King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 184

September 1995

On Relatively Injective and Weakly Injective S-Acts

Javed Ahsan, Liu Zhongkui

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

ON RELATIVELY INJECTIVE AND WEAKLY INJECTIVE S-ACTS

JAVED AHSAN AND LIU ZHONGKUI

1. Introduction and Preliminaries

Throughout this paper S will denote a given monoid, that is, a semigroup with identity. By the term S-act we shall mean a right unitary S-act over S. Let M be a fixed S-act. An S-act Q is M-injective if for each S-monomorphism g from an S-act N_s into M_s, every S-homomorphism $h : N_s \to Q_s$ can be extended to an S-homomorphism $h^* : M_s \to Q_s$ relative to g, i.e. $h^*g = h$. Thus Q is injective if and only if Q is M-injective for all S-acts M [2]. Similarly Q is quasi-injective if and only if Q is Q-injective [1, 4, 7]. S-acts which are S_S-injective in the above sense are called weakly injective [3]. In ring theory it is well known that R-injective R-modules are exactly injective. S-injective S-acts, however, need not be injective in the usual sense ([2], p. 272). An S-act A will be called *finitely injective* if for every S-monomorphism $f : X \to Y$ from a finitely generated S-act X to an S-act Y, every S-homomorphism $g : X \to A$ extends to an S-homomorphism $h : Y \to A$, that is, $g = hf$. If A is an S-subact of a right S-act B then A is *pure* in B if every finite system of equations with constants from A which is solvable in B is also solvable in A [6]. A right S-act is absolutely pure if it is pure within its injective hull. Normak [6] and independently Gould [5] have shown that an S-act is absolutely pure if and only if it is finitely M-injective for all finitely presented S-acts M. Thus the concept of finite injectivity lies strictly between absolute purity and injectivity. Generalizing weakly injective S-acts, a right S-act B is called *P-injective* (resp. *F-injective*) if B is weakly injective relative to the principal (resp. finitely generated) right ideal of S, that is, each right S-homomorphism from a principal (resp. finitely generated) right ideal of S to B extends to an S-homomorphism from S to B. Generalizing quasi-injective S-acts, we call an S-act M *P-quasi-injective*, in short PQI, if for any P-injective S-subact N of M, any right S-homomorphism of N into M extends to an endomorphism of M. A nonzero S-act is *totally irreducible* if the only right S-congruences of M are the universal congruence ω_M and the identity congruence i_M. Clearly every totally irreducible S-act is simple (that is, having no proper non zero subacts) and hence a PQI S-act. If $\{M_i : i \in I\}$ is a family of objects in the category of S-acts, then the product $\prod_{i\in I} M_i$ and coproduct $\amalg_{i\in I} M_i$ exist and are isomorphic, respectively, to the cartesian product and disjoint union of the sets M_i with suitable actions of S. If $\{M_i : i \in I\}$ is a family of S-acts such that every M_i contains a zero element, that is, a fixed one element S-subact θ_i, then the direct sum of $\{M_i : i \in I\}$ denoted by $\oplus_{i\in I} M_i$, is the subset of $\prod_{i\in I} M_i$ consisting of all $(m_i) \in \prod_{i\in I} M_i$ for which $\{i : m_i \neq \theta_i\}$ is finite. Then $\oplus_{i\in I} M_i$ is an S-act under componentwise action of S. An S-act Q will be called *$\sum$-injective* (resp. *$\sum$-quasi-injective*) if direct sum of arbitrary many copies of M is injective (resp. quasi-injective). A monoid S is *right noetherian* if every right ideal of S is finitely generated. S is called *regular* if for each $a \in S$, there exists $b \in S$ such that $a = aba$. One object of this paper is to show that in the category of S-acts, each P-injective (resp. F-injective) object is injective if and only if each P-injective (resp. F-injective) object is quasi-injective with a zero element. We also characterize monoids all of whose finitely injective S-acts are injective, and those monoids for which every S-act is finitely injective.

2. P-INJECTIVE S-ACTS

We need the following lemmas.

Lemma 2.1. *Let $\{M_i : i \in I\}$ be a family of S-acts. If each M_i is injective (resp. weakly injective, F-injective or P-injective), then the product $\prod_{i\in I} M_i$ is injective (resp. weakly injective, F-injective or P-injective).*

Proof. Follows from the definition of product in the category of S-acts. ■

The next lemma is due to Skornjakov [9].

Lemma 2.2. *Each direct sum of injective S-acts is injective if and only if S is right noetherian (see also Normak [6]).*

Lemma 2.3. *Let $\{M_i : i \in I\}$ be a family of P-injective (resp. F-injective) S-acts with zero elements. Then $\oplus_{i\in I} M_i$ is P-injective (resp. F-injective).*

Proof. Let aS be a principal right ideal of S ($a \in S$) and $f : aS \to \oplus_{i\in I} M_i$ be an S-homomorphism. Since the vector $f(a)$ involves only finitely many nonzero coordinates, $f(a)$ involves only finitely many M_i's, say $M_{i_1}, \ldots, M_{i_n}$. It is clear that finite direct sums of P-injective S-acts are P-injective. So $M_{i_1} \oplus \cdots \oplus M_{i_n}$ is P-injective. Since $Im(f) \leq M_{i_1} \oplus \cdots \oplus M_{i_n}$, there exists an S-homomorphism $g : S \to M_{i_1} \oplus \cdots \oplus M_{i_n}$ which extends f. We may regard g as a map whose image is in the larger S-act $\oplus_{i\in I} M_i$. Hence $\oplus_{i\in I} M_i$ is P-injective. The proof for the F-injective version is nearly the same. ■

Proposition 2.4. *Let M be an S-act and $E = E(M)$ be the injective hull of M. Then the following assertions are equivalent:*

(1) M is injective.

(2) M is a retract of E.

(3) M is a P-injective S-act with a zero element and $E \oplus M$ is PQI.

Proof. (1) $\Leftrightarrow$ (2): This is well known.

(3) $\Rightarrow$ (2): Let $D = E \oplus M$. Let $i : M \to D$ and $k : E \to D$ be the canonical injections, and let $j : M \to E$ be the inclusion map. Then $k \circ j : M \to D$ is an injection. Since $D = E \oplus M$ is PQI and M is P-injective, for the injections $i : M \to D$ and $k \circ j : M \to D$, there exists an S-endomorphism $\theta : D \to D$ such that $i = \theta \circ k \circ j$. Let $p : D \to M$ be the canonical projection. Then $p \circ i = 1_M$. Define $\varphi : E \to M$ by $\varphi = p \circ \theta \circ k$, then $\varphi \circ j = p \circ \theta \circ k \circ j = p \circ i = 1_M$. This shows that M is a retract of E.

(1) $\Rightarrow$ (3): Because M is injective, it contains a zero element [8], and is obviously P-injective. By Lemma 2.1, $E \oplus M$ $(= E \times M)$ is injective. Hence $D = E \oplus M$ is PQI. This completes the proof. ■

Theorem 2.5. *The following assertions in the category of S-acts are equivalent:*

(1) Every P-injective S-act is injective.

(2) Every P-injective S-act is $\sum$-injective with a zero element.

(3) Every P-injective S-act is $\sum$-quasi-injective with a zero element.

(4) Every P-injective S-act is a quasi-injective S-act with a zero element.

(5) Every P-injective S-act is a PQI S-act with a zero element.

Proof. (1) $\Rightarrow$ (2): Since every injective S-act contains a zero element, it follows from the hypothesis that every P-injective S-act contains a zero element. Let $\{M_i : i \in I\}$ be a family of injective S-acts. Then $\oplus_{i \in I} M_i$ is P-injective by Lemma 2.3. Hence by the hypothesis, $\oplus_{i \in I} M_i$ is injective. Hence by Lemma 2.2, S is right noetherian. Thus any P-injective S-act is injective with a zero element which is $\sum$-injective by Lemma 2.2, since S is right noetherian.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Obvious from the definitions.

(5) $\Rightarrow$ (1): Let M be a P-injective S-act. By the hypothesis, M contains a zero element. Let E be the injective hull of M. Then by Lemma 2.3, $E \oplus M$ is a P-injective S-act with zero. Hence by the hypothesis, $E \oplus M$ is a PQI S-act. Hence by Proposition 2.4, M is injective. This completes the proof. ■

Corollary 2.6. *Let S be a regular monoid with zero. Then the following assertions are equivalent:*

(1) Every S-act is injective.

(2) Every S-act is quasi-injective and S is right noetherian.

(3) Every S-act is PQI.

Proof. It follows from ([3], Thm. 2.1) and the above theorem. ■

Proposition 2.7. *Let S be a commutative monoid. Then the following assertions are equivalent:*

(1) Every PQI S-act is P-injective.

(2) S is regular.

Proof. (1) $\Rightarrow$ (2): Suppose every PQI S-act is P-injective. Then every totally irreducible S-act is P-injective. Hence by ([3], Corollary 2.4), S is regular.

(2) $\Rightarrow$ (1): Follows from ([3], Thm. 2.1). ■

Remark 2.1. *If S is a regular monoid (not necessarily commutative) then for each maximal right ideal K of S, the Rees factor S/K, considered as an S-act, is in fact weakly injective. For, let B be any right ideal of S and let $f : B \rightarrow S/K$ be an S-homomorphism. Then $f(B \cap K) = \{K\} =$ the zero element of S/K. For if $f(B \cap K) \neq K$, then there exists $a \in B \cap K$ such that $f(a) \notin K$. Since S is regular,* *there exists $b \in S$ such that $aba = a$. Hence $f(a) = f(aba) = f(ab)a \in (S/K)(K) \leq K$. But this is a contradiction. Now we consider only two possibilities:*

(i) $B \cup K = K$. Then $f(B) = f(B \cap K) = \{K\}$. Thus f can be extended.

(ii) $B \cup K = S$. Define $g : S \to S/K$ as follows:

$$g(x) = \begin{cases} f(x), & \text{if } x \in B \\ \{K\}, & \text{if } x \in K. \end{cases}$$

Then g is well-defined and $g|_B = f$. This shows that S/K is weakly injective.

Theorem 2.8. *The following assertions in the category of S-acts are equivalent:*

(1) Every weakly injective S-act is injective and S is right noetherian.

(2) Every F-injective S-act is injective.

(3) Every F-injective S-act is quasi-injective with a zero element.

Proof. (1) $\Rightarrow$ (2): Since S is right noetherian, every F-injective S-act is weakly injective, and hence injective by the hypothesis.

(2) $\Rightarrow$ (3): Since every injective S-act contains a zero element and is obviously quasi-injective, the desired implication is a consequence of the hypothesis.

(3) $\Rightarrow$ (2): Let M be an F-injective S-act. By the hypothesis, M contains a zero element. Let $E = E(M)$ be the injective hull of M. Then $D = M \oplus E$ contains a zero element and by Lemma 2.3, it is F-injective. Hence by the hypothesis, D is a quasi-injective S-act. Using the canonical injection and projection maps, we write $i : M \to M \oplus E$, $\pi : M \oplus E \to M$, $i' : E \to M \oplus E$, and $\pi' : M \oplus E \to E$ where $\pi \circ i = 1_M$ and $\pi' \circ i' = 1_E$. Let $\alpha : M \to E$ be the inclusion map. Then $i' \circ \alpha : M \to M \oplus E$ is an S-monomorphism and $i : M \to M \oplus E$ is an S-homomorphism. Since $M \oplus E$ is quasi-injective, there exists an S-homomorphism $g : M \oplus E \to M \oplus E$ such that $g \circ i' \circ \alpha = i$. Hence $1_M = \pi \circ i = \pi \circ g \circ i' \circ \alpha$. This shows that M is a retract of E. Hence M is injective.

(2) $\Rightarrow$ (1): Let $\{M_i : i \in I\}$ be a family of injective S-acts. Since every injective S-act is F-injective, $\oplus_{i\in I} M_i$ is F-injective by Lemma 2.3. Hence by the hypothesis, $\oplus_{i\in I} M_i$ is injective. Hence by Lemma 2.2, S is right noetherian. Clearly every weakly injective S-act is F-injective and hence injective by the hypothesis. This completes the proof. ■

We now consider finitely injective S-acts.

Theorem 2.9. *The following assertions in the category of S-acts are equivalent:*

(1) A is finitely injective.

(2) For every S-act B containing A, and for any finite set $\{a_1, \dots, a_n\}$ of elements of A there exists an S-homomorphism $\alpha : B \to A$ such that $\alpha(a_i) = a_i$ for $i = 1, \dots, n$.

(3) For any finite set $\{a_1, \dots, a_n\}$ of elements of A there exists an S-homomorphism $\alpha : E(A) \to A$ such that $\alpha(a_i) = a_i$ for $i = 1, \dots, n$, where $E(A)$ is the injective hull of A.

Proof. (1) $\Rightarrow$ (2): Let A' be an S-subact of A generated by the set $\{a_1, \dots, a_n\}$ and let $i : A' \to B$ and $j : A' \to A$ be the inclusion maps. Since A is finitely injective, there exists an S-homomorphism $\alpha : B \to A$ such that $\alpha \circ i = j$. Hence

$$\alpha(a_i) = \alpha \circ i(a_i) = j(a_i) = a_i$$

for $i = 1, \dots, n$.

(2) $\Rightarrow$ (3): This is clear.

(3) $\Rightarrow$ (1): Let $\lambda : X \to Y$ be an S-homomorphism from a finitely generated S-act X to an S-act Y, and let $\alpha : X \to A$ be an S-homomorphism. Moreover, let $i : A \to E(A)$ be the inclusion map and $\{x_1, \dots, x_n\}$ be the generating set of X. By the injectivity of $E(A)$, there exists an S-homomorphism $\alpha' : Y \to E(A)$ such that

$\alpha' \circ \lambda = i \circ \alpha$. By the hypothesis, there exists an S-homomorphism $f : E(A) \to A$ such that $f(\alpha(x_k)) = \alpha(x_k)$ for $k = 1, \ldots, n$. Let $\beta = f \circ \alpha'$. Then $\beta : Y \to A$ is an S-homomorphism such that $\beta \circ \lambda = \alpha$. Hence A is finitely injective. ■

Corollary 2.10. *Every finitely generated finitely injective S-act is injective.*

Proof. By the above theorem, every finitely generated finitely injective S-act is a retract of its injective hull and hence injective. ■

We now characterize monoids all of whose finitely injective S-acts are injective

Theorem 2.11. *The following assertions for a monoid S are equivalent:*

(1) Every finitely injective S-act is injective.

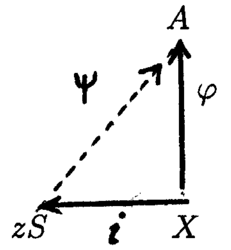
(2) Every finitely injective S-act is weakly injective.

(3) Every finitely injective S-act is quasi-injective.

(4) Every absolutely pure S-act is weakly injective.

(5) S is right noetherian.

Proof. (1) ⇒ (2), (1) ⇒ (3) and (4) ⇒ (3) are clear. (2) ⇒ (5) and (3) ⇒ (5) follow from ([1], Lemmas 1 and 3). We prove (5) ⇒ (1): Let A be a finitely injective S-act. Consider the diagram



where zS is generated by a single element z, $i : X \to zS$ is a monomorphism, and $\varphi : X \to A$ is an S-homomorphism. Let $I = \{s \in S : zs \in i(X)\}$. By the hypothesis,

I is generated by a finite set. Hence X is a finitely generated S-act. Since A is finitely injective, there exists an S-homomorphism $\psi : zS \to A$ such that $\psi \circ i = \varphi$. By ([8], Theorem 1) it follows that A is injective. Finally (5) $\Rightarrow$ (4) follows from the fact that every absolutely pure S-act is finitely S-injective, that is, F-injective. ([6], Prop. 4). ■

Corollary 2.12. *All absolutely pure S-acts are injective if and only if they are finitely injective and S is right noetherian.*

Proof. Follows from ([6], Prop. 8). ■

We shall call a monoid S *completely finitely injective* if all S-acts are finitely injective.

Theorem 2.13. *A Monoid S is completely finitely injective if and only if all finitely generated S-acts are injective.*

Proof. Necessity follows from Theorem 2.8. We prove sufficiency. Let A be an S-act and $i : X \to Y$ be an S-monomorphism from a finitely generated S-act X to an S-act Y. Moreover, let $\varphi : X \to A$ be an S-homomorphism. Then $\varphi(X) \subseteq A$ is also finitely generated. Then by the hypothesis, there exists an S-homomorphism $\psi : Y \to \varphi(X)$ such that $\psi \circ i = \varphi$. Hence A is finitely injective. ■

Corollary 2.14. *Every completely finitely injective monoid is regular and injective as a right S-act and its right ideals are linearly ordered with respect to inclusion.*

Proof. Since every finitely injective S-act is absolutely pure, the above corollary follows from ([6], Theorem 5). ■

Remark 2.2. *The monoid $S_1 = \{1, \alpha, x_1, x_2 : S_1 x_i = x_i;\ \alpha^2 = 1; x_i\alpha = x_j : \{i, j\} = \{1, 2\}\}$, an arbitrary group, and the monoid $S_2 = \{0, x, 1 : x^2 = 0\}$ show that the* *conditions in the last corollary are independent. Note that these conditions also imply that S contains a zero element.*

3. Λ-WEAKLY INJECTIVE S-ACTS

Now we consider another type of restrictive weakly injective S-acts. In the rest of this paper, S will denote a monoid with a two-sided zero element 0 and all S-acts are assumed to contain zero elements. Let A be a set of right ideals of S. An S-act M is called Λ-weakly injective if for every right ideal $I \in \Lambda$, every S-homomorphism from I into M can be extended to an S-homomorphism of S into M. Thus every weakly injective S-act is Λ-weakly injective. A right ideal I of S is *intersection large* in S if for each $0 \neq s \in S$, there exists $t \in S$ such that $0 \neq st \in I$. Thus I is intersection large in S if and only if the intersection of I with any nonzero right ideal of S is always nonzero.

Theorem 3.1. *Suppose that Λ satisfies the following condition: if $I \in \Lambda$ then $J \in \Lambda$ for all right ideals J of S with $J \supseteq I$. Then a right S-act M is Λ-weakly injective if and only if for any intersection large right ideal $I \in \Lambda$, every S-homomorphism from I into M can be extended to an S-homomorphism of S into M.*

Proof. The "only if" part is obvious. To prove the converse part, let us suppose that I is any right ideal in Λ and f be an S-homomorphism from I into M. Let Ω be the set of pairs (J, g) where J is a right ideal of S which contains I (hence $J \in \Lambda$) and g is an S-homomorphism from J into M which extends f. Ordering Ω in the obvious manner, by Zorn's lemma, Ω has a maximal element (J_1, g_1). If J_1 is not an intersection large ideal, then there exists a right ideal $0 \neq K \leq S$ such that

$K \cap J_1 = (0)$. Define a mapping $g^* : J_1 \cup K \to M$ by

$$g^*(\alpha) = \begin{cases} g_1(\alpha) & \text{if } \alpha \in J_1 \\ \theta \text{ (the zero element of } M) & \text{if } \alpha \in K. \end{cases}$$

It is easy to see that g^* is an S-homomorphism from $J_1 \cup K$ into M, which extends g_1 and hence f. Thus $(J_1 \cup K, g^*) \in \Omega$, which contradicts the fact that (J_1, g_1) is maximal in Ω. Hence J_1 must be an intersection large ideal. Hence by the hypothesis, g_1 can be extended to an S-homomorphism from S into M, which is therefore an extension of f. ■

Corollary 3.2. *An S-act M is weakly injective if and only if for any intersection large right ideal I of S, every S-homomorphism from I into M can be extended to an S-homomorphism of S into M.*

Proof. One direction is obvious. For the other direction, set $\Omega = \{I : I \text{ is a right ideal of } S\}$. Then the result follows from Theorem 3.1. ■

REFERENCES

1. J. Ahsan; Monoids characterized by their quasi-injective S-systems, Semigroup Forum, **36** (3) (1987), 285-292.
2. P. Berthiaume; The injective envelope of S-acts, Canad. Math. Bull., **10** (1967), 261-273.
3. J. K. Luedeman, F. R. McMorris, and Soon-Kiong Sim; Semigroups for which every totally irreducible S-system is injective, Commentationes Math. Univ. Carolinae, **19** (1) (1987), 27-35.
4. A. M. Lopez, Jr. and J. K. Luedeman; Quasi-injective S-systems and their S-endomorphism semigroups, Czechoslovak Math. J., **29** (104) (1979), 97-104.
5. V. Gould; The characterizations of monoids by properties of their S-systems, Semigroup Forum, **32** (1985), 251-265.
6. P. Normak; Purity in the category of M-sets, Semigroup Forum, **20** (1980), 157-170.
7. M. Satyanarayana; Quasi and weakly injective S-systems, Math. Nachr., **71** (1976), 183-190.
8. L. A. Skornjakov; On the homological classification of monoids, Siberian Math. J., **10** (1969), 843-846.
9. L. A. Skornjakov; Axiomatizability of the class of injective M-sets, Trudy Seminara im. I. G. Petrovskogo, **4** (1978), 233-239 (in Russian)..

Javed Ahsan
Department of Mathematical Sciences
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia

Liu Zhongkui
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P.R. China