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ON SEMI-WEAKLY SEMI-CONTINUOUS MAPPINGS

By

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Abstract: We introduce semi-weakly semi-continuous mappings and investigate some of their properties.

1. Introduction

In 1985, T. Noiri and B. Ahmad [Noiri & Ahmad; 1985] introduced the concept of semi-weakly continuous mappings and studied their several properties. The purpose of the present note is to introduce a new class of mappings called semi-weakly semi-continuous mappings and investigate some properties analogous to those given in [Noiri & Ahmad; 1985] and [Noiri; 1974] concerning semi-weakly continuous and weakly continuous mappings respectively.

2. Preliminaries

Let X be a topological space and let S be a subset of X. The closure and the interior of S are denoted by Cl(S) and Int(S) respectively. A subset S is said to be semi-open [Levine; 1963] if there exists an open set U such that $U \subseteq S \subseteq Cl(U)$. SO(X) will denote the class of all semi-open sets in a topological space X. The complement of a semi-open set is called semi-closed. The union of all semi-open subsets of X contained in S is called the semi-interior of S and denoted by sInt(S). The intersection of all semi-closed subsets of X containing S is called the semi-closure of S and denoted by sCl(S).

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Throughout this note, X and Y denote topological spaces, and by $f : X \to Y$ we denote a mapping f of a space X into a space Y.

3. Semi-weakly semi-continuous mappings

Definition 1. A mapping $f : X \to Y$ is called semi-weakly semi-continuous (briefly s.w.s.c.) if for each point $x \in X$ and each semi-open set $V \subseteq Y$ containing f(x), there exists a semi-open set $U \subseteq X$ containing x such that $f(U) \subseteq sCl(V)$.

Definition 2. A mapping $f : X \to Y$ is called an irresolute if and only if the inverse image of each semi-open set in Y, is a semi-open set in X.

Theorem 3. [Latif; 1993]. A mapping $f : X \to Y$ is called an irresolute if and only if for each point $x \in X$ and each semi-open set V containing f(x) there exists a semi-open set U containing x such that $f(U) \subseteq V$.

Theorem 4. Let $f : X \to Y$ be an irresolute. Then f is semi-weakly semi-continuous.

Proof. By using theorem 3, the result follows immediately.

The following example shows that the converse of theorem 4 may not be true in general.

Example 5. Let $X = \{1, 2, 3\}$. Let $T^* = \{\emptyset, \{1\}, X\}$ and $T = \{\emptyset, \{2\}, \{1, 2\}, X\}$ be topologies on X. Let $Id_X : (X, T^*) \to (X, T)$ be the identity map. Then Id_X is not an irresolute. We note that Id_X is semi-weakly semi-continuous because $sCl\{2\} = X$ in (X, T).

Definition 6. [Noiri, Ahmad; 1985]. A mapping $f : X \to Y$ is called semi-weakly continuous (briefly s.w.c.) if for each point $x \in X$ and each open set $V \subseteq Y$ containing f(x), there exists a semi-open set $U \subseteq X$ containing x such that $f(U) \subseteq sCl(V)$.

Theorem 7. Let $f : X \to Y$ be semi-weakly semi-continuous. Then f is semi-weakly continuous.

Proof. Note that every open set is a semi-open set.

The next example reveals that the converse of theorem 7 may not be true in general.

Example 8. Let $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3\}$. Let $T = \{\emptyset, \{1\}, X\}$ and $T^* = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, Y\}$ be topologies on X and Y respectively. Define $f : X \to Y$ by $f(1) = f(2) = 1$, $f(3) = 3$. be the identity map. Then clearly f is semi-weakly continuous. Note that $f(3) = 3 \in \{2, 3\} \in SO(Y)$. The semi-open sets in X containing 3 are only $\{1, 3\}$ and X. Now $f(\{1, 3\}) = \{1, 3\}$ and $sCl(\{2, 3\}) = \{2, 3\}$. Thus $f(\{1, 3\}) \not\subset sCl(\{2, 3\})$. Hence f is not semi-weakly semi-continuous.

Theorem 9. Let $f : X \to Y$ be semi-weakly continuous. Then

$$sCl[f^{-1}(V)] \subseteq f^{-1}[Cl(V)]$$

for each semi-open set $V \subseteq Y$.

Proof. Suppose there exists a point $x \in sCl[f^{-1}(V)] - f^{-1}[Cl(V)]$. Then $f(x) \notin Cl(V)$. Hence there exists an open set W containing f(x) such that $W \cap V = \emptyset$. Since V is semi-open, we have $V \cap sCl(W) = \emptyset$. Since f is semi-weekly continuous, there exists a semi-open set $U \subseteq X$ containing x such that $f(U) \subseteq sCl(W)$. Thus we obtain $f(U) \cap V = \emptyset$. On the other hand, since $x \in sCl[f^{-1}(V)]$, we have $U \cap f^{-1}(V) \neq \emptyset$ and hence $f(U) \cap V \neq \emptyset$. We have a contradiction. Therefore we have $sCl[f^{-1}(V)] \subseteq f^{-1}[Cl(V)]$.

Theorem 10. Let $f : X \to Y$ be semi-weakly semi-continuous. Then

$$sCl[f^{-1}(V)] \subseteq f^{-1}[sCl(V)].$$

for each semi-open set $V \subseteq Y$.

Proof. Suppose there exists a point $x \in sCl[f^{-1}(V)] - f^{-1}[sCl(V)]$. Then $f(x) \notin sCl(V)$. Hence there exists a semi-open set W containing f(x) such that $W \cap V = \emptyset$. Since V is semi-open, we have $V \cap sCl(W) = \emptyset$. Since f is semi-weekly semi-continuous, there exists a semi-open set $U \subseteq X$ containing x such that $f(U) \subseteq sCl(W)$. Thus we obtain $f(U) \cap V = \emptyset$.

On the other hand, since $x \in sCl[f^{-1}(V)]$, we have $U \cap f^{-1}(V) \neq \emptyset$ and hence $f(U) \cap V \neq \emptyset$. We have a contradiction. Thus we have $sCl[f^{-1}(V)] \subseteq f^{-1}[sCl(V)]$.

__Theorem 11__. Prove that a mapping $f : X \to Y$ is semi-weakly semi-continuous if and only if for every semi-open set T in Y $f^{-1}(T) \subseteq sInt[f^{-1}(sCl(T))]$.

Proof. Let $x \in X$ and T a semi-open set containing f(x). Then $f(x) \in f^{-1}(T) \subseteq sInt[f^{-1}(sCl(T))]$. Put $S = sInt[f^{-1}(sCl(T))]$. Then S is semi-open and $f(S) \subseteq sCl(T)$. This shows that f is semi-weakly semi-continuous.

Conversely, let T be a semi-open set of Y and $x \in f^{-1}(T)$. Then there exists a semi-open set S in X such that $x \in S$ and $f(S) \subseteq sCl(T)$. Therefore we have $x \in S \subseteq f^{-1}[sCl(T)]$ and hence $x \in sInt[f^{-1}(sCl(T))]$. This proves that $f^{-1}(T) \subseteq sInt[f^{-1}(sCl(T))]$.

__Theorem 12__. Let $f : X \to Y$ be a mapping and $g : X \to X \times Y$ be the graph mapping of f, given by $g(x) = (x, f(x))$ for every point $x \in X$. If g is semi-weakly semi-continuous, then f is semi-weakly semi-continuous.

Proof. Let $x \in X$ and T be any semi-open set containing f(x). Then by theorem 11 of [Levine; 1963], $X \times T$ is a semi-open set in $X \times Y$ containing g(x). Since g is semi-weakly semi-continuous, there exists a semi-open set S containing x such that $g(S) \subseteq sCl(X \times T)$. It follows from lemma 4 of [Noiri; 1978] that $sCl(X \times T) \subseteq X \times sCl(T)$. Since g is the graph mapping of f, we have $f(S) \subseteq sCl(T)$. This shows that f is semi-weakly semi-continuous.

__Definition 13__. A space (X, T) is semi-T_2 if and only if for every $x, y \in X$ such that $x \neq y$, there exist disjoint semi-open sets U and V such that $x \in U$ and $y \in V$.

__Theorem 14__. If $f : X \to Y$ is a semi-weakly semi-continuous mapping and Y is semi-T_2, then the graph G(f) is a semi-closed set of $X \times Y$.

Proof. Let $(x, y) \notin G(f)$. Then, we have $y \neq f(x)$. Since Y is semi-T_2, there exist disjoint semi-open sets S and T such that $f(x) \in S$ and $y \in T$. Since f is semi-weakly semi-continuous, there exists a semi-open set R containing x such that f(R)

$\subseteq$ sCl(S). Since S and T are disjoint, we have $T \cap sCl(S) = \emptyset$ and hence $T \cap f(R) = \emptyset$. This shows that $(R \times T) \cap G(f) = \emptyset$. It follows from theorem 2 and 11 in [Levine; 1963] that G(f) is semi-closed.

Definition 15. By a semi-weakly semi-continuous retraction, we mean a semi-weakly semi-continuous mapping $f : X \rightarrow A$, where $A \subseteq X$ and f/A is the identity mapping on A.

Theorem 16. Let $A \subseteq X$ and $f : X \rightarrow A$ be a semi-weakly semi-continuous retraction of X onto A. If X is a Hausdorff space then A is a semi-closed set in X.

Proof. Note that f is semi-weakly continuous by theorem 7. Now the result follows from theorem 4 of [Noiri & Ahmad; 1985].

4. S-Connected Spaces

Definition 17.[Thompson; 1981]. A space X is said to be S-connected if X can not be written as the disjoint union of two non-empty semi-open sets.

It is already known that S-connectedness is invariant under semi-continuous surjections. Our next result shows that S-connectedness is invariant under semi-weakly semi-continuous surjections.

Theorem 18. If X is an S-connected space and $f : X \rightarrow Y$ is a semi-weakly semi-continuous surjection, then Y is S-connected.

Proof. Suppose Y is not S-connected. Then there exist non-empty semi-open sets V_1 and V_2 of Y such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Hence we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjective, $f^{-1}(V_i) \neq \emptyset$ for $i = 1, 2$. By theorem 11, we have $f^{-1}(V_i) \subseteq sInt[f^{-1}(sCl(V_i))]$ because f is semi-weakly semi-continuous. Since V_i is semi-open and also semi-closed, we have $f^{-1}(V_i) \subseteq sInt[f^{-1}(V_i)]$. Hence $f^{-1}(V_i)$ is semi-open for $i=1, 2$. This implies that X is not S-connected. This is contrary to the hypothesis that X is S-connected. Therefore Y is S-connected.

Definition 19. A space X is called a Urysohn space if for every pair of distinct points x and y in X, there exist open sets U and V in X such that $x \in U$, $y \in V$ and $Cl(U) \cap Cl(V) = \emptyset$.

Theorem 20. If Y is Urysohn space and $f : X \to Y$ is a semi-weakly continuous injection, then X is semi-T_2 space.

Proof. For any distinct points $x_1, x_2 \in X$, we have $f(x_1) \neq f(x_2)$ because f is injective. Since Y is Urysohn, there exist open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $Cl(V_1) \cap Cl(V_2) = \emptyset$. Then $sCl(V_1) \cap sCl(V_2) = \emptyset$ since $sCl(V_j) \subseteq Cl(V_j)$ for $j = 1, 2$. Hence we have $sInt[f^{-1}(sCl(V_1))] \cap sInt[f^{-1}(sCl(V_2))] = \emptyset$. Since f is semi-weakly continuous, so by theorem 1 of [Noiri & Ahmad; 1985], we have $x_j \in f^{-1}(V_j) \subseteq sInt[f^{-1}(sCl(V_j))]$ for $j = 1, 2$. This implies that X is semi-T_2.

Definition 21. A topological space (X, T) is said to be s-Urysohn if for each pair x, y of distinct points in X, there exist $U, V \in S(X)$ such that $x \in U$, $y \in V$ and $Cl(U) \cap Cl(V) = \emptyset$.

Theorem 22. If Y is a s-Urysohn space and $f : X \to Y$ is a semi-weakly semi-continuous injection, then X is semi-T_2.

Proof. For any distinct points $x, y \in X$, we have $f(x) \neq f(y)$ because f is injective. Since Y is s-Urysohn, there exist semi-open sets U and V in Y such that $f(x) \in U$, $f(y) \in V$ and $Cl(U) \cap Cl(V) = \emptyset$. Hence we have $sInt[f^{-1}(sCl(U))] \cap sInt[f^{-1}(sCl(V))] = \emptyset$. Since f is semi-weakly semi-continuous, so by theorem 11, we have $x \in f^{-1}(U) \subseteq sInt[f^{-1}(sCl(U))]$ and $y \in f^{-1}(V) \subseteq sInt[f^{-1}(sCl(V))]$. This implies that X is semi-T_2.

Theorem 23. If X is an S-connected space and $f: X \to Y$ is an irresolute mapping with the semi-closed graph, then f is constant.

Proof. Suppose that f is not constant. Then there exist distinct points x, y in X such that $f(x) \neq f(y)$. Since the graph G(f) is semi-closed and (x, f(y)) is not in G(f), there exist semi-open sets U and V containing x and f(y), respectively, such that $f(U) \cap V = \emptyset$. Since f is irresolute, U and $f^{-1}(V)$ are disjoint non-empty semi-open sets. It follows from theorem 17 of [Thompson; 1981] that X is not S-connected. Therefore, f is constant.

Corollary 24. Let X be irreducible. If f : X → Y is an irresolute mapping with the semi-closed graph, then f is constant.

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