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SEMI-CONVERGENCE OF FILTERS AND NETS

By

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ABSTRACT:
In 1963, N. Levine introduced the concept of semi-open set and semi-continuity. Semi-convergence and semi-compactness were first introduced, investigated and characterized by C. Dorsett in 1978 and 1981 respectively. In this paper semi-convergence and semi-clusterence of filters are introduced, investigated and characterized.

Throughout, for a subset A of a topological space X, Cl(A) denotes the closure of A in X; no map is assumed to be continuous or surjective unless mentioned explicitly. Moreover X and Y denote topological spaces. For more details on nets and filters we refer the reader to [Willard; 1970].

DEFINITION. 1. Let (X, T) be a topological space and let $A \subseteq X$. Then A is semi-open if and only if there exists an open set U in X such that $U \subseteq A \subseteq Cl(U)$. Let SO(X) denote the class of all semi-open sets in a topological space X.

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REMARKS 2. N. Levine proved that a set A in a topological space X is semi-open if and only if A is contained in the closure of the interior of A in X. We note that every open set in a topological space X is a semi-open set but clearly a semi-open set may not be an open set in X. He also proved that the union of a collection of semi-open sets in a topological space is always semi-open. It is clear that a nowhere dense set in a space X is never semi-open in X and the complement of a nowhere dense set in X is always semi-open in X. In particular for any semi-open set S in a space X, the difference of the closure of S and S is not semi-open in X. The intersection of any family of semi-closed sets in a space X is always semi-closed in X. We observe that the intersection of two semi-open sets in a space X may not be a semi-open set in X. The semi-interior of a set A in a topological space X, denoted by sInt(A), is the union of all semi-open sets contained in A. We note that a set A of a space X is semi-open in X if and only if A = sInt(A).

DEFINITION. 3. If (X, T) is a topological space, $A \subseteq X$ and $x \in X$, then x is a semi-limit point of A if and only if every semi-open set containing x contains a point of A different from x. The union of A and the set of all semi-limit points of A is called the semi-closure of A and denoted by sCl(A).

THEOREM. 4. [Das; 1973]. If (X, T) is a topological space, $A \subseteq X$ and $x \in X$, then $x \in$ sCl(A) if and only if every semi-open set containing x intersects A.

DEFINITION. 5. Let X be a topological space. We say that a set $M_x \subseteq X$ is a semi-neighborhood of a point $x \in X$ if and only if there exists a semi-open set S such that $x \in S \subseteq M_x$.

DEFINITION. 6. Let (X, T) be a topological space. For each $x \in X$, let $S(x) = \{A \in SO(X) : x \in A\}$. Then S(x) has the finite intersection property. Thus S(x) is a filter subbasis on X. Let S_x be the filter generated by S(x), i.e., $S_x = \{A \subseteq X :$ there exists $\mu \subseteq S(x)$ such that μ is finite and $\cap\mu \subseteq A\}$. We will call S_x the semi-neighborhood filter at x.

DEFINITION. 7. Let (X, T) be a topological space. Let F be a filter on X. Let $x \in X$. We say that F semi-converges to x if and only if F contains S_x, that is, if and only if F is finer than the semi-neighborhood filter at x.

DEFINITION. 8. Let (X, T) be a topological space. Let F be a filter on X, and let $x \in X$. We say that F has x as a semi-cluster point (or, F semi-clusters at x) if and only if every $F \in$ F meets each $S \in S(x)$.

PROPOSITION. 9. Let (X, T) be a topological space. Let F be a filter on X, and let $x \in X$. Show that F has x as a semi-cluster point implies that $x \in \cap\{sCl(F) : F \in F\}$.

PROOF. Easy.

PROPOSITION. 10. If (X, T) is a topological space and F is a filter on X such that F semi-converges to x in X, then F converges to x.

PROOF. The straightforward proof is omitted.

The following example shows that the converse of proposition 10 is false.

EXAMPLE. 11. Let $X = \{1, 2, 3, 4\}$. Let $T = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, X\}$ be a topology on X. Consider the filter $F = \{\{1, 2\}, \{1, 2, 3\}, X\}$ on X. The neighborhood filter at 3 is $N_3 = \{\{1, 2, 3\}, X\}$. Clearly $N_3 \subseteq F$ implies F converges to 3. Now $Cl(\{1\}) = X$ implies that $\{1, 3\} \in S_x$. But $\{1, 3\} \notin F$. Hence F does not semi-converge to 3.

DEFINITION. 12. Let (X, T) be a topological space. Let F be a filter on X, and let $x \in X$. We say that F has x as a strong semi-cluster point (or F strongly semi-clusters at x) if and only if every $F \in$ F meets each $S \in S_x$.

PROPOSITION. 13. If (X, T) is a topological space and F is a filter on X such that F strongly semi-clusters at x in X, then F semi-clusters at x.

PROOF. Obvious.

The following example shows that the converse of proposition 13 is not true in general.

EXAMPLE. 14. Consider R with the usual metric. Let $\mathcal{F} = \{F \subseteq R : \{1/n : n = 1, 2, ...\} \cup \{-1/n : n = 1, 2, ...\} \subseteq F\}$. Then $\mathcal{F}$ is a filter on R. Clearly $\mathcal{F}$ semi-clusters at 0. Note that $\{0\} = (-1, 0] \cap [0, 1)$ being the intersection of two semi-open sets is in S_0. Also $A = \{1/n : n = 1, 2, ...\} \cup \{-1/n : n = 1, 2, ...\}$ belongs to $\mathcal{F}$. But $A \cap \{0\} = \emptyset$. Hence $\mathcal{F}$ does not have 0 as a strong semi-cluster point.

THEOREM. 15. Let (X, T) be a topological space. Let $\mathcal{F}$ be a filter on X, and let $x \in X$. Then $\mathcal{F}$ has x as a strong semi-cluster point if and only if there is a filter $\mathcal{G}$ finer than $\mathcal{F}$ which semi-converges to x.

PROOF. If $\mathcal{F}$ has x as a strong semi-cluster point, the collection $\mathcal{H} = \{S \cap F : S \in S_x, F \in \mathcal{F}\}$ is a filter base for a filter $\mathcal{G}$ which is finer than $\mathcal{F}$ and semi-converges to x. Conversely, if $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{G}$ semi-converges to x, then each $F \in \mathcal{F}$ and each $S \in S_x$, belong to $\mathcal{G}$ and hence meet, so $\mathcal{F}$ strongly semi-clusters at x.

THEOREM. 16. Let (X, T) be a topological space. Let $x \in X$ and $E \subseteq X$. Let $\mathcal{F}$ be a filter on X such that $E \in \mathcal{F}$ and $\mathcal{F}$ semi-converges to x. Then $x \in sCl(E)$.

PROOF. $\mathcal{F}$ semi-converges to x implies that x is a strong semi-cluster point and hence x is a semi-cluster point of $\mathcal{F}$. Since $E \in \mathcal{F}$. Thus clearly $x \in sCl(E)$.

The following example shows that the converse of the above theorem 16 may not hold.

EXAMPLE. 17. Consider R with the usual metric. $E = \{1/n : n = 1, 2, ...\} \cup \{-1/n : n = 1, 2, ...\}$. Then $0 \in sCl(E)$. Let $\mathcal{F} = \{A \subseteq R : E \subseteq A\}$. Then $\mathcal{F}$ is a filter on R. Clearly $\mathcal{F}$ semi-clusters at 0. Note that $\{0\} = (-1, 0] \cap [0, 1)$ being the intersection of two semi-open sets is in S_0 but $\{0\} \notin \mathcal{F}$ as $E \in \mathcal{F}$. Thus $\mathcal{F}$ does not semi-converge to 0.

DEFINITION. 18. If $\mathcal{F}$ is a filter on X and f : X → Y is a single-valued function where X and Y are topological spaces, then f($\mathcal{F}$) is the filter on Y having for a base the sets f(F), F ∈ $\mathcal{F}$.

DEFINITION. 19. Let f : X → Y be a single-valued function where X and Y are topological spaces. Then f : X → Y is called semi-continuous if and only if, for any V open in Y, $f^{-1}(V) \in SO(X)$.

THEOREM 20. [Latif; 1993]. Let f : X → Y be single-valued where X and Y are topological spaces. Then f : X → Y is semi-continuous if and only if, for each x in X and each neighborhood U of f(x), there is a semi-neighborhood V of x such that $f(V) \subseteq U$.

THEOREM. 21. Let f : X → Y be single-valued where X and Y are topological spaces. Then f is semi-continuous at $x^* \in X$ if and only if whenever $\mathcal{F}$ semi-converges to x* in X then f($\mathcal{F}$) converges to f(x*) in Y.

PROOF. Suppose f is semi-continuous at x* and $\mathcal{F}$ semi-converges to x*. Let V be any neighborhood of f(x*) in Y. Then for some semi-neighborhood U of x* in X, $f(U) \subseteq V$. Then since $U \in \mathcal{F}$, $V \in f(\mathcal{F})$. Hence f($\mathcal{F}$) converges to f(x*) in Y.
Conversely, suppose whenever $\mathcal{F}$ semi-converges to x* in X then f($\mathcal{F}$) converges to f(x*) in Y. Let $\mathcal{F}$ be the filter of all semi-neighborhoods of x* in X. Then each neighborhood V of f(x*) belongs to f($\mathcal{F}$). It follows that for some semi-neighborhood U of x*, $f(U) \subseteq V$. Thus f is semi-continuous at x*.

DEFINITION. 22. Let X and Y be topological spaces. A function f : X → Y is said to be irresolute if and only if for any semi-open subset S of Y, $f^{-1}(S)$ is semi-open in X.

THEOREM. 23. [Latif; 1993]. Let X and Y be topological spaces. Then a function f : X → Y is irresolute if and only if for each x in X and each semi-neighborhood U of f(x), there is a semi-neighborhood V of x such that $f(V) \subseteq U$.

<u>**THEOREM. 24**</u>. Let $f : X \to Y$ be single-valued where X and Y are topological spaces. Then f is an irresolute at $x^* \in X$ if and only if whenever a filter F on X semi-converges to x^* in X then f(F) semi-converges to $f(x^*)$ in Y.

PROOF. Suppose f is an irresolute at x^* and F semi-converges to x^*. Let V be any semi-neighborhood of $f(x^*)$ in Y. Then for some semi-neighborhood U of x^* in X, $f(U) \subseteq V$. Then since $U \in F$, $V \in f(F)$.

Conversely, suppose whenever F semi-converges to x^* in X then f(F) semi-converges to $f(x^*)$ in Y. Let F be the filter of all semi-neighborhoods of x^* in X. Then each semi-neighborhood V of $f(x^*)$ belongs to f(F), so for some semi-neighborhood U of x^*, $f(U) \subseteq V$. Thus f is an irresolute at x^*.

<u>**DEFINITION 25**</u>. Let (X, T) be a topological space. Let $(x_i : i \in I)$ be a net in X, and let $x \in X$. Then $(x_i : i \in I)$ semi-converges to x if and only if $(x_i : i \in I)$ is eventually in every semi-open set containing x.

<u>**THEOREM. 26**</u>. Let X be topological space. Then a filter F semi-converges to x in X if and only if the net based on F semi-converges to x.

PROOF. Suppose F semi-converges to x. If S is a semi-neighborhood of x, then $S \in F$. Pick $p \in S$. Then $(p, S) \in \Lambda_F$ and if $(q, T) \geq (p, S)$, then $q \in T \subseteq S$. Thus the net based on F semi-converges to x.

Conversely, suppose the net based on F semi-converges to x. Let S be a semi-neighborhood of x. Then for some $(p^*, F^*) \in \Lambda_F$, we have $(p, F) \geq (p^*, F^*)$ implies $p \in S$. But then $F^* \subseteq S$, otherwise, there is some $q \in F^* - S$, and then $(q, F^*) \geq (p^*, F^*)$, but $q \notin S$. Hence $S \in F$, so F semi-converges to x.

<u>**THEOREM. 27**</u>. A net $(x_i : i \in I)$ semi-converges to x in X if and only if the filter generated by $(x_i : i \in I)$ semi-converges to x.

PROOF. The net $(x_i : i \in I)$ semi-converges to x if and only if each semi-neighborhood of x contains a tail of $(x_i : i \in I)$. Since the tails of $(x_i : i \in I)$ are a base for the filter generated by $(x_i : i \in I)$, the result follows.

DEFINITION 28. Let (X, T) be a topological space. Let $(x_i : i \in I)$ be a net in X, and let $x \in X$. Then x is a semi-cluster point of $(x_i : i \in I)$ if and only if $(x_i : i \in I)$ is frequently in every semi-open set containing x.

THEOREM 29. The following conditions are equivalent for a topological space X.
(a) X is semi-compact.
(b) Every filter in X has a semi-cluster point.
(c) Every net in X has a semi-cluster point.

PROOF. (a) ⇒ (b). If F is a filter, then $F^* = \{sCl(S) : S \in F\}$ is a collection of semi-closed sets with the finite intersection property. Hence it is fixed by theorem 3.3 of [Dorsett; 1981] and each point in its intersection is a semi-cluster point.

(b) ⇒ (c). Given a net, its associated filter has a semi-cluster point; this is a semi-cluster point of the net, by definition.

(c) ⇒ (b). Interchange net and filter in the preceding part.

(b) ⇒ (a). Let C be a collection of semi-closed sets with finite intersection property. Let B be the set of all finite intersections of members of C. Then clearly B is a filterbase for a filter F and C is included in F . Let x be a semi-cluster point of F . Then $x \in \cap\{sCl(S) : S \in F\} \subseteq \cap\{sCl(S) : S \in C\} = \cap\{S : S \in C\}$. Thus C is fixed, and X is semi-compact by theorem 3.3 of [Dorsett; 1981].

THEOREM 30. Let (X, T) be a topological space, let $A \subseteq X$, and let $x \in X$. Then
(a) If there exists a filter in $A - \{x\}$ semi-converging to x, then x is a semi-limit point of A.
(b) If there exists a filter in A semi-converging to x, then $x \in sCl(A)$.
(c) If A is semi-closed, then no filter in A semi-converges to a point in $X - A$.

PROOF. (a) Let $\mathcal{F}$ be a filter in $A - \{x\}$ such that $\mathcal{F}$ semi-converges to x. Let S be a semi-open set with $x \in S$. Then as $\mathcal{F}$ semi-converges to x, so $(A - \{x\}) \cap S \neq \emptyset$. Hence, x is a semi-limit point of A.

(b) Obvious.

(c) If $\mathcal{F}$ is a filter in A semi-converging to a point x in $X - A$, then clearly $x \in sCl(A) = A$, a contradiction.

THEOREM. 31. Let (X, T) be a topological space, let $p \in X$. If p is a cluster point of a filter $\mathcal{F}$ on X, then at least one ultra filter containing $\mathcal{F}$ semi-converges to p.

PROOF. Let $\mathcal{E}$ denote the filter of all semi-neighborhoods of p. Then $\mathcal{F} \cup \mathcal{E}$ has the finite intersection property, and so it is embeddable in an ultra filter $\mathcal{A}$. Since $\mathcal{A}$ contains $\mathcal{E}$, it semi-converges to p.

Note. In particular, an ultra filter semi-converges to any semi-cluster point.

THEOREM. 32. Let $\{X_i : i \in I\}$ be a collection of topological spaces and let X be its product space. Then a filter $\mathcal{F}$ semi-converges to $x = (x_i : i \in I)$ in X implies that $\pi_i(\mathcal{F})$ semi-converges to x_i, for each $i \in I$.

PROOF. Suppose $\mathcal{F}$ semi-converges to x. Fix $i \in I$. Let S_i be a semi-open set in X_i such that $s_i \in S_i$. For every $j \in I$, let $T_j = S_i$ if $j = i$ and $T_j = X_j$ if $j \neq i$. Let S be the product of $\{T_j : j \in I\}$. Then clearly $x \in S$. Also note that S is semi-open in X by theorem 8 of [Noiri; 1973]. Then by hypothesis, $S \cap F \neq \emptyset$, for all $F \in \mathcal{F}$. Take any $F \in \mathcal{F}$. Then we can fix y $= (y_i : i \in I) \in S \cap F$. This implies that $y \in S$ and $y \in F$. Hence we have $\pi_i(y) = y_i \in \pi_i(S) = S_i$ and $\pi_i(y) = y_i \in \pi_i(F)$. So $y_i \in S_i \cap \pi_i(F)$. Therefore $S_i \cap \pi_i(F) \neq \emptyset$, for all $F \in \mathcal{F}$. Since $\{\pi_i(F) : F \in \mathcal{F}\}$ is a base for the filter $\pi_i(\mathcal{F})$. Hence $\pi_i(\mathcal{F})$ semi-converges to x_i.

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