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BILINEAR QUADRATIC OPTIMAL CONTROL: A RECURSIVE APPROACH

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Abstract — This paper deals with the time-varying bilinear quadratic optimal control problem. Using Adomian's decomposition method, we shall derive first a functional expansion for the input-output map of the system, then, transform the cost functional so that it yields, in a recursive manner, the optimal control. The optimal tracking problem is considered to illustrate the theory. An alternative method is derived which is proved to be more 'robust'.

Keywords: Time-varying Bilinear Systems, Optimal Control, Adomian's Decomposition, Functional Expansion.

Mathematics Subject Classification (1991): 49J15, 41A10, 49M27

1. Introduction

In recent years a great deal of effort has been devoted to the study of the optimal control of nonlinear systems. In particular the bilinear quadratic regulator problem was tackled with some success [1-10], however, the methods used led to difficult computations. In this paper, we shall first present a functional expansion for the input-output map of time-varying bilinear systems based on Adomian's decomposition method ([11] see also [12]). Then, we shall transform the original optimal control problem, namely a bilinear quadratic optimal control problem, into a recursive optimization problem so that at each step we obtain a term $u_*^{[k]}$ of the optimal control u_* so that ultimately $u_* = \sum_{k\geq 0} u_*^{[k]}$. Simultaneously, we get at each step, a term $x_*^{[k]}$ of the corresponding optimal state x_* and similarly, $x_* = \sum_{k\geq 0} x_*^{[k]}$.

To the author's knowledge, it is the first time that Adomian's decomposition method is used in optimal control. It is more systematic compared to ad-hoc techniques presented in [4,7,9] and less computationaly involved than existing methods.

The outline of the paper is as follows: In section 2, we shall present a functional expansion of the state of bilinear systems in function of the terms of a convergent series representation of the input. In section 3, we propose a recursive optimization approach for this class of problems and in section 4, we tackle the optimal tracking problem. In section 5, we shall derive an alternative method which is proved to be more robust in the sense that it yields the optimal control even if the previous method fails and a-fortiori the linear theory also fails. Finally in section 6, we draw some conclusions and give some ideas on future work.

2. Functional Expansion for Bilinear Systems

In this paper, we shall consider bilinear systems of the form

$$\frac{dx}{dt} = A(t)x + \sum_{j=1}^{p} u_j N_j(t)x + B(t)u \qquad , \qquad x(0) = x^0 \tag{2.1}$$

where $u(t) \in \mathbf{R}^p, x(t) \in \mathbf{R}^n$ for each $t \in [0,T]$ for some fixed $T > 0, u$ and x are respectively the input and the state, $A(t), B(t)$ and $N_j(t)$, $j = 1, \ldots, p$ are time-varying matrices with appropriate dimensions. If x is an n-vector, let $\mathcal{N}(t,x)$ denote the $n \times p$-matrix whose j^{th} column is equal to $N_j(t)x$ i.e., $\mathcal{N}(t,x) = [N_1(t)x| \ldots |N_p(t)x]$ and $\Phi(t,\tau)$ be the fundamental matrix solution of the equation $\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau) \quad , \quad \Phi(\tau,\tau) = I \quad$ (Identity).

For convenience, we shall not mention t explicitly in $A(t), B(t)$ and $N_j(t)$ and write A, B and N_j. We shall make the following assumption,

Assumption A.

(i) $\exists\, \alpha > 0,\ \rho > 0\ /\ |\Phi(t,\tau)| \le \rho e^{-\alpha(t-\tau)}, \quad 0 \le \tau \le t \le T$

(ii) $||B|| \le \beta$

(iii) $\exists L_1 > 0$ /
$|\mathcal{N}(t,x)u - \sum_{i=0}^{l-1}\sum_{r=0}^{i}\mathcal{N}(t,x^{[r]})u^{[i-r]}| \le C_1|x - \overline{x}_{l-1}| + C_2|u - \overline{u}_{l-1}|$,
$l \ge L_1 + 1$,where $\overline{x}_l = \sum_{i=0}^{l} x^{[i]}$, $\overline{u}_l = \sum_{i=0}^{l} u^{[i]}$.

(iv) $\widehat{\rho} = \rho C_1 < 1$

(iii) is due to the analyticity of $\mathcal{N}(t,x)u$.We claim the following theorem

Theorem 2.1. *Let $u \in L^\infty[0,T;\mathbf{R}^p]$ and suppose that u can be expanded in a uniformly and absolutely convergent series as $u = \sum_{l\ge 0} u^{[l]}$. Let assumption A be satisfied. Then, the solution x of (2.1) can be expanded in an absolutely and uniformly convergent series*

$$x = \sum_{i\ge 0} x^{[i]}(u^{[0]},\ldots,u^{[i]}) \tag{2.2}$$

where the $x^{[i]}, i \ge 0$ are defined by

$$\begin{cases} \frac{d}{dt}x^{[0]} = Ax^{[0]} + Bu^{[0]} & x^{[0]}(0) = x_0 \\ \frac{d}{dt}x^{[i+1]} = Ax^{[i+1]} + Bu^{[i+1]} + \sum_{r=0}^{i}\mathcal{N}(t,x^{[r]})u^{[i-r]} & x^{[i+1]}(0) = 0,\ i \ge 0 \end{cases} \tag{2.3}$$

$t \in [0,T], 0 \le T < \infty$.

Proof. Let assume, for the moment, that the solution x of (2.1) can be expanded in a convergent series i.e., $x = \sum_{i\ge 0} x^{[i]}$.
Replacing in equation (2.1) u and x with their expansions, we obtain

$$\frac{d}{dt}\sum_{i\ge 0} x^{[i]} = A\sum_{i\ge 0} x^{[i]} + \sum_{j=1}^{p}\sum_{m\ge 0} u_j^{[m]} N_j \sum_{r\ge 0} x^{[r]} + B\sum_{m\ge 0} u^{[m]} \tag{2.4}$$

which yields

$$\begin{aligned} \frac{d}{dt}\sum_{i\ge 0} x^{[i]} &= A\sum_{i\ge 0} x^{[i]} + \sum_{i\ge 0}\sum_{m+r=i}\sum_{j=1}^{p} u_j^{[m]} N_j x^{[r]} + B\sum_{i\ge 0} u^{[i]} \\ &= A\sum_{i\ge 0} x^{[i]} + \sum_{i\ge 0}\sum_{r=0}^{i}\sum_{j=1}^{p} u_j^{[i-r]} N_j x^{[r]} + B\sum_{i\ge 0} u^{[i]} \end{aligned} \tag{2.5}$$

We shall define the sequence of functions $\{x^{[i]}\}_{i\ge 0}$ as in (2.4).
Let $z_l = x - \overline{x}_l$. We have.

$$\frac{d}{dt}z_l = Az_l + B[u - \overline{u}_l] + [\mathcal{N}(t,x)u - \sum_{i=0}^{l-1}\sum_{r=0}^{i}\mathcal{N}(t,x^{[r]})u^{[i-r]}],\ z_l(0) = 0 \tag{2.6}$$

$$z_l(t) = \int_0^t \Phi(t,\tau)B[u - \overline{u}_l]d\tau + \int_0^t \Phi(t,\tau)[\mathcal{N}(\tau,x)u - \sum_{i=0}^{l-1}\sum_{r=0}^{i}\mathcal{N}(\tau,x^{[r]})u^{[i-r]}]d\tau \tag{2.7}$$

Hence, for $l \geq L_1 + 1$,

$$|z_l|e^{\alpha t} \leq \rho\beta\int_0^t e^{\alpha\tau}|u - \overline{u}_l|d\tau + \rho C_1\int_0^t e^{\alpha\tau}|z_{l-1}|d\tau + \rho C_2\int_0^t e^{\alpha\tau}|u - \overline{u}_{l-1}|d\tau \tag{2.8}$$

and for a given $\varepsilon > 0$, there exists $L_2 > 0$, such that for $l \geq L+1 = \max\{L_1, L_2\}+1$,

$$\widehat{z}_l \leq \widehat{\rho}\int_0^t \widehat{z}_{l-1}d\tau + \varepsilon \tag{2.9}$$

where $\widehat{z}_l = |z_l|e^{\alpha t}$, since $\overline{u}_l$ converges uniformly to u as l goes to infinity. We shall need the following lemma, whose proof is elementary,

Lemma 2.1. *If*

$$0 \leq \widehat{z}_l \leq \widehat{\rho}\int_0^t \widehat{z}_{l-1}d\tau + \varepsilon, \quad l \geq L+1$$

then

$$\widehat{z}_l \leq ||\widehat{z}_L||\frac{t^{l-L}}{(l-L)!} + \varepsilon\sum_{k=0}^{l-L-1}\widehat{\rho}^k\frac{t^k}{k!} \quad , \quad l \geq L+1.$$

Thus, for $l \geq L+1$,

$$\overline{z}_l \leq [||\overline{z}_L||\frac{t^{l-L}}{(l-L)!} + \varepsilon\sum_{k=0}^{l-L-1}\widehat{\rho}^k\frac{t^k}{k!}]e^{-\alpha t} \tag{2.10}$$

and clearly the left hand side of this inequality converges uniformly to zero as l goes to infinity. Therefore, $\overline{x}_l$ converges uniformly to x as l goes to infinity. Furthermore, the series $\sum_{i=0}^{\infty} x^{[i]}$ is absolutely convergent. Indeed,

$$\sum_{i=0}^{l}|x^{[i]}| \leq \rho\beta\int_0^t e^{-\alpha(t-\tau)}\sum_{i=0}^{l}|u^{[i]}|d\tau + \rho\int_0^t e^{-\alpha(t-\tau)}\sum_{i=0}^{l-1}|\sum_{r=0}^{i}\mathcal{N}(t,x^{[r]})u^{[i-r]}|d\tau \tag{2.11}$$

letting $\delta = \sum_{j=1}^{p}||N_j||$ and using Gronwall's lemma, we get,

$$\sum_{i=0}^{l}|x^{[i]}| \leq [\rho\beta\int_0^t e^{\alpha\tau}\sum_{i=0}^{l}|u^{[i]}|d\tau]\exp\{-\alpha t + \rho\delta\int_0^t\sum_{i=0}^{l-1}|u^{[i]}|d\tau\} < \infty \tag{2.12}$$

and hence, the series $\sum_{i=0}^{\infty} x^{[i]}$ is absolutely convergent which completes the proof of the theorem.

3. Recursive Bilinear Quadratic Optimal Control

In this section, we shall consider the bilinear quadratic regulator problem namely

$$\min_u J =: \frac{1}{2}x^T P(t)x_{|t=T} + \frac{1}{2}\int_0^T \left(x^T Q(t)x + u^T R(t)u\right) dt \qquad (3.1)$$

subject to the constraint (2.1), where $P(t)$ and $Q(t)$ are positive definite symmetric matrices and $R(t)$ is a positive definite symmetric matrix and z^T denotes the transpose of z. Here also, we shall write P, Q, R instead of $P(t), Q(t), R(t)$.

Let J_l denote the value of the cost functional when x is replaced by $\overline{x}_l$ and u by $\overline{u}_l$, that is

$$J_l = \frac{1}{2}\overline{x}_l^T P\overline{x}_{l|t=T} + \frac{1}{2}\int_0^T \left(\overline{x}_l^T Q\overline{x}_l + \overline{u}_l^T R\overline{u}_l\right) dt \qquad (3.2)$$

Suppose that the optimum values $u^{[0]}, \ldots, u^{[l-1]}$ and the corresponding $x^{[0]}, \ldots,$ $x^{[l-1]}$ have been obtained from the minimization of $J_0, \ldots, J_{l-1}$ with respect to $u^{[0]}, \ldots, u^{[l-1]}$ respectively. We shall seek the minimum of J_l with respect to $u^{[l]}$ if one exists, subject to the constraint:

$$\frac{d}{dt}x^{[l]} = Ax^{[l]} + Bu^{[l]} + \sum_{r=0}^{l-1} \mathcal{N}(t, x^{[r]})u^{[l-1-r]} \quad , \quad x^{[l]}(0) = 0 \qquad (3.3)$$

We shall make use of the following theorem which results from a minor variation to the standard linear quadratic optimal control [13, p175-176],

Theorem 3.1. *The solution to*

$$\min_w \ J := \frac{1}{2}(z-\overline{z})^T P(z-\overline{z})_{|t=T} + \frac{1}{2}\int_0^T \{(z-\overline{z})^T Q(z-\overline{z}) + (w-\overline{w})^T R(w-\overline{w})\}dt \qquad (3.4)$$

subject to the constraint

$$\frac{d}{dt}z = Az + Bw + F(t) \quad , \quad z(0) = z_0 \qquad (3.5)$$

is given by

$$w = \overline{w} - R^{-1}B^T\{S(z-\overline{z}) + v\} \qquad (3.6)$$

where $P(t)$ and $Q(t)$ are symmetric positive semidefinite matrices and $R(t)$ a symmetric positive definite matrix, S and v are solutions to the Riccati differential equation

$$\frac{dS}{dt} + SA + A^T S - SBR^{-1}B^T S + Q(t) = 0 \quad , \quad S(T) = P(T) \qquad (3.7)$$

and the linear differential equation

$$\frac{dv}{dt} + \{A^T - SBR^{-1}B^T\}v = S\{\frac{d\overline{z}}{dt} - A\overline{z} - B\overline{w} - F\} \quad , \quad v(T) = 0 \quad (3.8)$$

respectively. A and B are time-varying matrices whereas F is a time-varying vector function.

The objective function J_l can be written as,

$$\begin{aligned} J_l &= \frac{1}{2}(\overline{x}_{l-1} + x^{[l]})^T P(\overline{x}_{l-1} + x^{[l]})_{|t=T} + \\ &\frac{1}{2}\int_0^T \{(\overline{x}_{l-1} + x^{[l]})^T Q(\overline{x}_{l-1} + x^{[l]}) + (\overline{u}_{l-1} + u^{[l]})^T R(\overline{u}_{l-1} + u^{[l]})\}dt \end{aligned} \quad (3.9)$$

while the constraint is written,

$$\frac{dx^{[l]}}{dt} = Ax^{[l]} + Bu^{[l]} + F_{l-1}(x^{[0]}, ..., x^{[l-1]}, u^{[0]}, ..., u^{[l-1]}) \quad (3.10)$$

where $x^{[0]}(0) = x_0$, $x^{[l]}(0) = 0, l \geq 1$, $F_{-1} = 0$, and

$$F_{l-1}(x^{[0]}, ..., x^{[l-1]}, u^{[0]}, ..., u^{[l-1]}) = \sum_{r=0}^{l-1} N(x^{[r]})u^{[l-r-1]} \quad , \quad l \geq 1 \quad (3.11)$$

Now, according to theorem 3.1, the optimal control is given by,

$$u^{[l]} = -R^{-1}B^T\{Sx^{[l]} + v_{l-1}\} \quad , \quad l \geq 0 \quad , \quad v_{-1} = 0 \quad (3.12)$$

where S and v_i are solutions to the Riccati differential equation

$$\frac{dS}{dt} + SA + A^TS - SBR^{-1}B^TS + Q = 0 \quad , \quad S(T) = P(T) \quad (3.13)$$

and the linear differential equation

$$\frac{dv_i}{dt} + \{A^T - SBR^{-1}B^T\}v_i = -SF_i \quad , \quad v_i(T) = 0 \quad , \quad i \geq 0 \quad (3.14)$$

respectively where we have taken into account the differential equation satisfied by $x^{[i]}$.

Remark 3.1. The Riccati differential equation (3.13) is independent of the iteration step, so it can be solved once for all. Whereas equation (3.14) is a linear first order differential equation in which the coefficient of $\frac{dv_i}{dt}$ and the coefficient of v_i are independent of i. Only the forcing function depends upon the iteration step i. These equations will be integrated backward.

Hence, the first approximation to the optimal control law is given by $\overline{u}_0 = -R^{-1}B^T S\overline{x}_0$. Whereas the l^{th} approximation is given by

$$\overline{u}_{l-1} = -R^{-1}B^T(S\overline{x}_{l-1} + \overline{v}_{l-2}) \quad , \quad l \geq 2 \tag{3.15}$$

Remark 3.2. For linear systems, we obtain $\overline{u}_l = -R^{-1}B^T S\overline{x}_l$ and so, as l goes to infinity, we obtain $u = -R^{-1}B^T Sx$, where S is the solution of the Ricatti differential equation (3.13) with the boundary condition $S(T) = P(T)$ which agrees with the standard result of the linear quadratic regulator problem. Thus, we have proved in part the theorem 3.2 below.

Theorem 3.2. *The optimal control and state which minimize the objective function*

$$J = \frac{1}{2}x^T Px_{|t=T} + \frac{1}{2}\int_0^T \left(x^T Qx + u^T Ru\right) dt$$

subject to the constraint

$$\frac{dx}{dt} = Ax + Bu + \sum_{j=1}^{p} u_j N_j x, \quad , \quad x(0) = x_0$$

are given by the absolutely and uniformly convergent series $\sum_{l\geq 0} u^{[l]}$, *and* $\sum_{l\geq 0} x^{[l]}$, *where the* $u^{[l]}$*'s and* $x^{[l]}$*'s are defined by (3.12), (3.13) and (3.14).*

Before we prove this theorem, we shall need the following standard lemma.

Lemma 3.1. *Let* f *and* g *be two positive functions,* $g \in L^1[0,T]$. *If* u *is continuous on* $[0,T]$ *and satisfies*

$$0 \leq u(t) \leq f(t) + \int_t^T g(\tau)u(\tau)d\tau$$

for all $t \in [0,T]$ *then*

$$0 \leq u(t) \leq f(t) + \int_t^T f(\tau)g(\tau)e^{-\int_\tau^t g(\sigma)d\sigma}d\tau.$$

Proof:(of theorem 3.2)

Let $\Psi(t,\tau)$ be the $n \times n-$ matrix solution of

$$\frac{d}{dt}\Psi(t,\tau) + \left\{A^T - SBR^{-1}B^T\right\}\Psi(t,\tau) = 0 \ , \ \Psi(\tau,\tau) = I_{n\times n} \tag{3.16}$$

where $I_{n\times n}$ denotes the n by n identity matrix.

Let assume that $|\Psi(t,\tau)| \leq \beta e^{-\gamma(t-\tau)}$ for $0 \leq \tau \leq t \leq T$, and β and γ are positive constants.

Then the solution of (3.15) can be written as

$$v_i(t) = \int_t^T \Psi(t,\tau)S(\tau)F_i d\tau \tag{3.17}$$

Hence,

$$|v_i(t)| \leq \beta \int_t^T e^{-\gamma(t-\tau)}|S|.|F_i|d\tau \leq \beta\delta \int_t^T e^{-\gamma(t-\tau)}|S| \sum_{r=0}^{i} |x^{[r]}| \cdot |u^{[i-r]}|d\tau$$

$$\begin{aligned}
\sum_{i=0}^{l} |u^{[l]}| &\leq |R^{-1}| \cdot |B|(|S| \sum_{i=0}^{l} |x^{[l]}| + \beta\delta \int_t^T e^{-\gamma(t-\tau)}|S| \sum_{i=1}^{l}\sum_{r=0}^{i-1} |x^{[r]}| \cdot |u^{[l-1-r]}|d\tau) \\
&\leq |R^{-1}| \cdot |B|(|S| \sum_{i=0}^{l} |x^{[l]}| + \beta\delta \int_t^T e^{-\gamma(t-\tau)}|S| \sum_{i=0}^{l-1}\sum_{r=0}^{i} |x^{[r]}| \cdot |u^{[l-r]}|d\tau)
\end{aligned}$$

$$\sum_{i=0}^{l} |u^{[i]}| \leq |R^{-1}| \cdot |B|(|S| \sum_{i=0}^{l} |x^{[i]}| + \beta\delta \int_t^T e^{-\gamma(t-\tau)}|S| \sum_{i=0}^{l} |x^{[i]}| \sum_{i=0}^{l} |u^{[i]}|d\tau) \tag{3.18}$$

Using Lemma 2.1, we obtain

$$\begin{aligned}
\textstyle\sum_{i=0}^{l} |u^{[i]}| &\leq |R^{-1}| \cdot |B| \cdot |S| \textstyle\sum_{i=0}^{l} |x^{[i]}| + \beta\delta \int_t^T (|R^{-1}|.|B|.|S|. \textstyle\sum_{i=0}^{l} |x^{[i]}|)^2 \\
&\times \exp\{-\gamma(t-\tau) - \textstyle\int_\tau^t \beta\delta e^{-\gamma(\tau-\sigma)}|R^{-1}|.|B|.|S| \sum_{i=0}^{l} |x^{[i]}|d\sigma\} < \infty
\end{aligned} \tag{3.19}$$

if $\sum_{l\geq 0} |x^{[l]}| < \infty$. The previous inequality and equation (2.12) show that the series $\sum_{l\geq 0} u^{[l]}$ and $\sum_{l\geq 0} x^{[l]}$ converge together. This ends the proof of Theorem 3.2.

4. Optimal Tracking

In the previous section, we wanted to keep $x(t)$ as 'close' as possible to the origin while using the minimum control u. In this section, we shall consider the optimal tracking problem, that is making $x(t)$ to follow a prescribed trajectory $x_{ref}(t)$.

Explicitly, we shall solve

$$\min_u J =: \frac{1}{2}(x-x_{ref})^T P(x-x_{ref})_{|t=T} + \frac{1}{2}\int_0^T \left((x-x_{ref})^T Q(x-x_{ref}) + u^T Ru\right) dt \tag{4.1}$$

subject to the constraint (2.1), where again P and Q are positive semidefinite symmetric matrices and R is a positive definite symmetric matrix. P, Q and R are time-varying matrices.

With the same notation as in section 3 and along the same lines, we can state the following corollary,

Corollary 4.1. *The Optimal Control and State which minimize the objective function (4.1) subject to the constraint (2.1) are given by the absolutely and uniformly convergent series $\sum_{l\geq 0} u^{[l]}$ and $\sum_{l\geq 0} x^{[l]}$ whose terms $u^{[l]}$'s and $x^{[l]}$'s are given by*

$$u^{[l]} = -R^{-1}B^T(Sx^{[l]} + v_{l-1}) \tag{4.2}$$

where S and v_l satisfy

$$\frac{dS}{dt} + SA + A^TS - SBR^{-1}B^TS + Q = 0 \tag{4.3}$$

and

$$\frac{d}{dt}v_i + \left\{A^T - SBR^{-1}B^T\right\}v_i + S\sum_{r=0}^{i-1}\mathcal{N}(t,x^{[r]})u^{[i-1-r]} = Qx_{ref},, \quad i \geq 1 \tag{4.4}$$

together with the boundary conditions

$$\left\{\begin{array}{lcl} S(T) & = & P(T) \\ v_l(T) & = & -P(T)x_{ref}(T) \end{array}\right. \tag{4.5}$$

Remark 4.1. For linear systems, we obtain

$$\left\{\begin{array}{lclcl} \dot{v}_l & + & \left\{A^T - SBR^{-1}B^T\right\}v_l & = & Qx_{ref} \\ u^{[l]} & = & -R^{-1}B^T(Sx^{[l]} + v_{l-1}) & & \end{array}\right. \tag{4.6}$$

So, as l goes to infinity, we get $u = -R^{-1}B^T(Sx + v) = Qx_{ref}$ where v is solution of

$$\left\{\begin{array}{lclcl} \dot{v} & + & \left\{A^T - SBR^{-1}B^T\right\}v & = & Qx_{ref} \\ v(T) & = & -P(T)x_{ref}(T) & & \end{array}\right. \tag{4.7}$$

which agrees with the standard result.

5. An Alternative Method

From the above development, we can see that although the method introduced in previous sections provides an effective mean to compute the optimal control, it fails if $B = 0$. Indeed, in that case $\overline{u}_l$, and obviously $u = 0$ is not an optimal control.

We shall introduce in this section a modified version which overcomes this problem and therefore provides the desired optimal control.

Going back to (2.5), we shall define this time, the sequence of functions $\{x^{[i]}\}_{i\geq 0}$ as follows

$$\left\{\begin{array}{lclcl} \frac{d}{dt}x^{[0]} & = & Ax^{[0]}, & x^{[0]}(0) & = & x_0 \\ \frac{d}{dt}x^{[i+1]} & = & Ax^{[i+1]} + Bu^{[i]} + \sum_{r=0}^{i}\sum_{j=1}^{r} u_j^{[i-r]}N_jx^{[r]}, & x^{[i+1]}(0) & = & 0, \; i \geq 0 \end{array}\right. \tag{5.1}$$

that is

$$\begin{cases} \frac{d}{dt}x^{[0]} &= Ax^{[0]}, & x^{[0]}(0) &= x_0 \\ \frac{d}{dt}x^{[i+1]} &= Ax^{[i+1]} + [\mathcal{N}(t,x^{[0]}) + B]u^{[i]} & & \\ & +\sum_{r=1}^{i}\sum_{j=1}^{r} u_j^{[i-r]}N_j x^{[r]}, & x^{[i+1]}(0) &= 0,\ i \geq 0 \end{cases} \tag{5.2}$$

and so, we have the following theorem.

Theorem 5.1. *Let $u \in L^\infty[0,T,\mathbf{R}^p]$ such that u can be expanded in a uniformly and absolutely convergent series as $u = \sum_{l\geq 0} u^{[l]}$. Let assumption A be satisfied. Then the solution x of (2.1) can be expanded in an absolutely and uniformly convergent series*

$$x = x_0 + \sum_{i\geq 0} x^{[i+1]}(u^{[0]}, \ldots, u^{[i]})$$

where the $x^{[i]}$, $i \geq 0$ satisfy (5.2) , $t \in [0,T], 0 \leq T < \infty$.

Proof. Along the same lines as the proof of Theorem 2.1.

Next we shall redefine the objective function as follows:

$$J_l =: \frac{1}{2}\overline{x}_{l+1}^T P \overline{x}_{l+1}{}_{|t=T} + \frac{1}{2}\int_0^T \left(\overline{x}_{l+1}^T Q \overline{x}_{l+1} + \overline{u}_l^T R \overline{u}_l\right) dt \tag{5.3}$$

Suppose that the optimum values $u^{[0]}, \ldots, u^{[l-1]}$ and the corresponding $x^{[1]}, \ldots, x^{[l]}$ have been obtained from the minimization of $J_0, \ldots, J_{l-1}$ with respect to $u^{[1]}, \ldots, u^{[l-1]}$ respectively. We shall seek the minimum J_l with respect to $u^{[l]}$ subject to the constraint

$$\frac{dx^{[l+1]}}{dt} = Ax^{[l+1]} + [\mathcal{N}(t,x^{[0]}) + B]u^{[l]} + \sum_{r=1}^{l}\sum_{j=1}^{r} u_j^{[l-1]} N_j x^{[r]}\,,\ \ x^{[l+1]}(0) = 0\,,\ \ l \geq 0 \tag{5.4}$$

Using the same kind of development as in section 3, we obtain the following corollary,

Corollary 5.1. *The optimal control problem (5.3)will have the following solution:*

$$u^{[l]} = -R^{-1}[\mathcal{N}(t,x^{[0]}) + B]^T(Sx^{[l+1]} + v_l) \tag{5.5}$$

where S is the symmetric positive semidefinite matrix solution of the Ricatti equation

$$\frac{dS}{dt} + SA + A^TS - S[\mathcal{N}(t,x^{[0]}) + B]R^{-1}[\mathcal{N}(t,x^{[0]}) + B]^TS + Q = 0,\ \ S(T) = P(T) \tag{5.6}$$

and v_l is solution of the equation

$$\frac{dv_l}{dt} + \{A^T - S[\mathcal{N}(t,x^{[0]}) + B]R^{-1}[\mathcal{N}(t,x^{[0]}) + B]^T\}v_l \\ +S\sum_{i=1}^{i}\sum_{r=1}^{i}\mathcal{N}(t,x^{[r]})u^{[i-r]} = Qx_{ref}, \tag{5.7}$$

$v_l(T) = -P(T)x_{ref}(T)$, *together with (5.2).*

Remark 5.1. We can notice that even if $B(t) = 0$, we can still compute the optimal control, a point which underlines the efficiency of the method, (as a matter of fact $\mathcal{N}(t,x^{[0]}) + B(t) \neq 0$).

6. Conclusion

In this paper, we have solved the multidimensional bilinear quadratic optimal control problem. The method used, is recursive and consists on solving a matrix Ricatti differential equation once and updating the control using a linear differential equation with fixed dynamics and different forcing function at each step. The method has been used to solve successfully the tracking problem. An alternative method has also been derived to deal with situation covered neither by the previous method nor a-fortiori by the linear theory. This method compares favorably with the method in [4] in which one has to solve a Ricatti equation once and solving a sequence of linear differential equations with varying dynamics and forcing functions with increasing dimensions and the method in [7] and with the method in [9] which reduces to solving a sequence of time varying Lyapunov equations and is less computationally involved than other existing methods . We could have used Adomian's decomposition to solve directly the Hamilton-Jacobi-Bellman partial differential equation. However, this approach would have prevented us to gain any insight .We shall present in future papers extensions to the optimal control of nonlinear and distributed bilinear systems.

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