := \{(x,r) \in X \times R : \max\{\|x\|, |r|\} \leq 1\}.$$

This means that for each $v \in B_W$ there is some (x_v, r_v) such that

$$f(x_v) \leq r_v,$$

$$y + \eta\, v \in F(x_v),$$

$$\max\{\|x_v - x_0\|,\ |r_v - f(x_0)|\} \leq 1.$$

Hence

$$\psi(y + \eta v) \leq f(x_v) \leq r_v \leq 1 + f(x_0).$$

In this way we have shown that the convex function ψ is bounded from above over $y + \eta B_W$. We also know that $\psi(y) > -\infty$. It follows that $w \in W \mapsto \psi(w)$ is Lipschitz over some ball $y + \delta B_W$ centered at y. □

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