On Semi-Weakly Semi-Continuous Mappings

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By

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Abstract: We introduce semi-weakly semi-continuous mappings and investigate some of their properties.

1. Introduction

In 1985, T. Noiri and B. Ahmad [Noiri & Ahmad; 1985] introduced the concept of semi-weakly continuous mappings and studied their several properties. The purpose of the present note is to introduce a new class of mappings called semi-weakly semi-continuous mappings and investigate some properties analogous to those given in [Noiri & Ahmad; 1985] and [Noiri; 1974] concerning semi-weakly continuous and weakly continuous mappings respectively.

2. Preliminaries

Let X be a topological space and let S be a subset of X. The closure and the interior of S are denoted by Cl(S) and Int(S) respectively. A subset S is said to be semi-open [Levine; 1963] if there exists an open set U such that U \subseteq S \subseteq Cl(U). S\text{O}(X) will denote the class of all semi-open sets in a topological space X. The complement of a semi-open set is called semi-closed. The union of all semi-open subsets of X contained in S is called the semi-interior of S and denoted by sInt(S). The intersection of all semi-closed subsets of X containing S is called the semi-closure of S and denoted by sCl(S).

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Throughout this note, X and Y denote topological spaces, and by \( f : X \to Y \) we denote a mapping \( f \) of a space \( X \) into a space \( Y \).

3. Semi-weakly semi-continuous mappings

**Definition 1.** A mapping \( f : X \to Y \) is called semi-weakly semi-continuous (briefly s.w.s.c.) if for each point \( x \in X \) and each semi-open set \( V \subseteq Y \) containing \( f(x) \), there exists a semi-open set \( U \subseteq X \) containing \( x \) such that \( f(U) \subseteq sCl(V) \).

**Definition 2.** A mapping \( f : X \to Y \) is called an irresolute if and only if the inverse image of each semi-open set in \( Y \) is a semi-open set in \( X \).

**Theorem 3.** [Latif; 1993]. A mapping \( f : X \to Y \) is called an irresolute if and only if for each point \( x \in X \) and each semi-open set \( V \) containing \( f(x) \) there exists a semi-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

**Theorem 4.** Let \( f : X \to Y \) be an irresolute. Then \( f \) is semi-weakly semi-continuous.

**Proof.** By using theorem 3, the result follows immediately.

The following example shows that the converse of theorem 4 may not be true in general.

**Example 5.** Let \( X = \{ 1, 2, 3 \} \). Let \( T^* = \{ \emptyset, \{1\}, X \} \) and \( T = \{ \emptyset, \{2\}, \{1, 2\}, X \} \) be topologies on \( X \). Let \( Id_X : (X, T^*) \to (X, T) \) be the identity map. Then \( Id_X \) is not an irresolute. We note that \( Id_X \) is semi-weakly semi-continuous because \( sCl(\{2\}) = X \) in \( (X, T) \).

**Definition 6.** [Noiri, Ahmad; 1985]. A mapping \( f : X \to Y \) is called semi-weakly continuous (briefly s.w.c.) if for each point \( x \in X \) and each open set \( V \subseteq Y \) containing \( f(x) \), there exists a semi-open set \( U \subseteq X \) containing \( x \) such that \( f(U) \subseteq sCl(V) \).
Theorem 7. Let \( f : X \to Y \) be semi-weakly semi-continuous. Then \( f \) is semi-
weakly continuous.

Proof. Note that every open set is a semi-open set.

The next example reveals that the converse of theorem 7 may not be true in general.

Example 8. Let \( X = \{ 1, 2, 3 \} \) and \( Y = \{ 1, 2, 3 \} \). Let \( T = \{ \emptyset, \{ 1 \}, X \} \) and \( T^* = \{ \emptyset, \{ 1 \}, \{ 2 \}, \{ 1, 2 \}, Y \} \) be topologies on \( X \) and \( Y \) respectively. Define \( f : X \to Y \) by \( f(1) = f(2) = 1, f(3) = 3 \). be the identity map. Then clearly \( f \) is semi-
weakly continuous. Note that \( f(3) = 3 \in \{ 2, 3 \} \in SO(Y) \). The semi-open sets in \( X \)
containing \( 3 \) are only \( \{ 1, 3 \} \) and \( X \). Now \( f(\{ 1, 3 \}) = \{ 1, 3 \} \) and \( sCl(\{ 2, 3 \}) = \{ 2, 3 \} \). Thus \( f(\{ 1, 3 \}) \nsubseteq sCl(\{ 2, 3 \}) \). Hence \( f \) is not semi-weakly semi-continuous.

Theorem 9. Let \( f : X \to Y \) be semi-weakly continuous. Then
\[
sCl[f^{-1}(V)] \subseteq f^{-1}[Cl(V)]
\]
for each semi-open set \( V \subseteq Y \).

Proof. Suppose there exists a point \( x \in sCl[f^{-1}(V)] \setminus f^{-1}[Cl(V)] \). Then \( f(x) \nsubseteq Cl(V) \). Hence there exists an open set \( W \) containing \( f(x) \) such that \( W \cap V = \emptyset \).

Since \( V \) is semi-open, we have \( V \cap sCl(W) = \emptyset \). Since \( f \) is semi-weekly
continuous, there exists a semi-open set \( U \subseteq X \) containing \( x \) such that \( f(U) \subseteq sCl(W) \). Thus we obtain \( f(U) \cap V = \emptyset \). On the other hand, since \( x \in sCl[f^{-1}(V)] \), we have \( U \cap f^{-1}(V) \neq \emptyset \) and hence \( f(U) \cap V \neq \emptyset \). We have a contradiction. Therefore we have \( sCl[f^{-1}(V)] \subseteq f^{-1}[Cl(V)] \).

Theorem 10. Let \( f : X \to Y \) be semi-weakly semi-continuous. Then
\[
sCl[f^{-1}(V)] \subseteq f^{-1}[sCl(V)]
\]
for each semi-open set \( V \subseteq Y \).

Proof. Suppose there exists a point \( x \in sCl[f^{-1}(V)] \setminus f^{-1}[sCl(V)] \). Then \( f(x) \nsubseteq sCl(V) \). Hence there exists a semi-open set \( W \) containing \( f(x) \) such that \( W \cap V = \emptyset \). Since \( V \) is semi-open, we have \( V \cap sCl(W) = \emptyset \). Since \( f \) is semi-weekly
semi-continuous, there exists a semi-open set \( U \subseteq X \) containing \( x \) such that \( f(U) \subseteq sCl(W) \). Thus we obtain \( f(U) \cap V = \emptyset \).
On the other hand, since \( x \in s\text{Cl}(f^{-1}(V)) \), we have \( U \cap f^{-1}(V) \neq \emptyset \) and hence \( f(U) \cap V \neq \emptyset \). We have a contradiction. Thus we have \( s\text{Cl}(f^{-1}(V)) \subseteq f^{-1}[s\text{Cl}(V)] \).

**Theorem 11.** Prove that a mapping \( f : X \to Y \) is semi-weakly semi-continuous if and only if for every semi-open set \( T \) in \( Y \) \( f^{-1}(T) \subseteq s\text{Int}[f^{-1}(s\text{Cl}(T))] \).

**Proof.** Let \( x \in X \) and \( T \) a semi-open set containing \( f(x) \). Then \( f(x) \in f^{-1}(T) \subseteq s\text{Int}[f^{-1}(s\text{Cl}(T))] \). Put \( S = s\text{Int}[f^{-1}(s\text{Cl}(T))] \). Then \( S \) is semi-open and \( f(S) \subseteq s\text{Cl}(T) \). This shows that \( f \) is semi-weakly semi-continuous.

Conversely, let \( T \) be a semi-open set of \( Y \) and \( x \in f^{-1}(T) \). Then there exists a semi-open set \( S \) in \( X \) such that \( x \in S \) and \( f(S) \subseteq s\text{Cl}(T) \). Therefore we have \( x \in S \subseteq f^{-1}[s\text{Cl}(T)] \) and hence \( x \in s\text{Int}[f^{-1}(s\text{Cl}(T))] \). This proves that \( f^{-1}(T) \subseteq s\text{Int}[f^{-1}(s\text{Cl}(T))] \).

**Theorem 12.** Let \( f : X \to Y \) be a mapping and \( g : X \to X \times Y \) be the graph mapping of \( f \), given by \( g(x) = (x, f(x)) \) for every point \( x \in X \). If \( g \) is semi-weakly semi-continuous, then \( f \) is semi-weakly semi-continuous.

**Proof.** Let \( x \in X \) and \( T \) be any semi-open set containing \( f(x) \). Then by theorem 11 of [Levine; 1963], \( X \times T \) is a semi-open set in \( X \times Y \) containing \( g(x) \). Since \( g \) is semi-weakly semi-continuous, there exists a semi-open set \( S \) containing \( x \) such that \( g(S) \subseteq s\text{Cl}(X \times T) \). It follows from lemma 4 of [Noiri; 1978] that \( s\text{Cl}(X \times T) \subseteq X \times s\text{Cl}(T) \). Since \( g \) is the graph mapping of \( f \), we have \( f(S) \subseteq s\text{Cl}(T) \). This shows that \( f \) is semi-weakly semi-continuous.

**Definition 13.** A space \((X, T)\) is semi-T\(_2\) if and only if for every \( x, y \in X \) such that \( x \neq y \), there exist disjoint semi-open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \).

**Theorem 14.** If \( f : X \to Y \) is a semi-weakly semi-continuous mapping and \( Y \) is semi-T\(_2\), then the graph \( G(f) \) is a semi-closed set of \( X \times Y \).

**Proof.** Let \((x, y) \in G(f)\). Then, we have \( y \neq f(x) \). Since \( Y \) is semi-T\(_2\), there exist disjoint semi-open sets \( S \) and \( T \) such that \( f(x) \in S \) and \( y \in T \). Since \( f \) is semi-weakly semi-continuous, there exists a semi-open set \( R \) containing \( x \) such that \( f(R) \)
\( \subseteq \text{sCl}(S) \). Since \( S \) and \( T \) are disjoint, we have \( T \cap \text{sCl}(S) = \emptyset \) and hence \( T \cap f(R) = \emptyset \). This shows that \( (R \times T) \cap G(f) = \emptyset \). It follows from theorem 2 and 11 in [Levine; 1963] that \( G(f) \) is semi-closed.

**Definition 15.** By a semi-weakly semi-continuous retraction, we mean a semi-weakly semi-continuous mapping \( f : X \to A \), where \( A \subseteq X \) and \( f/A \) is the identity mapping on \( A \).

**Theorem 16.** Let \( A \subseteq X \) and \( f : X \to A \) be a semi-weakly semi-continuous retraction of \( X \) onto \( A \). If \( X \) is a Hausdorff space then \( A \) is a semi-closed set in \( X \).

**Proof.** Note that \( f \) is semi-weakly continuous by theorem 7. Now the result follows from theorem 4 of [Noiri & Ahmad; 1985].

### 4. S-Connected Spaces

**Definition 17.** [Thompson; 1981]. A space \( X \) is said to be S-connected if \( X \) cannot be written as the disjoint union of two non-empty semi-open sets.

It is already known that S-connectedness is invariant under semi-continuous surjections. Our next result shows that S-connectedness is invariant under semi-weakly semi-continuous surjections.

**Theorem 18.** If \( X \) is an S-connected space and \( f : X \to Y \) is a semi-weakly semi-continuous surjection, then \( Y \) is S-connected.

**Proof.** Suppose \( Y \) is not S-connected. Then there exist non-empty semi-open sets \( V_1 \) and \( V_2 \) of \( Y \) such that \( V_1 \cap V_2 = \emptyset \) and \( V_1 \cup V_2 = Y \). Hence we have \( f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \) and \( f^{-1}(V_1) \cup f^{-1}(V_2) = X \). Since \( f \) is surjective, \( f^{-1}(V_i) \neq \emptyset \) for \( i = 1, 2 \). By theorem 11, we have \( f^{-1}(V_i) \subseteq \text{sInt}(f^{-1}(\text{sCl}(V_i))) \) because \( f \) is semi-weakly semi-continuous. Since \( V_i \) is semi-open and also semi-closed, we have \( f^{-1}(V_i) \subseteq \text{sInt}(f^{-1}(V_i)) \). Hence \( f^{-1}(V_i) \) is semi-open for \( i = 1, 2 \). This implies that \( X \) is not S-connected. This is contrary to the hypothesis that \( X \) is S-connected. Therefore \( Y \) is S-connected.
Definition 19. A space $X$ is called a Urysohn space if for every pair of distinct points $x$ and $y$ in $X$, there exist open sets $U$ and $V$ in $X$ such that $x \in U$, $y \in V$ and $\text{Cl}(U) \cap \text{Cl}(V) = \emptyset$.

Theorem 20. If $Y$ is Urysohn space and $f : X \to Y$ is a semi-weakly continuous injection, then $X$ is semi-$T_2$ space.

Proof. For any distinct points $x_1, x_2 \in X$, we have $f(x_1) \neq f(x_2)$ because $f$ is injective. Since $Y$ is Urysohn, there exist open sets $V_1$ and $V_2$ in $Y$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $\text{Cl}(V_1) \cap \text{Cl}(V_2) = \emptyset$. Then $s\text{Cl}(V_1) \cap s\text{Cl}(V_2) = \emptyset$ since $s\text{Cl}(V_j) \subseteq \text{Cl}(V_j)$ for $j = 1, 2$. Hence we have $s\text{Int}[f^{-1}(s\text{Cl}(V_1))] \cap s\text{Int}[f^{-1}(s\text{Cl}(V_2))] = \emptyset$. Since $f$ is semi-weakly continuous, so by theorem 1 of [Noiri & Ahmad; 1985], we have $x_j \in f^{-1}(V_j) \subseteq s\text{Int}[f^{-1}(s\text{Cl}(V_j))]$ for $j = 1, 2$. This implies that $X$ is semi-$T_2$.

Definition 21. A topological space $(X, T)$ is said to be s-Urysohn if for each pair $x, y$ of distinct points in $X$, there exist $U, V \in S(X)$ such that $x \in U$, $y \in V$ and $\text{Cl}(U) \cap \text{Cl}(V) = \emptyset$.

Theorem 22. If $Y$ is a s-Urysohn space and $f : X \to Y$ is a semi-weakly semi-continuous injection, then $X$ is semi-$T_2$.

Proof. For any distinct points $x, y \in X$, we have $f(x) \neq f(y)$ because $f$ is injective. Since $Y$ is s-Urysohn, there exist semi-open sets $U$ and $V$ in $Y$ such that $f(x) \in U$, $f(y) \in V$ and $\text{Cl}(U) \cap \text{Cl}(V) = \emptyset$. Hence we have $s\text{Int}[f^{-1}(s\text{Cl}(U))] \cap s\text{Int}[f^{-1}(s\text{Cl}(V))] = \emptyset$. Since $f$ is semi-weakly semi-continuous, so by theorem 11, we have $x \in f^{-1}(U) \subseteq s\text{Int}[f^{-1}(s\text{Cl}(U))]$ and $y \in f^{-1}(V) \subseteq s\text{Int}[f^{-1}(s\text{Cl}(V))]$. This implies that $X$ is semi-$T_2$.

Theorem 23. If $X$ is an S-connected space and $f : X \to Y$ is an irresolute mapping with the semi-closed graph, then $f$ is constant.

Proof. Suppose that $f$ is not constant. Then there exist distinct points $x, y$ in $X$ such that $f(x) \neq f(y)$. Since the graph $G(f)$ is semi-closed and $(x, f(y))$ is not in $G(f)$, there exist semi-open sets $U$ and $V$ containing $x$ and $f(y)$, respectively, such that $f(U) \cap V = \emptyset$. Since $f$ is irresolute, $U$ and $f^{-1}(V)$ are disjoint non-empty semi-
open sets. It follows from theorem 17 of [Thompson; 1981] that X is not S-connected. Therefore, f is constant.

Corollary 24. Let X be irreducible. If f : X \rightarrow Y is an irresolute mapping with the semi-closed graph, then f is constant.

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