Semi-Convergence of Filters and Nets

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By

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ABSTRACT:
In 1963, N. Levine introduced the concept of semi-open set and semi-continuity. Semi-convergence and semi-compactness were first introduced, investigated and characterized by C. Dorsett in 1978 and 1981 respectively. In this paper semi-convergence and semi-clusterence of filters are introduced, investigated and characterized.

Throughout, for a subset A of a topological space X, Cl(A) denotes the closure of A in X; no map is assumed to be continuous or surjective unless mentioned explicitly. Moreover X and Y denote topological spaces. For more details on nets and filters we refer the reader to [Willard; 1970].

DEFINITION. 1. Let (X, T) be a topological space and let A ⊆ X. Then A is semi-open if and only if there exists an open set U in X such that U ⊆ A ⊆ Cl(U). Let SO(X) denote the class of all semi-open sets in a topological space X.

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REMARKS 2. N. Levine proved that a set $A$ in a topological space $X$ is semi-open if and only if $A$ is contained in the closure of the interior of $A$ in $X$. We note that every open set in a topological space $X$ is a semi-open set but clearly a semi-open set may not be an open set in $X$. He also proved that the union of a collection of semi-open sets in a topological space is always semi-open. It is clear that a nowhere dense set in a space $X$ is never semi-open in $X$ and the complement of a nowhere dense set in $X$ is always semi-open in $X$. In particular for any semi-open set $S$ in a space $X$, the difference of the closure of $S$ and $S$ is not semi-open in $X$. The intersection of any family of semi-closed sets in a space $X$ is always semi-closed in $X$. We observe that the intersection of two semi-open sets in a space $X$ may not be a semi-open set in $X$. The semi-interior of a set $A$ in a topological space $X$, denoted by $\text{sInt}(A)$, is the union of all semi-open sets contained in $A$. We note that a set $A$ of a space $X$ is semi-open in $X$ if and only if $A = \text{sInt}(A)$.

DEFINITION 3. If $(X, T)$ is a topological space, $A \subseteq X$ and $x \in X$, then $x$ is a semi-limit point of $A$ if and only if every semi-open set containing $x$ contains a point of $A$ different from $x$. The union of $A$ and the set of all semi-limit points of $A$ is called the semi-closure of $A$ and denoted by $\text{sCl}(A)$.

THEOREM 4. [Das; 1973]. If $(X, T)$ is a topological space, $A \subseteq X$ and $x \in X$, then $x \in \text{sCl}(A)$ if and only if every semi-open set containing $x$ intersects $A$.

DEFINITION 5. Let $X$ be a topological space. We say that a set $M_x \subseteq X$ is a semi-neighborhood of a point $x \in X$ if and only if there exists a semi-open set $S$ such that $x \in S \subseteq M_x$.

DEFINITION 6. Let $(X, T)$ be a topological space. For each $x \in X$, let $S(x) = \{A \in \text{SO}(X) : x \in A\}$. Then $S(x)$ has the finite intersection property. Thus $S(x)$ is a filter subbasis on $X$. Let $S_x$ be the filter generated by $S(x)$, i.e., $S_x = \{A \subseteq X : \text{there exists } \mu \subseteq S(x) \text{ such that } \mu \text{ is finite and } \cap \mu \subseteq A\}$. We will call $S_x$ the semi-neighborhood filter at $x$. 
DEFINITION 7. Let \((X, T)\) be a topological space. Let \(F\) be a filter on \(X\). Let \(x \in X\). We say that \(F\) semi-converges to \(x\) if and only if \(F\) contains \(S_x\), that is, if and only if \(F\) is finer than the semi-neighborhood filter at \(x\).

DEFINITION 8. Let \((X, T)\) be a topological space. Let \(F\) be a filter on \(X\), and let \(x \in X\). We say that \(F\) has \(x\) as a semi-cluster point (or, \(F\) semi-clusters at \(x\)) if and only if every \(F \in F\) meets each \(S \in S(x)\).

PROPOSITION 9. Let \((X, T)\) be a topological space. Let \(F\) be a filter on \(X\), and let \(x \in X\). Show that \(F\) has \(x\) as a semi-cluster point implies that \(x \in \bigcap (sCl(F) : F \in F)\).

PROOF. Easy.

PROPOSITION 10. If \((X, T)\) is a topological space and \(F\) is a filter on \(X\) such that \(F\) semi-converges to \(x\) in \(X\), then \(F\) converges to \(x\).

PROOF. The straightforward proof is omitted.

The following example shows that the converse of proposition 10 is false.

EXAMPLE 11. Let \(X = \{1, 2, 3, 4\}\). Let \(T = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, X\}\) be a topology on \(X\). Consider the filter \(F = \{(1, 2), \{1, 2, 3\}, X\}\) on \(X\). The neighborhood filter at \(3\) is \(N_3 = \{(1, 2, 3), X\}\). Clearly \(N_3 \subseteq F\) implies \(F\) converges to \(3\). Now \(Cl(1) = X\) implies that \(\{1, 3\} \in S_x\). But \(\{1, 3\} \notin F\). Hence \(F\) does not semi-converge to \(3\).

DEFINITION 12. Let \((X, T)\) be a topological space. Let \(F\) be a filter on \(X\), and let \(x \in X\). We say that \(F\) has \(x\) as a strong semi-cluster point (or \(F\) strongly semi-clusters at \(x\)) if and only if every \(F \in F\) meets each \(S \in S_x\).

PROPOSITION 13. If \((X, T)\) is a topological space and \(F\) is a filter on \(X\) such that \(F\) strongly semi-clusters at \(x\) in \(X\), then \(F\) semi-clusters at \(x\).

PROOF. Obvious.
The following example shows that the converse of proposition 13 is not true in general.

**EXAMPLE. 14.** Consider $\mathbb{R}$ with the usual metric. Let $F = (F \subseteq \mathbb{R} : \{1/n : n = 1, 2, \ldots\} \cup \{-1/n : n = 1, 2, \ldots\} \subseteq F)$. Then $F$ is a filter on $\mathbb{R}$. Clearly $F$ semi-clusters at $0$. Note that $\{0\} = (-1, 0] \cap [0, 1)$ being the intersection of two semi-open sets is in $S_0$. Also $A = \{1/n : n = 1, 2, \ldots\} \cup \{-1/n : n = 1, 2, \ldots\}$ belongs to $F$. But $A \cap \{0\} = \emptyset$. Hence $F$ does not have $0$ as a strong semi-cluster point.

**THEOREM 15.** Let $(X, T)$ be a topological space. Let $F$ be a filter on $X$, and let $x \in X$. Then $F$ has $x$ as a strong semi-cluster point if and only if there is a filter $G$ finer than $F$ which semi-converges to $x$.

**PROOF.** If $F$ has $x$ as a strong semi-cluster point, the collection $H = \{S \cap F : S \in S_x, F \in F\}$ is a filter base for a filter $G$ which is finer than $F$ and semi-converges to $x$. Conversely, if $F \subseteq G$ and $G$ semi-converges to $x$, then each $F \in F$ and each $S \in S_x$, belong to $G$ and hence meet, so $F$ strongly semi-clusters at $x$.

**THEOREM 16.** Let $(X, T)$ be a topological space. Let $x \in X$ and $E \subseteq X$. Let $F$ be a filter on $X$ such that $E \in F$ and $F$ semi-converges to $x$. Then $x \in sCl(E)$.

**PROOF.** $F$ semi-converges to $x$ implies that $x$ is a strong semi-cluster point and hence $x$ is a semi-cluster point of $F$. Since $E \in F$. Thus clearly $x \in sCl(E)$.

The following example shows that the converse of the above theorem 16 may not hold.

**EXAMPLE. 17.** Consider $\mathbb{R}$ with the usual metric. $E = \{1/n : n = 1, 2, \ldots\} \cup \{-1/n : n = 1, 2, \ldots\}$. Then $0 \in sCl(E)$. Let $F = \{A \subseteq \mathbb{R} : E \subseteq A\}$. Then $F$ is a filter on $\mathbb{R}$. Clearly $F$ semi-clusters at $0$. Note that $\{0\} = (-1, 0] \cap [0, 1)$ being the intersection of two semi-open sets is in $S_0$ but $\{0\} \not\in F$ as $E \in F$. Thus $F$ does not semi-converge to $0$. 
**Definition 18.** If $F$ is a filter on $X$ and $f : X \to Y$ is a single-valued function where $X$ and $Y$ are topological spaces, then $f(F)$ is the filter on $Y$ having for a base the sets $f(F)$, $F \in F$.

**Definition 19.** Let $f : X \to Y$ be a single-valued function where $X$ and $Y$ are topological spaces. Then $f : X \to Y$ is called semi-continuous if and only if, for any $V$ open in $Y$, $f^{-1}(V) \in SO(X)$.

**Theorem 20.** [Latif; 1993]. Let $f : X \to Y$ be single-valued where $X$ and $Y$ are topological spaces. Then $f : X \to Y$ is semi-continuous if and only if, for each $x$ in $X$ and each neighborhood $U$ of $f(x)$, there is a semi-neighborhood $V$ of $x$ such that $f(V) \subseteq U$.

**Theorem 21.** Let $f : X \to Y$ be single-valued where $X$ and $Y$ are topological spaces. Then $f$ is semi-continuous at $x^* \in X$ if and only if whenever $F$ semi-converges to $x^*$ in $X$ then $f(F)$ converges to $f(x^*)$ in $Y$.

**Proof.** Suppose $f$ is semi-continuous at $x^*$ and $F$ semi-converges to $x^*$. Let $V$ be any neighborhood of $f(x^*)$ in $Y$. Then for some semi-neighborhood $U$ of $x^*$ in $X$, $f(U) \subseteq V$. Then since $U \in F$, $V \in f(F)$. Hence $f(F)$ converges to $f(x^*)$ in $Y$.

Conversely, suppose whenever $F$ semi-converges to $x^*$ in $X$ then $f(F)$ converges to $f(x^*)$ in $Y$. Let $F$ be the filter of all semi-neighborhoods of $x^*$ in $X$. Then each neighborhood $V$ of $f(x^*)$ belongs to $f(F)$. It follows that for some semi-neighborhood $U$ of $x^*$, $f(U) \subseteq V$. Thus $f$ is semi-continuous at $x^*$.

**Definition 22.** Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is said to be irresolute if and only if for any semi-open subset $S$ of $Y$, $f^{-1}(S)$ is semi-open in $X$.

**Theorem 23.** [Latif; 1993]. Let $X$ and $Y$ be topological spaces. Then a function $f : X \to Y$ is irresolute if and only if for each $x$ in $X$ and each semi-neighborhood $U$ of $f(x)$, there is a semi-neighborhood $V$ of $x$ such that $f(V) \subseteq U$.
**THEOREM 24.** Let \( f : X \to Y \) be single-valued where \( X \) and \( Y \) are topological spaces. Then \( f \) is an irresolute at \( x^* \in X \) if and only if whenever a filter \( F \) on \( X \) semi-converges to \( x^* \) in \( X \) then \( f(F) \) semi-converges to \( f(x^*) \) in \( Y \).

**PROOF.** Suppose \( f \) is an irresolute at \( x^* \) and \( F \) semi-converges to \( x^* \). Let \( V \) be any semi-neighborhood of \( f(x^*) \) in \( Y \). Then for some semi-neighborhood \( U \) of \( x^* \) in \( X \), \( f(U) \subseteq V \). Then since \( U \in F \), \( V \in f(F) \).

Conversely, suppose whenever \( F \) semi-converges to \( x^* \) in \( X \) then \( f(F) \) semi-converges to \( f(x^*) \) in \( Y \). Let \( F \) be the filter of all semi-neighborhoods of \( x^* \) in \( X \). Then each semi-neighborhood \( V \) of \( f(x^*) \) belongs to \( f(F) \), so for some semi-neighborhood \( U \) of \( x^* \), \( f(U) \subseteq V \). Thus \( f \) is an irresolute at \( x^* \).

**DEFINITION 25.** Let \((X, T)\) be a topological space. Let \((x_i : i \in I)\) be a net in \( X \), and let \( x \in X \). Then \((x_i : i \in I)\) semi-converges to \( x \) if and only if \((x_i : i \in I)\) is eventually in every semi-open set containing \( x \).

**THEOREM 26.** Let \( X \) be topological space. Then a filter \( F \) semi-converges to \( x \) in \( X \) if and only if the net based on \( F \) semi-converges to \( x \).

**PROOF.** Suppose \( F \) semi-converges to \( x \). If \( S \) is a semi-neighborhood of \( x \), then \( S \in F \). Pick \( p \in S \). Then \((p, S) \in \Lambda F \) and if \((q, T) \geq (p, S) \), then \( q \in T \subseteq S \). Thus the net based on \( F \) semi-converges to \( x \).

Conversely, suppose the net based on \( F \) semi-converges to \( x \). Let \( S \) be a semi-neighborhood of \( x \). Then for some \((p^*, F^*) \in \Lambda F \), we have \((p, F) \geq (p^*, F^*) \) implies \( p \in S \). But then \( F^* \subseteq S \), otherwise, there is some \( q \in F^* - S \), and then \((q, F^*) \geq (p^*, F^*) \), but \( q \notin S \). Hence \( S \in F \), so \( F \) semi-converges to \( x \).

**THEOREM 27.** A net \((x_i : i \in I)\) semi-converges to \( x \) in \( X \) if and only if the filter generated by \((x_i : i \in I)\) semi-converges to \( x \).
PROOF. The net \((x_i : i \in I)\) semi-converges to \(x\) if and only if each semi-neighborhood of \(x\) contains a tail of \((x_i : i \in I)\). Since the tails of \((x_i : i \in I)\) are a base for the filter generated by \((x_i : i \in I)\), the result follows.

**DEFINITION 28.** Let \((X, T)\) be a topological space. Let \((x_i : i \in I)\) be a net in \(X\), and let \(x \in X\). Then \(x\) is a semi-cluster point of \((x_i : i \in I)\) if and only if \((x_i : i \in I)\) is frequently in every semi-open set containing \(x\).

**THEOREM 29.** The following conditions are equivalent for a topological space \(X\).

(a) \(X\) is semi-compact.

(b) Every filter in \(X\) has a semi-cluster point.

(c) Every net in \(X\) has a semi-cluster point.

**PROOF.** (a) \(\Rightarrow\) (b). If \(F\) is a filter, then \(F^* = \{\text{sCl}(S) : S \in F\}\) is a collection of semi-closed sets with the finite intersection property. Hence it is fixed by theorem 3.3 of [Dorsett; 1981] and each point in its intersection is a semi-cluster point.

(b) \(\Rightarrow\) (c). Given a net, its associated filter has a semi-cluster point; this is a semi-cluster point of the net, by definition.

(c) \(\Rightarrow\) (b). Interchange net and filter in the preceding part.

(b) \(\Rightarrow\) (a). Let \(C\) be a collection of semi-closed sets with finite intersection property. Let \(B\) be the set of all finite intersections of members of \(C\). Then clearly \(B\) is a filterbase for a filter \(F\) and \(C\) is included in \(F\). Let \(x\) be a semi-cluster point of \(F\). Then \(x \in \bigcap\{\text{sCl}(S) : S \in F\} \subseteq \bigcap\{\text{sCl}(S) : S \in C\} = \bigcap\{S : S \in C\}\). Thus \(C\) is fixed, and \(X\) is semi-compact by theorem 3.3 of [Dorsett; 1981].

**THEOREM 30.** Let \((X, T)\) be a topological space, let \(A \subseteq X\), and let \(x \in X\). Then

(a) If there exists a filter in \(A - \{x\}\) semi-converging to \(x\), then \(x\) is a semi-limit point of \(A\).

(b) If there exists a filter in \(A\) semi-converging to \(x\), then \(x \in \text{sCl}(A)\).

(c) If \(A\) is semi-closed, then no filter in \(A\) semi-converges to a point in \(X - A\).
PROOF. (a) Let $F$ be a filter in $A - \{x\}$ such that $F$ semi-converges to $x$. Let $S$ be a semi-open set with $x \in S$. Then as $F$ semi-converges to $x$, so $(A - \{x\}) \cap S \neq \emptyset$. Hence, $x$ is a semi-limit point of $A$.

(b) Obvious.

(c) If $F$ is a filter in $A$ semi-converging to a point $x$ in $X - A$, then clearly $x \in sCl(A) = A$, a contradiction.

**Theorem 3.1.** Let $(X, T)$ be a topological space, let $p \in X$. If $p$ is a cluster point of a filter $F$ on $X$, then at least one ultra filter containing $F$ semi-converges to $p$.

PROOF. Let $E$ denote the filter of all semi-neighborhoods of $p$. Then $F \cup E$ has the finite intersection property, and so it is embeddable in an ultra filter $A$. Since $A$ contains $E$, it semi-converges to $p$.

Note. In particular, an ultra filter semi-converges to any semi-cluster point.

**Theorem 3.2.** Let $\{X_i : i \in I\}$ be a collection of topological spaces and let $X$ be its product space. Then a filter $F$ semi-converges to $x = (x_i : i \in I)$ in $X$ implies that $\pi_i(F)$ semi-converges to $x_i$, for each $i \in I$.

PROOF. Suppose $F$ semi-converges to $x$. Fix $i \in I$. Let $S_i$ be a semi-open set in $X_i$ such that $x_i \in S_i$. For every $j \in I$, let $T_j = S_i$ if $j = i$ and $T_j = X_j$ if $j \neq i$. Let $S$ be the product of $\{T_j : j \in I\}$. Then clearly $x \in S$. Also note that $S$ is semi-open in $X$ by theorem 8 of [Noiri; 1973]. Then by hypothesis, $S \cap F \neq \emptyset$, for all $F \in F$. Take any $F \in F$. Then we can fix $y = (y_i : i \in I) \in S \cap F$. This implies that $y \in S$ and $y \in F$. Hence we have $\pi_i(y) = y_i \in \pi_i(S) = S_i$ and $\pi_i(y) = y_i \in \pi_i(F)$. So $y_i \in S_i \cap \pi_i(F)$. Therefore $S_i \cap \pi_i(F) \neq \emptyset$, for all $F \in F$. Since $\{\pi_i(F) : F \in F\}$ is a base for the filter $\pi_i(F)$. Hence $\pi_i(F)$ semi-converges to $x_i$. 
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