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CHARACTERIZATIONS OF FV-RINGS BY QUASI-CONTINUOUS MODULES

J. AHSAN AND L. ZHONGKUI

1. Introduction

A ring R is called a left V-ring if every simple left R-module is injective. It is proved in Hiremath [9] that V-rings are precisely those in which every finitely cogenerated left R-module is injective. This result is generalized by Garcia Hernandez and Gomez Pardo in [6] in which V-rings relative to left Gabriel topologies on R are studied. It is proved in [6] that a ring R is an FV-ring if and only if every F-torsionfree and F-finitely cogenerated left R-module is dense in its injective envelope. In this paper, we give the characterizations of FV-rings by proving that, if F is a left Gabriel topology on R such that all quasi-continuous left R-modules are F-injective, then R is an FV-ring if and only if every F-torsionfree and F-finitely cogenerated left R-module is dense in its some essential extensions which are quasi-continuous. As a corollary we deduce that a ring R is a V-ring if and only if every finitely cogenerated left R-module is continuous (if and only if every finitely cogenerated left R-module is quasi-continuous). Also we consider a particular case when F is a strongly semiprime left Gabriel topology on R, and obtain new characterizations of FV-rings which, for trivial F, give characterizations of left V-rings. For example, we show that R is a left V-ring if and only if every finitely quasi-copresented left R-module is quasi-injective, where a left R-module M is called finitely quasi-copresented if M and $E_q(M)/M$ are finitely cogenerated where $E_q(M)$ is the quasi-injective envelope of M.

2. Preliminaries

All rings considered in this paper are associative with identity and unless otherwise mentioned all modules are unitary left R-modules. For any module ${}_RM$ we denote by $E(M)$, $E_q(M), E_{vN}(M)$ and $Soc(M)$ the injective envelope, the quasi-injective envelope (see [5]), the vN-injective envelope (see [13]) and the socle of M, respectively.

Let M be a left R-module. Recall that M is called a *CS-module* provided every submodule of M is essential in a direct summand of M, or equivalently, every maximal essential extension of a submodule M is a direct summand of M (see [2] or [3]). M is called continuous if M is a CS-module and every submodule isomorphic to a direct summand is a direct summand (see [10] or [11]). M is called *quasi-continuous* (or *π-injective* in [8]) if it is a CS-module and for any direct summands A and B of M with $A \cap B = 0$, $A \oplus B$ is also a direct summand of M (see [11] or [15]).

The following lemma and its simple proof appeared in [15].

Lemma 2.1. *Let M be a quasi-continuous R-module, $M = B \oplus A$. Then B is A-injective; that is, any homomorphism from a submodule of A to B extends to a homomorphism from A to B.*

Following [11] in which vN-injective modules were investigated, a module M is called *vN-injective* if $f(M) \leq M$ for any von Neumann regular element $f \in End_R(E(M))$.

Lemma 2.2. *(cf.* [11] *and* [14]*) For any R-module, we have the following implications:*

injective $\Rightarrow$ quasi-injective $\Rightarrow$ vN-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous.

Also all implications are one-sided.

Let R be a ring and F a left Gabriel topology on R. According to Garcia Hernandez and Gomez Pardo [6], we say that R is an FV-ring if every simple object of the category (R,F)-Mod is injective. Note that if $F=\{R\}$, then FV-rings are precisely the left V-rings. On the other hand, if F is a perfect Gabriel topology, then the inclusion functor $j:(R,F)\text{-Mod}\to R_F\text{-Mod}$ is an equivalence; hence, R is an FV-ring if and only if R_F is a left V-ring. If F is the left Goldie topology then the category (R,F)-Mod is a spectral category, so clearly R is an FV-ring.

An R-module M is called *F-finitely cogenerated* if M_F is a finitely cogenerated object of (R,F)-Mod (see [6]). We record below some well-known results often referred to in the proof of our theorems.

Lemma 2.3. *([6]) Let M be an F-torsionfree R-module. If M is F-finitely cogenerated, then every family of F-torsionfree modules which cogenerates M, does cogenerate it finitely.*

Lemma 2.4. *([6]) Let M be an F-torsionfree R-module. Then M is F-finitely cogenerated if and only if there exist F-cocritical modules $C_1,\dots,C_n$ such that $E(M)\cong\oplus_{i=1}^{n}E(C_i)$.*

By analogy with the concept of σ-cyclic modules defined by Ahsan in [1], we have:

Definition 2.1. *Let F be a left Gabriel topology on R and $M\in R$-Mod. M is called an F-cocyclic R-module if there exists an F-cocritical module C such that the following sequence is exact:*

$$0\to M_F\to E(C_F).$$

M is called a σ-F-cocyclic R-module if M is a direct sum of finitely many F-cocyclic modules.

Every F-cocritical module clearly is F-cocyclic, and so is σ-F-cocyclic. If $F = \{R\}$, then F-cocyclic modules and σ-F-cocyclic modules just are cocyclic modules defined in [9] and [12] and σ-cocyclic modules defined in [12], respectively.

Lemma 2.5. *Let M be an R-module, if M is σ-F-cocyclic, then M is F-finitely cogenerated.*

Proof. It is easy to see that $M = N_1 \oplus \cdots \oplus N_n$, where every N_i is F-cocyclic. Hence there exist F-cocritical modules $C_1, \ldots, C_n$ such that the following sequence is exact:

$$0 \to M_F \to \oplus_{i=1}^{n} E(C_{iF}).$$

The result follows now since every C_{iF} is a simple object of category (R, F)-Mod. ■

3. Main results

Theorem 3.1. *Let F be a left Gabriel topology on R. If every quasi-continuous R-module is F-injective, then the following conditions are equivalent.*

(i) R is an FV-ring.

(ii) Every F-torsionfree and F-finitely cogenerated R-module is dense in its some essential extensions which are quasi-continuous.

(iii) Every F-torsionfree and σ-F-cocyclic R-module is dense in its some essential extensions which are quasi-continuous.

(iv) For every F-torsionfree and F-finitely cogenerated R-module M, there exists a quasi-continuous R-module N with essential submodule M such that $Rad_F(N) = 0$.

(v) For every F-torsionfree and σ-F-cocyclic R-module M, there exists a quasi-continuous R-module N with essential submodule M such that $Rad_F(N) = 0$.

Proof. (i) $\Rightarrow$ (ii). This follows from Theorem 2.2 of [6], which shows that if R is an FV-ring, then every F-torsionfree and F-finitely cogenerated R-module is dense in its injective envelope.

(ii) $\Rightarrow$ (iii). This follows from Lemma 2.5.

(iii) $\Rightarrow$ (i). The inclusion functor $i : (R,F)$-Mod $\rightarrow$ R-Mod has a left adjoint $a : R$-Mod $\rightarrow$ (R,F)-Mod which is exact and assigns to each $M \in R$-Mod its module of quotients M_F. Let C be a simple object of the category (R,F)-Mod. Then $i(C) \in R$-Mod. It is clear that $i(C)$ and $E(i(C))$ are F-cocyclic since $i(C)$ is an F-cocritical module. Hence $D = i(C) \oplus E(i(C))$ is a σ-F-cocyclic module. The fact that D is F-torsionfree is also clear. By hypothesis, there exists an essential extension N of D, such that N is quasi-continuous and D is dense in N. Then it follows that

$$C \oplus E(C) \cong i(C)_F \oplus E(i(C))_F \cong D_F = N_F.$$

By assumption, N is an F-injective module. It is also clear that N is F-torsionfree. Therefore N is F-closed. So we have that $D \cong i(N_F) \cong N$ is a quasi-continuous R-module. By Lemma 2.1, $i(C)$ is $E(i(C))$-injective; that is, any homomorphism from a submodule of $E(i(C))$ to $i(C)$ extends to a homomorphism from $E(i(C))$ to $i(C)$. Thus it is easy to see that $i(C) = E(i(C))$, which means that $i(C)$ is an injective R-module. By Proposition X.1.4 of [16], C is an injective object of (R,F)-Mod. Therefore R is an FV-ring.

(ii) $\Rightarrow$ (iv). Let $M \in R$-Mod be F-torsionfree and F-finitely cogenerated. By condition (ii), there exists an essential extension N of M such that N is quasi-continuous and M is dense in N. By Proposition 1.2 of [6] we have

$$(Rad_F(N))_F = Rad(N_F) = Rad(M_F).$$

Since M is F-finitely cogenerated, there exists F-cocritical R-modules $S_1, \dots, S_n$ such the following sequence is exact:

$$0 \to M_F \to \oplus_{i=1}^n E(S_{iF}).$$

By the results proved above, when condition (ii) holds, R is an FV-ring. Therefore every simple object of (R, F)-Mod is injective. It follows that $E(S_{iF}) = S_{iF}$, $i = 1, \dots, n$, which implies that

$$Rad(\oplus_{i=1}^n E(S_{iF})) = \oplus_{i=1}^n Rad(E(S_{iF})) = 0.$$

Therefore $Rad(M_F) = 0$. Thus $Rad_F(N) = t(Rad_F(N))$, where t is a left exact radical corresponding to the Gabriel topology F. This means that $Rad_F(N)$ is an F-torsion module. On the other hand, $Rad_F(N)$ is an F-saturated submodule of N, so is F-torsionfree. Thus we have $Rad_F(N) = 0$, as required.

(iv) $\Rightarrow$ (v). This follows from Lemma 2.5.

(v) $\Rightarrow$ (iii). Let M be an F-torsionfree and σ-F-cocyclic R-module. Then there exists an essential extension N of M such that N is quasi-continuous and $Rad_F(N) = 0$. It follows from Proposition 1.3 of [6] that N is cogenerated by a class of F-cocritical modules.

Since M is σ-F-cocyclic, by Lemma 2.5, M is F-finitely cogenerated. Thus there exist F-cocritical R-modules $S_1, \dots, S_n$ such that $E(M) \cong \oplus_{i=1}^n E(S_i)$. Therefore N can be embedded in $\oplus_{i=1}^n E(S_i)$ since N is an essential extension of M. This means that N is F-finitely cogenerated. By Lemma 2.3, there exist a finite number of F-cocritical R-modules $C_1, \dots, C_n$ such that the sequence $0 \to N \to \oplus_{i=1}^n C_i$ is exact, which implies that the following sequence is exact:

$$0 \to N_F \to \oplus_{i=1}^n C_{iF}$$

Therefore N_F is a semisimple object of (R, F)-Mod. On the other hand, M is an essential submodule of N, so M_F is essential in N_F by Lemma 0.1 of [6]. It is then easy to see that $M_F = N_F$, in other words M is dense in N, and we are done. ■

The following Corollary is a generalization of [13].

Corollary 3.2. *Let F be a left Gabriel topology on R. If every continuous R-module is F-injective, then the following conditions are equivalent.*

(i) R is an FV-ring.

(ii) Every F-torsionfree and F-finitely cogenerated R-module is dense in its vN-injective envelope.

(iii) Every F-torsionfree and F-finitely cogenerated R-module is dense in its quasi-injective envelope.

(iv) Every F-torsionfree and F-finitely cogenerated R-module is dense in its injective envelope.

(v) For every F-torsionfree and F-finitely cogenerated R-module M,
$Rad_F(E_{vN}(M)) = 0$.

(vi) For every F-torsionfree and F-finitely cogenerated R-module M,
$Rad_F(E_q(M)) = 0$.

(vii) For every F-torsionfree and F-finitely cogenerated R-module M,
$Rad_F(E(M)) = 0$.

Proof. This follows from Lemma 2.2, from Theorem 2.2 of [6], and from the similar proof of Theorem 3.1. ■

Corollary 3.3. *The following conditions are equivalent.*

(i) R is a left V-ring.

(ii) Every finitely cogenerated R-module is quasi-continuous (continuous, vN-injective, quasi-injective, injective, respectively).

(iii) Every σ-cocyclic R-module is quasi-continuous (continuous, vN-injective, quasi-injective, injective, respectively).

4. A PARTICULAR CASE

Let $M \in R$-Mod. Recall that M is *finitely copresented* if and only if M and $E(M)/M$ are finitely cogenerated (see [4]). By [6], M is called *F-finitely copresented* if M_F is a finitely copresented object of (R, F)-Mod. It is proved in [6] that if M is F-torsionfree, then M is F-finitely copresented if and only if M and $E(M)/M$ are F-finitely cogenerated. We give the following definition.

Definition 4.1. *Let F be a left Gabriel Topology on R and $M \in R$-Mod. We say M is F-finitely quasi-copresented (F-finitely vN-copresented, respectively) if M and $E_q(M)/M$ ($E_{vN}(M)/M$, respectively) are F-finitely cogenerated. When $F = \{R\}$, F-finitely quasi-copresented R-modules and F-finitely vN-copresented R-modules are called finitely quasi-copresented and finitely vN-copresented, respectively.*

Lemma 4.1. *For any R-module, we have the following implications.*

F-finitely copresented $\Rightarrow$ F-finitely quasi-copresented $\Rightarrow$ F-finitely vN-copresented $\Rightarrow$ F-finitely cogenerated.

Proof. . It follows from Definition 4.1 and Lemma 2.2. ■

The following example shows that there exists an F-finitely quasi-copresented R-module which is not F-finitely copresented.

Example 4.1. *In a personal communication to V. Camillo and M. F. Yuosif, Patrick Smith pointed out that there are modules E with $Soc(E)$ simple and essential and*

$E/Soc(E)$ not finite dimensional (see [2]). Set $M = Soc(E)$. Then it is clear that M is finitely quasi-copresented. On the other hand, there exists a homomorphism $f : E \to E(M)$ such that $f|_M = i$, where $i : M \to E(M)$ is the inclusion mapping. Since $M = Soc(E)$ is essential in E, it follows that f is monomorphism. Thus E/M is isomorphic to a submodule of $E(M)/M$. Hence $E(M)/M$ is not finite dimensional, which implies that $E(M)/M$ is not finitely cogenerated, and thus M is not finitely copresented.

Lemma 4.2. *([6]) Let F be a strongly semiprime left Gabriel topology on R. If $M \in R$-Mod is not an F-torsion module, then M has a nonzero F-torsionfree and F-finitely cogenerated quotient.*

Theorem 4.3. *Let F be a strongly semiprime left Gabriel topology on R. Suppose that every quasi-continuous R-module is F-injective. Then the following conditions are equivalent.*

(i) R is an FV-ring.

(ii) Every F-torsionfree and F-finitely vN-copresented R-module is dense in its vN-injective envelope.

(iii) Every F-torsionfree and F-finitely quasi-copresented R-module is dense in its quasi-injective envelope.

(iv) For every left ideal I of R, if R/I is F-torsionfree and F-finitely cogenerated, then there exists a quasi-continuous R-module N with essential submodule R/I such that $Rad_F(N) = 0$.

Proof. . (i) ⇒ (ii) and (i) ⇒ (iii) follow from Lemma 4.1 above and Theorem 2.2 of [6].

(ii) $\Rightarrow$ (i). From Theorem 3.1 we see that it is enough to prove that for every F-torsionfree and σ-F-cocyclic R-module M, there exists a quasi-continuous R-module N with essential submodule M such that M is dense in N. Let M be an F-torsionfree and σ-F-finitely cogenerated R-module. It is easy to see that there exists a quasi-continuous R-module N with essential submodule M ($N = E(M)$, or $E_q(M)$ or $E_{vN}(M)$, for example).

Suppose that $N = E_{vN}(M)$ and M is not a dense submodule of N. Then $(N/M)_F \neq 0$ and so N/M is not an F-torsionfree module. By Lemma 4.2, N/M has a nonzero F-finitely cogenerated and F-torsionfree quotient, L, which is of the form $N/K \cong (N/M)/(K/M)$ for some K such that $M \leq K \leq N$.

From Lemma 2.5 it is clear that M is F-finitely cogenerated, and so, by Lemma 2.4 there exist F-cocritical R-modules $S_1, \dots, S_n$ such that $E(M) \cong \oplus_{i=1}^n E(S_i)$. Because $K \leq N$ can be embedded to $E(M)$, we have the following exact sequence:

$$0 \rightarrow K_F \rightarrow \oplus_{i=1}^n E(S_{iF}),$$

which implies that K is F-finitely cogenerated. Now it is clear that K is F-finitely vN-copresented since $N/K \cong L$ is F-finitely cogenerated. Therefore, by condition (ii), K is dense in its vN-injective envelope. It is easy to see that $E_{vN}(M) = E_{vN}(K)$, so $K_F = (E_{vN}(M))_F$. Hence $E_{vN}(M)/K$ is an F-torsion module. On the other hand, $E_{vN}(M)/K \cong L$ is F-torsionfree. Therefore $L = 0$, which is a contradiction. Hence M is dense in $E_{vN}(M)$, as required.

(iii) $\Rightarrow$ (i). It is similar to (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iv). It follows from Theorem 3.1

(iv) $\Rightarrow$ (i). Since F is strongly semiprime, the class of F-torsionfree modules is cogenerated by the direct sum of the injective envelopes of F-cocritical modules. Thus it suffices to prove that if C is F-cocritical, then $E(C)$ is F-semisimple; that is, $E(C)_F$ is a semisimple object of (R, F)-Mod.

Let M be a cyclic submodule of $E(C)$. Then, since $E(C)$ is F-finitely cogenerated, M is F-finitely cogenerated also. Clearly, $M \cong R/I$, where I is a left ideal of R. From condition (iv), there exists a quasi-continuous module N with essential submodule M such that $Rad_F(N) = 0$. Therefore, by Proposition 1.3 of [6], N is cogenerated by F-cocritical modules. It is clear that N is F-finitely cogenerated since N can be embedded in $E(M)$. Therefore N is a submodule of a finite direct sum of F-cocritical modules by Lemma 2.3, which implies that N is F-semisimple. It then follows that M is F-semisimple. It is clear that $E(C)$ is a quotient of the direct sum of its cyclic submodules. Since the class of F-semisimple modules is closed under quotient modules and direct sums (see [7]), it follows that $E(C)$ is F-semisimple. ■

Corollary 4.4. *Let F be a strongly semiprime left Gabriel topology on R. Suppose that every quasi-continuous R-module is F-injective. Then the following conditions are equivalent.*

(i) R is an FV-ring.

(ii) $Rad_F(E_q(R/I)) = 0$ ($Rad_F(E_{vN}(R/I)) = 0$, respectively) for every left ideal I of R such that R/I is F-torsionfree and F-finitely cogenerated.

(iii) For every left ideal I of R, if R/I is F-torsionfree and F-finitely cogenerated, then there exists a continuous R-module N with essential submodule R/I such that $Rad_F(N) = 0$.

Proof. It follows from Theorem 4.3, Corollary 3.1, and Lemma 2.2. ■

In [6], it is proved that R is a left V-ring if and only if every finitely copresented R-module is injective. Here we have:

Corollary 4.5. *The following conditions are equivalent.*

(i) R is a left V-ring.

(ii) Every finitely quasi-copresented R-module is quasi-injective.

(iii) Every finitely vN-copresented R-module is vN-injective.

Recall that a ring R is called left *conoetherian* if every quotient of finitely cogenerated R-module is finitely cogenerated. It is proved in [12] that R is a left V-ring if and only if every finitely copresented R-module is quasi-injective and R is a conoetherian ring. Here we have:

Corollary 4.6. *R is a left V-ring if and only if every finitely copresented R-module is quasi-continuous and R is left conoetherian.*

Proof. $\Rightarrow$). It follows from [9].

$\Leftarrow$). Suppose that R is left conoetherian. Then it follows that every finitely cogenerated R-module is finitely copresented by [9]. Hence every finitely cogenerated module is quasi-continuous. Thus R is a left V-ring by Corollary 3.3. ■

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