Duo rings and finite representation type

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Abstract

In this note, we prove through a short and direct argument, that for a left pure semisimple ring $R$, $R$ is left (resp. right) serial if and only if it is left (resp. right) duo. As a corollary we obtain that if $R$ is a counter-example to the pure semisimple conjecture, then it cannot be a duo ring.

1. Introduction. Throughout, $R$ is a ring with identity and $J = J(R)$ is its Jacobson radical; and, unless stated otherwise, all modules are left unital. Consider the following conditions.

(i) $R$ is artinian serial.
(ii) $R$ has finite representation type.
(iii) $R$ is left pure semisimple, that is, every $R$–module is pure-injective.

It is well-known that the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are always true, and that, if $R$ is commutative, their converses hold as well (see for example [3, 12]). The equivalence of $(iii)$ and $(ii)$ for arbitrary rings, known as the pure semisimple conjecture, has so far eluded all attempts of proof, but has been established for
several types of non-commutative rings (e.g. Artin algebras) mainly through the
use of dualities and homological methods. We refer to [1, 4, 5, 6, 12] for more
details.

In another direction, various authors have described classes of rings for which
duoness implies that they are serial (see for example [7, 10]). The aim of this note
is to discuss by using a direct, non-homological argument, the above conditions
for left (or right) duo rings. As a corollary, we obtain that (iii)⇒(i) for duo rings,
providing yet another evidence as to the truth of the pure semisimple conjecture.
Let us first recall some definitions. Following Warfield [8], we say that a module
is serial if its submodules are linearly ordered by inclusion. The ring $R$ is left
(resp. right) serial if it is a direct sum of serial modules, and is serial if it is both
left and right serial. If $R$ has only finitely many non-isomorphic indecomposable
left modules, it is said to have finite representation type. (It is known that finite
representation type is left-right symmetric.) We say that $R$ is left (resp. right)
duo if each left (resp. right) ideal is two-sided, and that it is duo if it is both left
and right duo.

2. Results. The main result is the following

**Theorem.** Let $R$ be a left pure semisimple ring. Then $R$ is left (resp. right) duo
if and only if it is left (resp. right) serial.

Proof. Since $R$ is left pure semisimple, $RR$ is $\Sigma$ — pure-injective and so, by [11],
$R$ is semiprimary. Suppose first that $R$ is left (resp. right) duo, then by [2], $R$ is
a finite direct sum of left duo local rings $R_n$. Furthermore, $R$ is easily seen to be
left pure semisimple if, and only if, each \( R_n \) is left pure semisimple. We may thus assume that \( R \) itself is a left (resp. right) duo local ring with maximal ideal \( J \). If \( R/J \) is finite, then \( R/J \) is clearly a field, and we infer from [6] that \( R \) is serial. Suppose therefore that \( R/J \) is infinite, and let \( H \) be an infinite subset of \( R/J \) whose elements are distinct modulo \( J \). The proof of the 'only if' part is complete once we show that the two-sided ideals of \( R \) are linearly ordered with respect to inclusion. Let \( u, v \) be in \( J \), we prove that either \( u \in RuR \) or \( v \in RuR \). For each \( h \in H \), set

\[
R_h = \begin{cases} 
R_r & \text{if } R \text{ is left duo} \\
r_hR & \text{if } R \text{ is right duo} 
\end{cases}
\]

where \( r_h = u - vh \), and let \( M_h = R/R_h \), \( P = \prod_{h \in H} M_h \), \( S = \bigoplus_{h \in H} M_h \). Denote by \( q_h (h \in H) \) the canonical composition \( P \xrightarrow{\text{proj}} M_h \xrightarrow{\text{incl}} S \), let \( \mu \in P \) be given by \( \mu(h) = 1 + R_h (h \in H) \) and consider the following system (1) of equations

\[
x + r_h y_h = q_h(v\mu) \quad (h \in H)
\]

with unknowns \( x, (y_h)_{h \in H} \). If (1)' is the system obtained from (1) by restricting \( h \) to a finite subset \( \{h_1, h_2, ..., h_n\} \) of \( H \), and if \( w_{ij} \) (\( 1 \leq i, j \leq n, i \neq j \)) are elements of \( R \) with \( (h_i - h_j)w_{ij} = 1 \) (recall that when \( i \neq j \), \( h_i - h_j \notin J \) and so \( h_i - h_j \) is a unit of \( R \)), then \( x = \sum_{j=1}^{n} q_{hi}(v\mu), y_{hi} = \sum_{j=1, j \neq i}^{n} w_{ij}q_{hi}(\mu) \) (\( 1 \leq i \leq n \)) is easily seen to be a solution of (1)' in \( S \). Since \( S \) is pure-injective, the system (1) is solvable by \( a, (b_h)_{h \in H} \) in \( S \), say. Now, \( a \) and \( (b_h)_{h \in H} \) have finite support, and so there exist \( h_0 \in H \) and \( c, d \in R \) such that \( v = r_{h_0}c + dr_{h_0} \). Next, as \( R \) is left (resp. right) duo, we obtain that \( v \in R_h \), and, since \( R \) is local, this means \( v \in RuR \) or \( u \in RuR \).

For the 'if' part, observe first that the direct product of left duo rings is again
left duo, and so $R$ may be assumed to be a local left serial ring. Since $R$ is perfect, it follows easily that it is left artinian, and has therefore a unique composition series of left ideals $R \supseteq J \supseteq J^2 \supseteq \ldots \supseteq J^k = 0$, for some $k$. It is clear then that $R$ is left duo. A symmetric argument shows that a right serial perfect ring is right duo.

The theorem immediately yields

**Corollary 1.** For any ring $R$, the following statements are equivalent.

(i) $R$ is an artinian serial ring.

(ii) $R$ is a left pure semisimple duo ring.

(iii) $R$ is duo and has finite representation type.

Remark. Combining [9, Theorem 3] and [4, Corollary 5.3], we obtain that a left pure-semisimple duo ring $R$ with $J^2 = 0$ has finite representation type. Corollary 1 shows that the condition $J^2 = 0$ is not necessary.

**Corollary 2.** Let $R$ be a local left pure semisimple one-sided duo ring. Then either $R$ is serial or $J^2 = 0$.

Proof. Use Theorem and [5, Theorem 2.2].

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References

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