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A RECURSIVE APPROACH

by

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Abstract — In this paper, we shall generalize our result on the optimal control of bilinear systems to nonlinear systems. Adomian's decomposition is used to derive series expansions of the optimal control and state.

Keywords: Nonlinear Systems, Optimal Control, Adomian's Decomposition.

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1. Introduction

The theory of optimal control of nonlinear systems has always been an active area of research [2,4,8,9,10,11] to cite a few. However, the methods used led to difficult computations. In a recent paper [5], we solved successfuly and in a simple manner the bilinear quadratic optimal control problem. The idea was to transform the original optimal control problem into a recursive optimization problem, so that at each step we obtain a term of a series representation of the optimal control and a term of the corresponding series representation of the optimal state, i.e., $u^* = \sum_{k\geq 0} u_k^*$, $x^* = \sum_{k\geq 0} x_k^*$. These series were shown to be absolutely and uniformly convergent on $[0,T]$, for any fixed but arbitrary $T > 0$, to the solution of the original problem.

Continuing our effort to provide an effective mean to compute the optimal control, we propose , in this paper to generalize the results in [5] to the optimal control of nonlinear systems.

The outline of the paper is as follows, in section 2, we shall derive a functional series representation of the state in terms of the functions involved in the series representation of the input. We shall prove that if the series representation of the input is absolutely and uniformly convergent so is the series representation of the state. In section 3, we state and solve the optimal control of nonlinear systems.

2. The functional expansion for nonlinear input-output maps

In this section, we shall present a functional expansion for nonlinear i/o maps suitable for the computation of the optimal control.

Consider the nonlinear system defined by,

$$\begin{cases} \frac{dx}{dt} = f(x,u) \quad , \quad t \in [0,T] \\ x(0) = x_0 \end{cases} \tag{2.1}$$

where f is an analytic function from $\mathbf{R}^q \times \mathbf{R}^p$ to $\mathbf{R}^q$ such that $f(0,0) = 0$, u is an p-dimensional input vector and x is an q-dimensional state vector.

We shall use Adomian's decomposition method [1] (see also [5], [6],[7]) and seek a series expansion of the state of (2.1) in the form,

$$x(t) = \sum_{n\geq 0} x^{[n]}(t) \tag{2.2}$$

while

$$u(t) = \sum_{k\geq 0} u^{[k]}(t) \tag{2.3}$$

Thus,

$$f(x,u) = f(\sum_{n\geq 0} x^{[n]}, \sum_{k\geq 0} u^{[k]}) = F_0(x^{[0]}) + \sum_{m\geq 1} F_m(x^{[0]}, ..., x^{[m]}, u^{[0]}, ..., u^{[m-1]}) \tag{2.4}$$

where,

$$\begin{cases} F_0(x^{[0]}) = f(x^{[0]},0) \\ F_m(x^{[0]},...,x^{[m]};u^{[0]},...,u^{[m-1]}) = \\ \sum_{p=1}^{m} \sum_{\substack{k_1+...+k_p=m \\ k_1\geq 1...k_p\geq 1}} f_p(x^{[0]},0)(x^{[k_1]},u^{[k_1-1]};...;x^{[k_p]},u^{[k_p-1]}),\ m\geq 1 \end{cases} \tag{2.5}$$

where the $f_p(x^{[0]},0)(.,.;...;.,.)$ are the p-linear maps appearing in the Taylor's series expansion of f around $(x^{[0]},0)$,

$$f(x,u) = f(x^{[0]},0) + \sum_{p\geq 1} f_p(x^{[0]},0)\underbrace{(x-x^{[0]},u;...;x-x^{[0]},u)}_{p-times} \tag{2.6}$$

Remark 2.1: The F_m are polynomials in $x^{[1]},...,x^{[m]},u^{[0]},...,u^{[m-1]}$, $m\geq 1$.

Going back to (2.1), replacing x, u, $f(x,u)$ by their respective series expansions, we get,

$$\frac{d}{dt}\sum_{n\geq 0} x^{[n]} = F_0(x^{[0]}) + \sum_{m\geq 1} F_m(x^{[0]},...,x^{[m]},u^{[0]},...,u^{[m-1]}) \tag{2.7}$$

We shall define the sequence $\{x^{[n]}\}_{n\geq 0}$ by,

$$\begin{cases} \frac{d}{dt}x^{[0]} = f(x^{[0]},0) \quad , \quad x^{[0]}(0) = x_0 \\ \frac{d}{dt}x^{[m+1]} = F_{m+1}(x^{[0]},...,x^{[m+1]},u^{[0]},...,u^{[m]}), \\ x^{[m+1]}(0) = 0, m\geq 0 \end{cases} \tag{2.8}$$

We shall make the following assumption,

Assumption A:

(i) $||B|| \leq \beta$

(ii) $\exists L_1 > 0 / \ |f(x,u) - \sum_{i=0}^{l-1} F_i| \leq C_1|x - \overline{x}_{l-1}| + C_2|u - \overline{u}_{l-1}| \quad , \quad l \geq L_1 + 1$

where $\overline{x}_l = \sum_{i=0}^{l} x^{[i]}$ and $\overline{u}_l = \sum_{i=0}^{l} u^{[i]}$

(iii) $\widehat{\rho} = \rho C_1 < 1$

The inequality in (ii) is due to the analyticity of $f(x,u)$.

We claim the following theorem:

Theorem 2.1. *Let $u \in L^\infty[0,T,\mathbf{R}^p]$ such that it can be expanded in a uniformly and absolutely convergent series as $\sum_{k\geq 0} u^{[k]}$. Let assumption A be satisfied. Then the state x of the system (2.1) has an absolutely and uniformly convergent series representation $\sum_{n\geq 0} x^{[n]}$ whose terms are computed recursively using (2.8) and (2.5).*

Proof. That the sum $\sum_{n\geq 0} x^{[n]}$ is the solution to (2.1) is trivial. What remains to be proved is the fact that the series $\sum_{n\geq 0} x^{[n]}$ is uniformly and absolutely convergent when $\sum_{k\geq 0} u^{[k]}$ is.

Let $z_l = x - \overline{x}_l$. We have,

$$\frac{d}{dt} z_l = f(x,u) - \sum_{i=0}^{l-1} F_i \quad , \quad z_l(0) = 0 \tag{2.9}$$

A straightforward application of Lemma (2.1)[5] and Gronwall's Lemma together with assumption A, will yield the result (proof follows along the lines of the proof of theorem (2.1)[5]).

3. Optimal control of nonlinear systems

In this section we shall consider the optimization problem

$$\min_u J =: h(x)|_{t=T} + \int_0^T g(x,u)dt \tag{3.1}$$

subject to the constraint (2.1), where h and g are any convex analytic functions.

Let J_l denote the value of the cost functionnal when x is replaced by $\overline{x}_{l+1}$ and u by $\overline{u}_l$ that is,

$$J_l =: h(\overline{x}_{l+1})|_{t=T} + \int_0^T g(\overline{x}_{l+1}, \overline{u}_l)dt \tag{3.2}$$

Suppose that the optimum values $u^{[0]}, ..., u^{[l-1]}$ and the corresponding $x^{[1]}, ..., x^{[l]}$ have been obtained from the minimization of $J_0, ..., J_{l-1}$ with respect to $u^{[0]}, ..., u^{[l-1]}$ (under the corresponding constraint) respectively. We shall seek the minimum of J_l with respect to $u^{[l]}$ subject to the constraint

$$\begin{gathered} \tfrac{d}{dt}x^{[l+1]} = F_{l+1}(x^{[0]}, ..., x^{[l+1]}, u^{[0]}, ..., u^{[l]}), \\ x^{[l+1]}(0) = 0, l \geq 0 \end{gathered} \tag{3.3}$$

We have,

$$F_{l+1}(x^{[0]},...,x^{[l+1]};u^{[0]},...,u^{[l]}) = \frac{\partial f}{\partial x}(x^{[0]},0)x^{[l+1]} + \frac{\partial f}{\partial u}(x^{[0]},0)u^{[l]} + \widehat{F}_{l+1}, l \geq 0 \tag{3.4}$$

where,

$$\widehat{F}_{l+1} = \sum_{p=2}^{l+1} \sum_{\substack{k_1+...+k_p=l+1 \\ k_1\geq 1...k_p\geq 1}} f_p(x^{[0]},0)(x^{[k_1]},u^{[k_1-1]};...;x^{[k_p]},u^{[k_p-1]}) \tag{3.5}$$

$\overline{F}_{l+1}$ is independent of $x^{[l+1]}$ and $u^{[l]}$.

The Hamiltonian of the problem is

$$H_l = g(\overline{x}_{l+1},\overline{u}_l) + \lambda_l^T\{\frac{\partial f}{\partial x}(x^{[0]},0)x^{[l+1]} + \frac{\partial f}{\partial u}(x^{[0]},0)u^{[l]} + \widehat{F}_{l+1}\} \tag{3.6}$$

therefore, the necessary conditions for optimality are,

$$\frac{\partial}{\partial u^{[l]}}H_l = 0 \quad , \quad \frac{\partial}{\partial x^{[l+1]}}H_l + \frac{d\lambda_l^T}{dt} = 0 \tag{3.7}$$

together with the boundary condition

$$\lambda_l^T = \frac{\partial h}{\partial x}(\overline{x}_{l+1}) \quad \text{at} \quad t = T \tag{3.8}$$

which read,

$$\frac{\partial g}{\partial u}(\overline{x}_{l+1},\overline{u}_l) + \lambda_l^T \frac{\partial f}{\partial u}(x^{[0]},0) = 0 \tag{3.9}$$

$$\frac{\partial g}{\partial x}(\overline{x}_{l+1},\overline{u}_l) + \lambda_l^T \frac{\partial f}{\partial x}(x^{[0]},0) + \frac{d\lambda_l^T}{dt} = 0 \tag{3.10}$$

Therefore, locally, around $(\overline{x}_l, \overline{u}_{l-1})$, we have

$$u^{[l]} = -R_l^{-1}\{W_l^T x^{[l+1]} + (\frac{\partial f}{\partial u}(x^{[0]},0))^T \lambda_l\} \tag{3.11}$$

$$\frac{d\lambda_l}{dt} = -(\frac{\partial f}{\partial x}(x^{[0]},0))^T \lambda_l - Q_l x^{[l+1]} - W_l u^{[l]} \tag{3.12}$$

together with

$$\lambda_l = P_l x^{[l+1]} \qquad \text{at} \qquad t = T \tag{3.13}$$

where,

$$\left\{ \begin{aligned} R_l &= \tfrac{\partial^2}{\partial u^2} g(\overline{x}_l, \overline{u}_{l-1}) \\ W_l &= \tfrac{\partial^2}{\partial x \partial u} g(\overline{x}_l, \overline{u}_{l-1}) \\ Q_l &= \tfrac{\partial^2}{\partial x^2} g(\overline{x}_l, \overline{u}_{l-1}) \\ P_l &= \tfrac{\partial^2 h}{\partial x^2}(\overline{x}_l) \end{aligned} \right. \tag{3.14}$$

provided, R_l is non singular.

Remark 3.1: This result agrees with the bilinear quadratic case presented in [5] where $W_l = 0$, $R_l = R$ and $\frac{\partial f}{\partial u}(x^{[0]},0) = B + x^{[0]T}N$.

Standard computations and arguments show that the optimal control is given by,

Theorem 3.1. *The optimal control problem (3.1) and (2.1) has a suboptimal solution given by a uniformly and absolutely convergent series (2.2) and (2.3) whose terms are given by*

$$u^{[l]} = -R_l^{-1}\{W_l^T + B^T S\}x^{[l+1]} - R_l^{-1}B^T v_l \tag{3.15}$$

together with (2.5) and (2.8), where S is the solution of the Ricatti differential equation

$$\begin{gathered} \tfrac{d}{dt}S + SA_l + A_l^T S - SBR_l^{-1}B^T S + \widehat{Q}_l = 0 \\ S = P_l \qquad \text{at} \qquad t = T \end{gathered} \tag{3.16}$$

and v_l is the solution of the differential equation,

$$\begin{gathered} \tfrac{d}{dt}v_l + \{A^T - SBR_l^{-1}B^T - W_l R_l^{-1}B^T\}v_l + S\widehat{F}_{l+1} = 0 \\ v_l = 0 \qquad \text{at} \qquad t = T \end{gathered} \tag{3.17}$$

where R_l, W_l, Q_l, P_l are defined in (3.14) and A, B, A_l and $\widehat{Q}_l$ are defined below,

$$A = \frac{\partial f}{\partial x}(x^{[0]}, 0) \tag{3.18}$$

$$B = \frac{\partial f}{\partial u}(x^{[0]}, 0) \tag{3.19}$$

$$A_l = A - BR_l^{-1}W_l^T \tag{3.20}$$

$$\widehat{Q}_l = Q_l - W_l R_l^{-1} W_l^T \tag{3.21}$$

and provided R_l is non singular for all $l \geq 1$.

Proof. follows along the lines of the proof of theorem (3.2)[5].

4. Conclusion

In this paper we have extended our result [5] concerning the quadratic bilinear optimal control to the optimal control of nonlinear systems.The method used is promising and simple to implement. It consists of solving a matrix Ricatti differential equation and linear differential equation at each step. The method is more systematic as compared to ad-hoc techniques presented for example in [4]. We shall see in a futur paper how well this approach compares with existing methods by providing some simulation results.

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