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ON A CONNECTION BETWEEN THE GENERALIZED INCOMPLETE GAMMA FUNCTIONS AND THEIR EXTENSIONS

By

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Abstract

In this paper we have proved that the generalized incomplete gamma functions and their extensions are mutually related through integral and differential representations.

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1. INTRODUCTION

Chaudhry and Zubair considered the generalized gamma functions [4]

$$\gamma(\alpha, x; b) = \int_0^x t^{\alpha-1} e^{-t-bt^{-1}}\, dt, \quad \text{and} \tag{1}$$

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1} e^{-t-bt^{-1}}\, dt, \tag{2}$$

found useful in a variety of transient heat conduction problems [4, 5, 13, 14].

The extensions

$$\gamma_\nu(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}(b/t)dt, \quad \text{and} \tag{3}$$

$$\Gamma_\nu(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}(b/t)dt, \quad (b > 0,\ x > 0,\ -\infty < \alpha < \infty), \tag{4}$$

of the generalized incomplete gamma functions (1) – (2) were introduced in connection with the generalization of the inverse Gaussian distribution [6]. It is to be noted that

$$\Gamma_0(\alpha, x; b) = \Gamma(\alpha, x; b), \quad \text{and} \tag{5}$$

$$\gamma_0(\alpha, x; b) = \gamma(\alpha, x; b). \tag{6}$$

Some applications of the functions (3) – (4) to the representation of Laplace and K–transforms were shown in [6]. Several properties of these functions including decomposition formula, recurrence relations and special cases were also discussed. It was shown that when $\nu = n$ is an integer, the functions (3) – (4) can be simplified in terms of the generalized incomplete gamma functions (1) – (2). As a matter of fact, it was shown that

$$\Gamma_n(\alpha, x; b) = \sum_{m=0}^{n} \frac{(2b)^{-m}}{m!} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \Gamma(\alpha+m, x; b). \tag{7}$$

For non–integral values of ν we were not able to develop a relationship between the functions (1) – (2) and (3) – (4) and it was left as an open problem. The present paper is a continuation of our earlier work [4, 6].

In this paper we have found interesting relationships between the functions (1) – (2) and their extensions (3) – (4) for non–integral values of ν. Following Erdélyi [8, 9], we shall define the Laplace, Hankel and K–transforms of a function $f(t)$, $(0<t<\infty)$, respectively, as follows:

$$L\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt, \tag{8}$$

$$H_\nu\{f(t); y\} = \int_0^\infty f(t) J_\nu(yt)(yt)^{1/2} dt, \quad \text{and} \tag{9}$$

$$R_\nu\{f(t); y\} = \int_0^\infty f(t) K_\nu(yt)(yt)^{1/2} dt. \tag{10}$$

2. SOME PRELIMINARIES

In this section we recall some results from [6].

Theorem (2.1). *Let* $H(t) = \begin{cases} 1 & if\, t > 0 \\ 0 & if\, t < 0 \end{cases}$ *be the Heaviside unit step function and*

$$\begin{aligned} f(t) &= t^{-\alpha-1}e^{-b/t}H\left(t-\frac{1}{x}\right), \quad b>0,\ x>0, \quad \textit{then} && (11)\\ R_\nu\{f(t);y\} &= \left(\frac{\pi}{2}\right)^{1/2} b^{-\alpha}\gamma_{\nu-\frac{1}{2}}(\alpha, bx; by), && (12)\end{aligned}$$

and

$$L\left\{t^{-\alpha-1}e^{-b/t}H\left(t-\frac{1}{x}\right);y\right\} = b^{-\alpha}\gamma(\alpha, bx; by) \quad (b>0,\ x>0). \tag{13}$$

Theorem (2.2)

$$\text{Let} \quad f(t) = t^{-\alpha-1}e^{-b/t}H\left(\frac{1}{x}-t\right)H(t), \quad (b>0,\ x\geq 0,\ t>0). \tag{14}$$

Then

$$R_\nu\{f(t);y\} = \left(\frac{\pi}{2}\right)^{1/2} b^{-\alpha}\Gamma_{\nu-\frac{1}{2}}(\alpha, bx; by), \tag{15}$$

and

$$L\left\{t^{-\alpha-1}e^{-b/t}H\left(\frac{1}{x}-t\right)H(t);y\right\} = b^{-\alpha}\Gamma(\alpha, bx; by). \tag{16}$$

3. INTEGRAL REPRESENTATIONS

According to (7), the extension $\Gamma_\nu(\alpha, x; b)$ can be simplified in terms of the generalized gamma functions $\Gamma(\alpha, x; b)$ for integral values of ν. In this section we shall prove that these functions are related to each other through the integral representations for all $\nu > -1$. Some special cases of these results are found interesting.

Theorem (3.1)

$$\Gamma_\nu(\alpha, x; y) = \frac{2^{-\nu}y^{-\nu}}{\Gamma(\nu+1)}\int_y^\infty (\xi^2 - y^2)^\nu \Gamma(\alpha-\nu-1, x; \xi)d\xi, \quad (y \geq 0,\ \nu > -1). \tag{17}$$

Proof. Let

$$f(t) = t^{-\alpha-1} e^{-1/t} H\left(\frac{1}{x} - t\right) H(t). \tag{18}$$

Then, according to (15),

$$R_\nu\{f(t); y\} = \left(\frac{\pi}{2}\right)^{1/2} \Gamma_{\nu-\frac{1}{2}}(\alpha, x; y). \tag{19}$$

Moreover, according to (16) we have

$$L\left\{t^{\frac{1}{2}+\nu} f(t); \xi\right\} = \Gamma\left(\alpha - \nu - \frac{1}{2}, x; \xi\right). \tag{20}$$

However, according to [9, p. 122]

$$R_\nu\{f(t); y\} = \frac{\pi^{1/2} 2^{-\nu} y^{\frac{1}{2}-\nu}}{\Gamma\left(\frac{1}{2}+\nu\right)} \int_y^\infty (\xi^2 - y^2)^{\nu-\frac{1}{2}} L\left\{t^{\frac{1}{2}+\nu} f(t); \xi\right\} d\xi, \quad \operatorname{Re} \nu > -\frac{1}{2}. \tag{21}$$

From (19) – (21) we get

$$\left(\frac{\pi}{2}\right)^{1/2} \Gamma_{\nu-\frac{1}{2}}(\alpha, x; y) = \frac{\pi^{1/2} 2^{-\nu} y^{\frac{1}{2}-\nu}}{\Gamma\left(\frac{1}{2}+\nu\right)} \int_y^\infty (\xi^2 - y^2)^{\nu-\frac{1}{2}} \Gamma\left(\alpha - \nu - \frac{1}{2}, x; \xi\right) d\xi. \tag{22}$$

Multiplying both sides in (22) by $\left(\frac{2}{\pi}\right)^{1/2}$ and replacing ν by $\nu + \frac{1}{2}$, our proof is complete.

Corollary.

$$\Gamma(\alpha, x; y) = \int_y^\infty \Gamma(\alpha - 1, x; \xi) d\xi, \quad y \geq 0. \tag{23}$$

Proof. This follows from (17) when we substitute $\nu = 0$. It should be noted that (23) can be proved directly from the definition (2). In particular when we take $y = 0$ in (23) we get an interesting relation

$$\Gamma(\alpha, x) = \int_0^\infty \Gamma(\alpha - 1, x; \xi) d\xi, \tag{24}$$

between the classical incomplete gamma function $\Gamma(\alpha, x)$ and the generalized gamma function $\Gamma(\alpha - 1, x; \xi)$. Several special cases of (24) can be listed. For example, the substitution $\alpha = 0$ in (24) leads to

$$-\operatorname{Ei}(-x) = \int_0^\infty \Gamma(-1, x; \xi) d\xi. \tag{25}$$

The substitution $\alpha = \frac{1}{2}$ in (24) leads to [4]

$$\int_0^\infty \left[e^{-2\sqrt{\xi}} \mathrm{Erfc}\left\{ \sqrt{x} - \sqrt{\frac{\xi}{x}} \right\} - e^{2\sqrt{\xi}} \mathrm{Erfc}\left\{ \sqrt{x} + \sqrt{\frac{\xi}{x}} \right\} \right] \frac{d\xi}{\sqrt{\xi}} = 2\mathrm{Erfc}(\sqrt{x}). \tag{26}$$

Theorem (3.2)

$$\gamma_\nu(\alpha, x; y) = \frac{2^{-\nu} y^{-\nu}}{\Gamma(\nu+1)} \int_y^\infty (\xi^2 - y^2)^\nu \gamma(\alpha - \nu - 1, x; \xi) d\xi, \quad (y \geq 0, \nu > -1). \tag{27}$$

Proof. This is similar to the proof of (17). In particular, substituting $\nu = 0$ in (27), we get

$$\gamma(\alpha, x; y) = \int_y^\infty \gamma(\alpha - 1, x; \xi) d\xi, \tag{28}$$

which can be verified directly from (1).

The substitution $y = 0$ and $\alpha = \frac{1}{2}$ in (28) leads to

$$\int_0^\infty \gamma\left(-\frac{1}{2}, x; \xi\right) d\xi = \sqrt{\pi}\mathrm{Erf}[\sqrt{x}]. \tag{29}$$

Theorem (3.3)

$$\Gamma_\nu(\alpha+\mu, x; b) = 2^{1-\mu}[\Gamma(\mu)]^{-1} b^{\nu+1} \int_b^\infty \xi^{-\mu-\nu} (\xi^2 - b^2)^{\mu-1} \Gamma_{\nu+\mu}(\alpha, x; \xi) d\xi, \quad (\nu \geq -1, \mu > 0, b > 0). \tag{30}$$

Proof. Let $f(t) = t^{-\alpha-1} e^{-1/t} H\left(\frac{1}{x} - t\right) H(t)$. Then, according to (15), we have,

$$g(y;\nu) = R_\nu\{f(t); y\} = \left(\frac{\pi}{2}\right)^{1/2} \Gamma_{\nu-\frac{1}{2}}(\alpha, x; y), \quad \text{and} \tag{31}$$

$$R_\nu\left\{t^{-\mu} f(t); b\right\} = \left(\frac{\pi}{2}\right)^{1/2} \Gamma_{\nu-\frac{1}{2}}(\alpha+\mu, x; b). \tag{32}$$

However, according to the result [9, p. 126(7)]

$$R_\nu\left\{t^{-\mu} f(t); b\right\} = 2^{1-\mu}[\Gamma(\mu)]^{-1} b^{\nu+\frac{1}{2}} \int_b^\infty \xi^{\frac{1}{2}-\mu-\nu} (\xi^2 - b^2)^{\mu-1} g(\xi; \nu+\mu) d\xi, \quad \left(b > 0, \mu > 0, \nu > -\frac{1}{2}\right). \tag{33}$$

From (31) – (33), we get

$$\Gamma_{\nu-\frac{1}{2}}(\alpha+\mu,x;b)=2^{1-\mu}[\Gamma(\mu)]^{-1}b^{\nu+\frac{1}{2}}\int_b^\infty \xi^{\frac{1}{2}-\mu-\nu}(\xi^2-b^2)^{\mu-1}\Gamma_{\nu+\mu-\frac{1}{2}}(\alpha,x;\xi)d\xi. \tag{34}$$

Replacing ν by $\nu+\frac{1}{2}$ in (34), our proof is complete.

Corollary.

$$\Gamma(\alpha+\mu,x;b)=\frac{2^{1-\mu}}{\Gamma(\mu)}\int_b^\infty \xi^{1-\mu}(\xi^2-b^2)^{\mu-1}\Gamma_{\mu-1}(\alpha,x;\xi)d\xi, \quad (\mu>0, b\geq 0). \tag{35}$$

Proof. This follows from (30) when we take $\nu=-1$ and use the fact that

$$\Gamma(\alpha,x;b)=\Gamma_{-1}(\alpha,x;b)=\Gamma_0(\alpha,x;b). \tag{36}$$

In particular, substituting $\mu=1$ in (35), we get

$$\Gamma(\alpha+1,x;b)=\int_b^\infty \Gamma(\alpha,x;\xi)d\xi, \tag{37}$$

which can be verified directly from (2).

The substitution $b=0$ in (35) leads to

$$\Gamma(\alpha+\mu,x)=\frac{2^{1-\mu}}{\Gamma(\mu)}\int_0^\infty \xi^{\mu-1}\Gamma_{\mu-1}(\alpha,x;\xi)d\xi, \quad \mu>0, \tag{38}$$

where $\Gamma(\alpha,x)$ is the classical incomplete gamma function.

Theorem (3.4)

$$\gamma_\nu(\alpha+\mu,x;y)=2^{1-\mu}[\Gamma(\mu)]^{-1}y^{\nu+1}\int_y^\infty (\xi^2-y^2)^{\mu-1}\gamma_{\nu+\nu}(\alpha,x;\xi)d\xi, \quad (\mu>0,\nu>-1,y\geq 0). \tag{39}$$

Proof: Let

$$f(t)=t^{-\alpha-1}e^{-1/t}H\left(t-\frac{1}{x}\right), \quad x>0. \tag{40}$$

Then, following the steps of Theorem (3.3), we get the proof of (39). In particular, the substitution $\nu = -1$ in (39) leads to

$$\gamma(\alpha+\mu, x; y) = \frac{2^{1-\mu}}{\Gamma(\mu)} \int_y^\infty \xi^{1-\mu}(\xi^2 - y^2)^{\mu-1} \gamma_{\mu-1}(\alpha, x; \xi) d\xi \quad (x > 0, \mu > 0, y \geq 0). \tag{41}$$

4. DIFFERENTIAL REPRESENTATIONS

The properties of the K–transforms and the relations (11) – (14) could be exploited to prove the differential representations of the generalized incomplete gamma functions and their extensions. In this section we prove these representations.

Theorem (4.1)

$$\gamma_\nu(\alpha - m, x; y) = y^\nu \left(-\frac{1}{y}\frac{\partial}{\partial y}\right)^m \left[y^{m-\nu}\gamma_{\nu-m}(\alpha, x; y)\right], \quad (x > 0, m = 0, 1, 2, 3, \ldots,). \tag{42}$$

Proof. Let $f(t) = t^{-\alpha-1}e^{-1/t}H\left(t - \frac{1}{x}\right), \quad x > 0$. Then, according to (12)

$$g(y;\nu) = R_\nu\{f(t); y\} = \left(\frac{\pi}{2}\right)^{1/2} \gamma_{\nu-\frac{1}{2}}(\alpha, x; y), \quad \text{and} \tag{43}$$

$$R_\nu\left\{t^m f(t); y\right\} = \left(\frac{\pi}{2}\right)^{1/2} \gamma_{\nu-\frac{1}{2}}(\alpha - m, x; y). \tag{44}$$

However, according to [9, p. 125(4)]

$$R_\nu\left\{t^m f(t); y\right\} = y^{\nu+\frac{1}{2}} \left(-\frac{1}{y}\frac{\partial}{\partial y}\right)^m \left\{y^{m-\nu-\frac{1}{2}} g(y; \nu - m)\right\}. \tag{45}$$

Therefore, from (43) – (45), we get

$$\gamma_{\nu-\frac{1}{2}}(\alpha - m, x; y) = y^{\nu+\frac{1}{2}} \left(-\frac{\partial}{y\partial y}\right)^m \left\{y^{m-\nu-\frac{1}{2}} \gamma_{\nu-m-\frac{1}{2}}(\alpha, x; y)\right\}. \tag{46}$$

Replacing ν by $\nu + \frac{1}{2}$ in (46), our proof of (42) is complete. In particular the substitution $\nu = m$ in (42) leads to

$$\gamma_m(\alpha - m, x; y) = y^m \left(-\frac{1}{y}\frac{\partial}{\partial y}\right)^m \left\{\gamma(\alpha, x; y)\right\}, \quad (m = 0, 1, 2, 3, \ldots,). \tag{47}$$

Theorem (4.2)

$$\Gamma_\nu(\alpha-m,x;y)=y^\nu\left(-\frac{\partial}{y\partial y}\right)^m\left[y^{m-\nu}\Gamma_{\nu-m}(\alpha,x;y)\right],\quad (m=0,1,2,3,\ldots,). \tag{48}$$

Proof. If we take $f(t)=t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x}-t\right)H(t)$, and follow the steps of the proof of Theorem (4.1), we get the proof of (48). In particular the substitution $\nu=m$ in (48) leads to

$$\Gamma_m(\alpha-m,x;y)=y^m\left(-\frac{\partial}{y\partial y}\right)^m[\Gamma(\alpha,x;y)],\quad (m=0,1,2,3,\ldots,). \tag{49}$$

5. FUNCTIONAL RECURRENCE RELATIONS

Theorem (5.1)

$$\gamma_\nu(\alpha+1,x;y)=\frac{y}{2\nu+1}\left[\gamma_{\nu+1}(\alpha,x;y)-\gamma_{\nu-1}(\alpha,x;y)\right]. \tag{50}$$

Proof. Let $f(t)=t^{-\alpha-1}e^{-1/t}H\left(t-\frac{1}{x}\right)$, $\quad x>0$. Then, according to (12)

$$R_\nu\left\{t^{-1}f(t);y\right\}=\left(\frac{\pi}{2}\right)^{1/2}\gamma_{\nu-\frac{1}{2}}(\alpha+1,x;y)=g(y;\nu). \tag{51}$$

However, according to [9, p. 125(5)]

$$R_\nu\left\{t^{-1}f(t);y\right\}=\frac{y}{2\nu}\left[g(y,\nu+1)-g(y;\nu-1)\right]. \tag{52}$$

From (51) – (52), we get

$$\gamma_{\nu-\frac{1}{2}}(\alpha+1,x;y)=\frac{y}{2\nu}\left[\gamma_{\nu+\frac{1}{2}}(\alpha,x;y)-\gamma_{\nu-\frac{3}{2}}(\alpha,x;y)\right]. \tag{53}$$

Replacing ν by $\nu+\frac{1}{2}$ in (53), our proof is complete.

Theorem (5.2)

$$\Gamma_\nu(\alpha+1,x;y)=\frac{y}{2\nu+1}\left[\Gamma_{\nu+1}(\alpha,x;y)-\Gamma_{\nu-1}(\alpha,x;y)\right],\quad (x\geq 0,y>0). \tag{54}$$

Proof. This is similar to the proof of Theorem (5.1).

6. $C^{(\nu)}(\alpha,x;y)$ AND $S^{(\nu)}(\alpha,x;y)$ FUNCTIONS

The Hankel and K–transforms are related to each other via [9, p. 121]

$$H_\nu\{f(t);y\} = \frac{1}{\pi}\left[e^{\frac{i}{2}(\nu+\frac{1}{2})\pi}R_\nu\{f(t);iy\} + e^{-\frac{i}{2}(\nu+\frac{1}{2})\pi}R_\nu\{f(t);-iy\}\right] \tag{55}$$

Taking $f(t)$ as defined by (18) and replacing ν by $\nu+\frac{1}{2}$ in (55) and using (15), we get

$$H_{\nu+\frac{1}{2}}\left\{t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x}-t\right)H(t);y\right\} = \frac{1}{\sqrt{2\pi}}\left[e^{\frac{i}{2}(\nu+1)}\Gamma_\nu(\alpha,x;iy) + e^{-\frac{i}{2}(\nu+1)}\Gamma_\nu(\alpha,x;-iy)\right] \tag{56}$$

Substituting $x=0$ in (56) and using the relation

$$H(\infty - t) = 1, \tag{57}$$

we get

$$H_{\nu+\frac{1}{2}}\left\{t^{-\alpha-1}e^{-1/t};y\right\} = \frac{1}{\sqrt{2\pi}}\left[e^{\frac{i}{2}(\nu+1)}\Gamma_\nu(\alpha,0;iy) + e^{-\frac{i}{2}(\nu+1)}\Gamma_\nu(\alpha,0;-iy)\right] \tag{58}$$

According to [9, p. 30(15)]

$$H_{\nu+\frac{1}{2}}\left\{t^{-3/2}e^{-1/t};y\right\} = 2\sqrt{y}J_{\nu+\frac{1}{2}}(\sqrt{2y})K_{\nu+\frac{1}{2}}(\sqrt{2y}). \tag{59}$$

Substituting $\alpha=\frac{1}{2}$ in (58) and using (59), we get an interesting relation

$$e^{\frac{i}{2}(\nu+1)}\Gamma_\nu\left(\frac{1}{2},0;iy\right) + e^{-\frac{i}{2}(\nu+1)}\Gamma_\nu\left(\frac{1}{2},0;-iy\right) = 2\sqrt{2\pi y}\,J_{\nu+\frac{1}{2}}(\sqrt{2y})K_{\nu+\frac{1}{2}}(\sqrt{2y}). \tag{60}$$

In particular, for $\nu=-1$ in (60) and using $\Gamma_{-1}(\alpha,x;b)=\Gamma(\alpha,x;b)$, we get

$$\Gamma\left(\frac{1}{2},0;iy\right) + \Gamma\left(\frac{1}{2},0;-iy\right) = 2\sqrt{2\pi y}\,J_{-1/2}(\sqrt{2y})K_{-1/2}(\sqrt{2y}) = 2\sqrt{\pi}\,e^{-\sqrt{2y}}\cos(\sqrt{2y}), \tag{61}$$

which can be verified directly from (2). Similarly, the substitution $\nu=0$ in (60) leads to

$$e^{\frac{i}{2}\pi}\Gamma\left(\frac{1}{2},0;iy\right) + e^{-\frac{i}{2}\pi}\Gamma\left(\frac{1}{2},0;-iy\right) = 2\sqrt{\pi}\,e^{-\sqrt{2y}}\sin(\sqrt{2y}). \tag{62}$$

Therefore, it seems natural to introduce a new pair of functions defined by

$$C^{(\nu)}(\alpha,x;y) = \frac{1}{2}\left[e^{\frac{i}{2}(\nu+1)\pi}\Gamma_\nu(\alpha,x;iy)+e^{-\frac{i}{2}(\nu+1)\pi}\Gamma_\nu(\alpha,x;-iy)\right], \quad \text{and} \tag{63}$$

$$S^{(\nu)}(\alpha,x;y) = \frac{1}{2i}\left[e^{\frac{i}{2}(\nu+1)\pi}\Gamma_\nu(\alpha,x;iy)-e^{-\frac{i}{2}(\nu+1)\pi}\Gamma_\nu(\alpha,x;-iy)\right]. \tag{64}$$

The identities (60) – (62) can now be written as

$$\begin{aligned}
C^{(\nu)}\left(\frac{1}{2},0;y\right) &= 2\sqrt{2\pi y}\,J_{\nu+\frac{1}{2}}(\sqrt{2y}\,)K_{\nu+\frac{1}{2}}(\sqrt{2y}\,),\\
C^{(-1)}\left(\frac{1}{2},0;y\right) &= 2e^{-\sqrt{2y}}\cos(\sqrt{2y}\,), \quad \text{and}\\
C^{(0)}\left(\frac{1}{2},0;y\right) &= 2e^{-\sqrt{2y}}\sin(\sqrt{2y}\,).
\end{aligned}$$

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