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PERFECT MAPS IN TOPOLOGICAL SPACES

by

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Abstract. We give a characterization of perfect maps in topological spaces in terms of compact sets.

Properties of perfect maps in metric spaces have been studied by Vainstein [3] in 1947. In [2], Garg and Goel have recently proved that in metric spaces, a map is perfect if and only if it preserves compactness of sets both ways. In this note, we will extend this result from metric spaces to Hausdorff topological spaces. We will also note that the result is not true even in T_1-spaces.

Throughout, for a subset A of a topological space X, Cl(A) denotes the closure of A in X; no map is assumed to be continuous or surjective unless mentioned explicitly.

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We will use the following Theorem in order to prove our next main result.

Definition.1. Let (X, T) and (Y, T^*) be topological spaces. We will say that a mapping $f: X \rightarrow Y$ is compact (compact-preserving) if inverse image (image) of each compact set is compact.

Definition.2. Let (X, T) and (Y, T^*) be topological spaces. We say that a mapping $f: X \rightarrow Y$ is perfect if it is continuous, closed, and compact fibers $f^{-1}(y)$, $y \in Y$.

Theorem.3. Let (X, T) be a topological space. Let A be a subset of X and x be an element of X. Then $x \in Cl(A)$ if and only if there exists a net $(x_i : I \in I)$ in E with $x_i \rightarrow x$.

Proof. Willard [4 , 11.7 p.75].

Theorem.4. Let (X, T) and (Y, T^*) be any two Hausdorff topological spaces. Then prove that a map $f: X \rightarrow Y$ is perfect if and only if it is compact and compact-preserving.

Proof.

If f is perfect, then it is compact by [1, 11.17 p.58] and is compact-preserving since f is continuous. Conversely, assume f is compact and compact-preserving. For continuity of f, let F be a closed subset of Y and suppose that $x \in Cl[f^{-1}(F)] - f^{-1}(F)$. Then by Theorem.3, there exists a net $(x_i : i \in I)$ in $f^{-1}(F)$ such that $x_i \rightarrow x$. Since f is a compact-preserving map and the set $H = \{x_i : i \in I\} \cup \{x\}$ is compact, it follows that the set $f(H)$ is compact and therefore, the set $f(H) \cap F$ is compact. Then f is compact implies the set $f^{-1}[f(H) \cap F]$ is compact, and so closed being a compact subset of a T_2-space. But this is a contradiction, since it is easy to see that $x \in Cl[f^{-1}(f(H) \cap F)] - f^{-1}[f(H) \cap F]$.

For the closedness of f, let F be a closed subset of X and let $y \in Cl[f(F)] - f(F)$. Then by Theorem.3, there exists a net $(x_i : i \in I)$ in F such that $f(x_i) \rightarrow y$. Since f is a compact map and the set $K = \{f(x_i) : i \in I\} \cup \{y\}$ is compact, it follows that the set $f^{-1}(K)$ is compact and therefore, the set $F \cap f^{-1}(K)$ is compact. Then f is compact-preserving implies the set $f[F \cap f^{-1}(K)]$ is compact, and so closed being a compact subset of a T_2-space. But this is a contradiction, since it is easy to see that

$$y \in Cl[f(F \cap f^{-1}(K))] - f[F \cap f^{-1}(K)].$$

Finally f is perfect, since the compactness of fibers follows from the compactness of f.

Corollary.5. For any two arbitrary Hausdorff topological spaces X and Y, a bijection $f : X \rightarrow Y$ is a homeomorphism if and only if it is compact and compact-preserving.

The next two examples show that the above Theorem.4, does not hold even any one is a T_1-space.

Example.6. Let X be the set of all natural numbers with cofinite topology T_c. Let $Y = \{1, 2\}$ be with discrete topology T_d. Let $f : X \rightarrow Y$ be defined by $f(1) = 1$, $f(n) = 2$ for $n \geq 2$. Then f is compact and compact-preserving. But f is not a perfect map as f is not continuous.

Example.7. Let $X = \{1, 2\}$ be with topology $T = \{\emptyset, \{1\}, X\}$. Let Y be the set of all natural numbers with cofinite topology T_c. Let $f : X \rightarrow Y$ be defined by $f(1) = 1$, $f(2) = 2$. Then f is compact and compact-preserving. But f is not a perfect map as f is not continuous.

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