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TWO INTEGRALS INVOLVING A CLASS OF GENERALIZED INCOMPLETE GAMMA FUNCTIONS

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ABSTRACT

The closed form evaluation of the integrals

$$\int_0^\infty \Gamma_\nu(\alpha, x; b) b^{s-1} db \quad \text{and} \quad \int_0^\infty \gamma_\nu(\alpha, x; b) b^{s-1} db,$$

involving the extensions of the generalized incomplete gamma functions, is presented. Some special cases of these results are discussed.

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1. Introduction

The family of the classical incomplete gamma functions

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt, \quad \text{Re } \alpha > 0, \tag{1}$$

and

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt, \tag{2}$$

plays an important role in the analytic study of a considerable number of problems in applied mathematics, statistics, nuclear physics and engineering. Chaudhry and Zubair have recently considered the generalized incomplete gamma functions

$$\gamma(\alpha, x; b) = \int_0^x t^{\alpha-1} e^{-t-bt^{-1}}, \tag{3}$$

and

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1} e^{-t-bt^{-1}}, \quad (\text{Re } b \geq 0, \ x > 0), \tag{4}$$

found useful in a variety of transient heat conduction problems [4, 8, 9, 10].

It is to be noted that

$$\gamma(\alpha, x; 0) = \gamma(\alpha, x) \tag{5}$$

and

$$\Gamma(\alpha, x; 0) = \Gamma(\alpha, x). \tag{6}$$

Moreover, the generalized incomplete gamma functions (3) – (4) satisfy the decomposition formula

$$\gamma(\alpha, x; b) + \Gamma(\alpha, x; b) = 2b^{\alpha/2} K_\alpha(2\sqrt{b}), \quad \text{Re } b > 0. \tag{7}$$

By letting $b \to 0^+$ in (7), we get the classical decomposition formula

$$\gamma(\alpha, x) + \Gamma(\alpha, x) = \Gamma(\alpha), \tag{8}$$

satisfied by the incomplete gamma functions. The function $\Gamma(\alpha, x; b)$ can be expressed in terms of complementary error functions for $\alpha = \frac{1}{2} \pm n,\ n = 0, 1, 2, 3, \ldots$. As a matter of fact [1, 2, 3],

$$\Gamma\left(\frac{1}{2}, x; b\right) = \frac{\sqrt{\pi}}{2}\left[e^{-2\sqrt{b}}\mathrm{Erfc}\left(\sqrt{x} - \sqrt{\frac{b}{x}}\right) + e^{2\sqrt{b}}\mathrm{Erfc}\left(\sqrt{x} + \sqrt{\frac{b}{x}}\right)\right], \tag{9}$$

and

$$\Gamma\left(-\frac{1}{2}, x; b\right) = \frac{\sqrt{\pi}}{2\sqrt{b}}\left[e^{-2\sqrt{b}}\mathrm{Erfc}\left(\sqrt{x} - \sqrt{\frac{b}{x}}\right) - e^{2\sqrt{b}}\mathrm{Erfc}\left(\sqrt{x} + \sqrt{\frac{b}{x}}\right)\right]. \tag{10}$$

For the general formula of $\Gamma\left(\frac{1}{2} + n; x; b\right)$, we refer to [2].

Chaudhry and Zubair [5] considered the extensions

$$\gamma_\nu(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_0^x t^{\alpha - \frac{3}{2}} e^{-t} K_{\nu + \frac{1}{2}}(b/t)\,dt, \tag{11}$$

and

$$\Gamma_\nu(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_x^\infty t^{\alpha - \frac{3}{2}} e^{-t} K_{\nu + \frac{1}{2}}(b/t)\,dt, \quad (\mathrm{Re}\ b \geq 0, -\infty < \alpha < \infty,\ x \geq 0), \tag{12}$$

in connection with the generalization of the inverse Gaussian distribution. It can be seen that

$$\gamma_0(\alpha, x; b) = \gamma(\alpha, x; b), \tag{13}$$

and

$$\Gamma_0(\alpha, x; b) = \Gamma(\alpha, x; b). \tag{14}$$

In general, for $\nu = n$, the function $\Gamma_n(\alpha, x; b)$ can be simplified in terms of the generalized gamma function (4) to give

$$\Gamma_n(\alpha, x; b) = \sum_{m=0}^{n} \frac{(2b)^{-m}}{m!} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \Gamma(\alpha+m, x; b). \tag{15}$$

In this paper, we find closed form evaluation of the integrals

$$\int_0^\infty \Gamma_\nu(\alpha, x; b) b^{s-1} db \text{ and } \int_0^\infty \gamma_\nu(\alpha, x; b) b^{s-1} db.$$

Some new identities involving the Mellin transform and its inversion are proved as special cases of these results.

2. Evaluation of the Integrals $\int_0^\infty \Gamma_\nu(\alpha, x; b) b^{s-1} db$ and $\int_0^\infty \gamma_\nu(\alpha, x; b) b^{s-1} db$

Theorem (2.1)

$$\int_0^\infty \Gamma_\nu(\alpha, x; b) b^{s-1} db = \left(\frac{2}{\pi}\right)^{1/2} 2^{s-\frac{3}{2}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) \Gamma(s+\alpha, x),$$

$$(\mathrm{Re}(s-\nu) > 0, \quad \mathrm{Re}(s+\alpha) > 0, \; x \geq 0). \tag{16}$$

Proof. Let us define $f(t)$ and $g(t)$, $t > 0$, as follows:

$$f(t) = t^{\alpha-\frac{3}{2}} e^{-t} H(t-x), \tag{17}$$

and

$$g(t) = K_{\nu+\frac{1}{2}}(t), \tag{18}$$

where $H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$ is the Heaviside unit step function.

Taking the Mellin transform of $f(t)$ and using [6, p. 312(3)], we get

$$M\{f(t); s\} = \Gamma\left(\alpha + s - \frac{3}{2}, x\right). \tag{19}$$

Similarly, taking the Mellin transform of $g(t)$ and using [6, p. 331(26)], we get

$$M\{g(t);s\} = 2^{s-2}\Gamma\left(\frac{s-\nu-1/2}{2}\right)\Gamma\left(\frac{s+\nu+1/2}{2}\right), \quad \left(\text{Re } s > \left|\text{Re}\left(\nu+\frac{1}{2}\right)\right|\right). \tag{20}$$

However, it follows from (12) and (17) – (18) that

$$\Gamma_\nu(\alpha,x;b)b^{-1/2} = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty f(t)g(b/t)dt. \tag{21}$$

Taking the Mellin transform of both sides in (21) and using [6, p. 308(14)], we get

$$\int_0^\infty \Gamma_\nu(\alpha,x;b)b^{s-\frac{3}{2}}db = \left(\frac{2}{\pi}\right)^{1/2} M\{f(t);s+1\}M\{g(t);s\}. \tag{22}$$

From (19) – (20) and (22) we get

$$\int_0^\infty \Gamma_\nu(\alpha,x;b)b^{s-\frac{3}{2}}db = \left(\frac{2}{\pi}\right)^{1/2} 2^{s-2}\Gamma\left(\frac{s-\nu-1/2}{2}\right)\left(\frac{s+\nu+1/2}{2}\right)\Gamma\left(\alpha+s-\frac{1}{2},x\right). \tag{23}$$

Replacing s by $s+\frac{1}{2}$ in (23) yields the proof of (16).

Theorem (2.2)

$$\int_0^\infty \gamma_\nu(\alpha,x;b)b^{s-1}db = \left(\frac{2}{\pi}\right)^{1/2} 2^{s-2}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu+1}{2}\right)\gamma(\alpha+s,x),$$

$$(\text{Re}(s-\nu) > 0,\ \text{Re}(\alpha+s) > 0,\ x > 0). \tag{24}$$

Proof. This is similar to the proof of Theorem (2.1) when we take $f(t) = t^{\alpha-\frac{3}{2}}e^{-t}H(x-t)H(t)$ and $g(t)$, the same as defined by (18).

3. Applications (Special Cases)

In this section some applications and special cases of (16) and (24) are given. It is to be noted that the inverse Mellin transform operator will be defined according to [6, p. 307] as follows

$$M^{-1}\{g(s);x\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)x^{-s}ds. \qquad (25)$$

Corollary (3.1)

$$\int_0^\infty \Gamma(\alpha,x;b)b^{s-1}db = \Gamma(s)\Gamma(\alpha+s,x), \quad (\text{Re } s>0,\ x\geq 0). \qquad (26)$$

Proof. Substituting $\nu=0$ in (16) and using (14), we get

$$\int_0^\infty \Gamma(\alpha,x;b)b^{s-1}db = \left(\frac{2}{\pi}\right)^{1/2} s^{s-\frac{3}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma(\alpha+s,x). \qquad (27)$$

However, according to the Legendre duplication formula [7, p. 946(8.335)(1)]

$$\Gamma(s) = \left(\frac{2}{\pi}\right)^{1/2} 2^{s-\frac{3}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right). \qquad (28)$$

From (27) and (28) our proof of (26) is complete.

Remark

The result (26) is extremely important in the sense that it provides a closed form representation of the Mellin inverse transform of the product of the Euler gamma function and the Legendre incomplete gamma function. This result can be written in operational form to give

$$M^{-1}\{\Gamma(s)\Gamma(\alpha+s,x);b\} = \Gamma(\alpha,x;b) \qquad (29)$$

which does not seem to be known in the literature. Moreover, it seems natural to consider (29) as a definition of the generalized incomplete gamma function. Several special cases of (29) can be considered. We list here some of these relations for interest.

$$M^{-1}\{\Gamma(s)\Gamma(\alpha+s);b\} = 2b^{\alpha/2}K_\alpha(2\sqrt{b}), \quad \text{Re } b>0, \qquad (30)$$

which follows from (29) when we take $x = 0$ and use (7).

$$M^{-1}\{\Gamma^2(s); b\} = 2k_0(2\sqrt{b}), \quad \text{Re } b > 0, \tag{31}$$

which follows from (30) when we take $\alpha = 0$.

$$M^{-1}\left\{\Gamma(s)\Gamma\left(\frac{1}{2}+s,x;b\right),b\right\} = \frac{\sqrt{\pi}}{2}\left[e^{-2\sqrt{b}}\text{Erfc}\left(\sqrt{x}-\sqrt{\frac{b}{x}}\right)+e^{2\sqrt{b}}\text{Erfc}\left(\sqrt{x}+\sqrt{\frac{b}{x}}\right)\right],$$
$$(\text{Re } s > 0), \tag{32}$$

and

$$M^{-1}\left\{\Gamma(s)\Gamma\left(-\frac{1}{2}+s,x;b\right),b\right\} = \frac{\sqrt{\pi}}{2\sqrt{b}}\left[e^{-2\sqrt{b}}\text{Erfc}\left(\sqrt{x}-\sqrt{\frac{b}{x}}\right)-e^{2\sqrt{b}}\text{Erfc}\left(\sqrt{x}+\sqrt{\frac{b}{x}}\right)\right]. \tag{33}$$

Corollary (3.2)

$$\int_0^\infty \gamma(\alpha,x;b)b^{s-1}db = \Gamma(s)\gamma(\alpha+s,x), \quad (\text{Re } s > 0,\ \text{Re}(\alpha+s) > 0, x > 0). \tag{34}$$

Proof. This is similar to the proof of (26). In particular, the substitution $s = 1$ in (26) and (34) yields

$$\int_0^\infty \Gamma(\alpha,x;b)db = \Gamma(\alpha+1,x), \tag{35}$$

and

$$\int_0^\infty \gamma(\alpha,x;b)db = \gamma(\alpha+1,x). \tag{36}$$

Similarly, we can write (33) in operational form to give

$$M^{-1}\{\Gamma(s)\gamma(\alpha+s,x);b\} = \gamma(\alpha,x;b), \tag{37}$$

that does not seem to be known in the literature.

Remark

The following result which is a consequence of (16), does not seem to be known in the literature. It provides an interesting representation of a finite series of product of complete and incomplete gamma functions.

Corollary (3.3)

$$\sum_{m=0}^{n} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \frac{\Gamma(s-m)}{2^m m!} \Gamma(\alpha+s-m,x)$$
$$= \left(\frac{2}{\pi}\right)^{1/2} 2^{s-\frac{3}{2}} \Gamma\left(\frac{s-n}{2}\right) \Gamma\left(\frac{s+n+1}{2}\right) \Gamma(s+\alpha,x), \quad (\mathrm{Re}(s-n)>0, x\geq 0) \tag{38}$$

Proof. This follows from (16) when we substitute $\nu = n$ and use (15) and (26).

Remark

It should be noted that the functions $\gamma_\nu(\alpha,x;b)$ and $\Gamma_\nu(\alpha,x;b)$ satisfy the decomposition formula [5]

$$\gamma_\nu(\alpha,x;b) + \Gamma_\nu(\alpha,x;b) = 2^{\alpha-2}\pi^{-1}b^{1/2} G_{04}^{40}\left(\frac{b^2}{16}\middle| \begin{matrix} \frac{1}{2}\left(\frac{\nu+1}{2}\right), -\frac{1}{2}\left(\frac{\nu+1}{2}\right) \\ \frac{1}{2}\left(\frac{\alpha+1}{2}\right), \frac{1}{2}\left(\frac{\alpha-1}{2}\right) \end{matrix}\right),$$
$$(\mathrm{Re}\ b>0,\ -\infty<\alpha<\infty). \tag{39}$$

The substitution $x=0$ in (16) leads to an interesting identity

$$\int_0^\infty G_{04}^{40}\left(\frac{b^2}{16}\middle| \begin{matrix} \frac{1}{2}\left(\frac{\nu+1}{2}\right), -\frac{1}{2}\left(\frac{\nu+1}{2}\right) \\ \frac{1}{2}\left(\frac{\alpha+1}{2}\right), \frac{1}{2}\left(\frac{\alpha-1}{2}\right) \end{matrix}\right) b^{s-\frac{1}{2}}\, db = \pi^{1/2}2^{s-\alpha+1}\,\Gamma(s+\alpha)\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu+1}{2}\right),$$
$$(\mathrm{Re}(s+\alpha)>0,\ \ \mathrm{Re}(s-\nu)>0). \tag{40}$$

Several special cases of (40) can be considered by specializing the parameters ν, α and s.

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