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SEMI-OPEN AND REGULARLY CLOSED SETS IN COMPACT METRIC SPACES

by

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ABSTRACT

This paper deals with the finite and countable intersections of semi-open sets and regularly closed sets in compact metric spaces. We prove that a set in a compact metric space is the intersection of two semi-open sets if and only if it is the union of an open set and a nowhere dense set. We also show that a set in a compact metric space is the intersection of finitely many regularly closed sets if and only if it is the union of a regularly closed set and a closed nowhere dense set. We also prove that a countable intersection of semi-open sets is a closed set minus a set of the first category.

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Topological space, Metric space, Open set, Closed set, Interior, Semi-open set, Nowhere dense set, Union, Intersection, Countable, Regularly closed set, Compact space, First category.

INTRODUCTION

In 1963, N. Levine introduced the concept of semi-open set by defining a subset A of a topological space X to be semi-open if there exists an open set U in X such that A contains U and A is contained in the closure of U in X. He proved that a set A in a topological space X is semi-open if and only if A is contained in the closure of the interior of A in X. $SO(X)$ will denote the class of all semi-open sets in a topological space X. We note that every open set in a topological space X is a semi-open set but a semi-open set may not be an open set in X. N. Levine also proved that the union of a collection of semi-open sets in a topological space is always semi-open. However the intersection of even two semi-open sets may not be a semi-open set. It is clear that a nowhere dense set in a space X is always not semi-open in X and the complement of a nowhere dense set in X is always semi-open in X. In particular for any semi-open set S in a space X, the difference of the closure of S and S is not semi-open in X.

MAIN RESULTS

Let (X, d) be a metric space. Let A be a subset of X and a be any point in X. Let δ be any arbitrary positive real number. Then for simplicity we define the following sets.

$$\overline{O_\delta}(A) = \{x \in X: d(x, A) \leq \delta\},$$
$$O_\delta(a) = \{x \in X: d(x, a) < \delta\},$$
$$\overline{O_\delta}(a) = \{x \in X: d(x, a) \leq \delta\}.$$

In the following our first result gives an attractive characterization of the finite intersection of semi-open sets in a topological space.

THEOREM.1. The intersection of a finite number of semi-open sets in a space X is the union of an open set and a nowhere dense set in X.

PROOF: Suppose $A = S_1 \cap S_2 \cap \ldots \cap S_n$ where each S_i is semi-open in X. Then for each $i \in \{1, 2, \ldots, n\}$, there exists G_i open in X such that

$$G_i \subseteq S_i \subseteq Cl(G_i).$$

Let $T_i = S_i - G_i$, for each $i \in \{1, 2, \ldots, n\}$. Then each $S_i = G_i \cup T_i$ and each T_i is nowhere dense in X. Now

$$A = \cap_{i=1}^{n} (G_i \cup T_i)$$
$$= [\cap_{i=1}^{n} G_i] \cup [\cup_{i=1}^{2^n-1} (\cap_{i=1}^{n} E_{(i,k)})]$$

where for all $i \in \{1, 2, 3, \ldots, 2^n - 1\}$, $E_{(i,k)} = G_k$ or T_k, for $k = 1, 2, \ldots, n$, and $E_{(i,k)} = T_k$ for some $k \in \{1, 2, \ldots, n\}$. We notice that $\cap_{i=1}^{n} G_i$ is open, and all other sets in the union are nowhere dense being contained in a nowhere dense set, whence, as is well known, their union is nowhere dense.

Now our next goal is to prove that the converse of the above theorem is also true in a compact metric space. For this we will make use of the result that a closed nowhere dense set in a compact metric space X is contained in the frontier of two disjoint open sets in X. To establish this latter fact, we need the following result.

THEOREM.2. Let (X, ρ) be a metric space. Let A be a compact nowhere dense subset of X. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every $a \in A$, there exists $x_a \in X$ such that $x_a \in O_\varepsilon(a) - O_\delta^-(A)$.

PROOF: Let $\varepsilon > o$. Put $\gamma = \{O_{\varepsilon/2}(a) : a \in A\}$. Then clearly γ is an open covering of A. By hypothesis A is compact, so there exist $a_1, a_2, \ldots, a_n \in A$ such that $A \subseteq \cup_{k=1}^{n} O_{\varepsilon/2}(a_k)$. Since A is nowhere dense in X, $Cl[X - Cl(A)] = X$. Since A is a compact subset of a metric space X, so A is closed in X. Hence $Cl(X - A) = X$. Thus if $a_k \in Cl(X - A)$, for every $k \in \{1, 2, \ldots, n\}$. Then $\rho(a_k, X - A) = 0$, for every $k \in \{1, 2, \ldots, n\}$. So for each $k \in \{1, 2, \ldots, n\}$, there exists $x_k \in X - A$ such that $\rho(a_k, x_k) < \varepsilon/2$. Since $\{x_1, x_2, \ldots, x_n\} \cap A = \Phi$ and A is closed in X. We have $\rho(x_k, A) > 0$, for every $k \in \{1, 2, \ldots, n\}$. Put $\delta = (1/2) \min \{ \rho(x_k, A) : k = 1, 2, \ldots, n\}$. Then $\delta > 0$. Now let $a \in A$. Then $a \in \cup_{k=1}^{n} O_{\varepsilon/2}(a_k)$. So there exists $k_a \in \{1, 2, \ldots, n\}$ such that $a \in O_{\varepsilon/2}(a_k)$. Then $\rho(a, a_{k_a}) < \varepsilon/2$. Also we have $\rho(a_{k_a}, x_{k_a}) < \varepsilon/2$. Thus, by the triangle inequality, we have $\rho(a, x_{k_a}) < \varepsilon$. Also we have, $2\delta \leq \rho(x_{k_a}, A) \leq \rho(x_{k_a}, a)$. Thus $2\delta \leq \rho(x_{k_a}, A) \leq \rho(a, x_{k_a}) < \varepsilon$.

Hence, we have a $\delta > 0$ such that for every $a \in A$, there exists $x_{k_a} = x_a \in X$ such that $\delta < 2\delta \leq \rho(x_a, A) \leq \rho(a, x_a) < \varepsilon$.

i.e., $x_a \in O_\varepsilon(a) - O_\delta^-(A)$.

COROLLARY.3. Let (X, ρ) be a compact metric space. Let A be a nonempty closed nowhere dense subset of X. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every $a \in A$, there exists x_a in $O_\varepsilon(a) - \overline{O}_\delta(A)$.

Now let us obtain the key fact that we need to prove the converse of Theorem 1 in a compact metric space.

THEOREM.4. Let (X, ρ) be a compact metric space. Let A be a closed nowhere dense subset of X. Then there exist disjoint open sets G_1 and G_2 in X such that

$$A \subseteq Cl(G_i) - G_i, \quad \text{for } i = 1, 2.$$

PROOF: By Corollary 3, for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$, such that for each $a \in A$, there exists x_a in $O_\varepsilon(a) - \overline{O}_{\delta_\varepsilon}(A)$. Choose $\varepsilon_1 = 1$ and, once ε_n has been chosen, choose $\varepsilon_{n+1} = \min\{\frac{1}{n+1}, \delta_{\varepsilon_n}, \frac{\varepsilon_n}{2}\}$. Then $\varepsilon_n \to 0$ as $n \to \infty$ and for any $n \in N$, and for any $a \in A$, there exists $x_{a,n} \in O_{\varepsilon_n}(a) - \overline{O}_{\delta_{\varepsilon_n}}(A)$, whence $x_{a,n} \in O_{\varepsilon_n}(a) - \overline{O}_{\varepsilon_{n+1}}(A)$ as $\varepsilon_{n+1} \le \delta_{\varepsilon_n}$.

Define $S_n = O_{\varepsilon_n}(a) - \overline{O}_{\varepsilon_{n+1}}(A)$, for every $n \in N$. Then S_n is open in X. Now for every $a \in A$, for each $n \in N$, there exists $x_{a,n} \in O_{\varepsilon_n}(a) - \overline{O}_{\varepsilon_{n+1}}(A)$. This implies that $x_{a,n} \in S_n$ and $\rho(a, x_{a,n}) < \varepsilon_n$.

Let $G_1 = \cup_{n=0}^{\infty} S_{2n+1}$, $G_2 = \cup_{n=1}^{\infty} S_{2n}$. Take any $n \in N$ and let $x \in S_n$. Then $x \notin \overline{O}_{\varepsilon_{n+1}}(A)$. So we have $\rho(x, A) > \varepsilon_{n+1}$. Since $\{\varepsilon_n : n \in N\}$ is a strictly decreasing sequence of positive real numbers, so $\rho(x, A) > \varepsilon_{n+k}$, for all $k \in N$. Thus $x \notin O_{\varepsilon_{n+k}}(A)$, for all $k \in N$. This implies that $x \notin S_{n+k}$, for all $k \in N$. Thus $S_n \cap S_{n+k} = \phi$, for all $k \in N$. Hence $\{S_n : n \in N\}$ is a pairwise disjoint sequence of open sets in X. Thus G_1 and G_2 are mutually disjoint open sets in X.

Now for all $n \in N$, $A \subseteq \overline{O}_{\varepsilon_{n+1}}(A) \subseteq O_{\varepsilon_n}(A)$. Thus $A \cap S_n = \Phi$, for all $n \in N$. Hence, $A \cap G_i = \Phi$, for $i = 1, 2$.

Now we show that $A \subseteq Cl(G_1)$. Suppose $A - Cl(G_1) \neq \Phi$. Fix $a \in A - Cl(G_1)$. Then $a \notin Cl(G_1)$. So there exists $\varepsilon > 0$ such that $G_1 \cap O_\varepsilon(a) = \phi$.

Hence $S_{2n+1} \cap O_\varepsilon(a) = \phi$, for all $n \in N$.

Take $m \in N$ such that $\varepsilon_{2m+1} < \varepsilon$. Then $[O_{\varepsilon_{2m+1}}(a) - \overline{O}_{\varepsilon_{2m+2}}(A)] \cap O_\varepsilon(a) = \phi$.

So $O_{\varepsilon_{2m+1}}(a) - \overline{O}_{\varepsilon_{2m+2}}(A) = \phi$.

Hence, $x_{a,2m+1} \notin O_{\varepsilon_{2m+1}}(a) - \overline{O}_{\varepsilon_{2m+2}}(A)$ which is a contradiction.

Thus $A \subseteq Cl(G_1)$. Consequently, $A \subseteq Cl(G_1) - G_1$.

Similarly, $A \subseteq Cl(G_2) - G_2$.

Now we state and prove our aimed theorem in a compact metric space.

THEOREM.5. A set A in a compact metric space X is the intersection of finitely many semi-open sets if and only if A is the union of an open set and a nowhere dense set.

PROOF: NECESSITY. Follows from Theorem 1.

SUFFICIENCY. Suppose $A = H \cup N$ where H is open and N is nowhere dense. Since N is nowhere dense in X, there exist two disjoint open sets G_1 and G_2 in X such that $Cl(N) \subseteq Cl(G_i) - G_i$, for $i = 1, 2$. Let $S_i = G_i \cup N$, for $i = 1, 2$. Then S_1 and S_2 are semi-open in X. Also $S_1 \cap S_2 = N$. But then $H \cup S_1$ and $H \cup S_2$ are semi-open. Also

$$(H \cup S_1) \cap (H \cup S_2) = H \cup (S_1 \cap S_2) = H \cup N.$$

So $(H \cup S_1) \cap (H \cup S_2) = A$.

Now our next task is to derive certain properties of finite and countable intersections of regularly closed sets in compact metric spaces.

DEFINITION.6. Let (X, T) be a topological space. A closed subset of X is called regularly closed in X if and only if it is the closure of its interior.

The following lemma gives a nice representation of a closed nowhere dense set in a compact metric space X in terms of regularly closed sets in X. It will be useful to us in what follows.

LEMMA.7. In a compact metric space (X, ρ), a closed nowhere dense set A can be written as $F_1 \cap F_2$ where F_1 and F_2 are regularly closed.

PROOF: Let $S_1, S_2, \ldots$ be the sequence of disjoint open sets as in the proof of Theorem 4. Let $G_1 = \cup_{n=0}^{\infty} S_{4n+1}$, $G_2 = \cup_{n=0}^{\infty} S_{4n+3}$.

We note that $\{S_{4n+1} : n \in N\}$ is a locally finite family of subsets of X. Thus $Cl(G_1) = \cup_{n=0}^{\infty} Cl(S_{4n+1})$.

We also claim that $Int[Cl(G_1)] = \cup_{n=0}^{\infty} Int[Cl(S_{4n+1})]$. For if $x \in Int[Cl(G_1)]$, then $O_\varepsilon(x) \subseteq \cup_{n=0}^{\infty} Cl(S_{4n+1})$ for some $\varepsilon > 0$.

Since $\{Cl(S_{4n+1}) : n \in N\}$ is a pairwise disjoint sequence of closed sets. So there exists a unique $m \in N$ such that $x \in Cl(S_{4m+1})$ and then also easily using normality of X, there exists $\varepsilon^* > 0$ such that $O_{\varepsilon^*}(x) \subseteq Cl(S_{4m+1})$.

Hence, $x \in \cup_{n=0}^{\infty} Int[Cl(S_{4n+1})]$. So $Int[Cl(G_1)] \subseteq \cup_{n=0}^{\infty} Int[Cl(S_{4n+1})]$.

Note the reverse inclusion is obvious.

Now we also note that $\{Int[Cl(S_{4n+1})] : n \in N\}$ is a locally finite family of sets. Hence $Cl[Int[Cl(G_1)]] = \cup_{n=0}^{\infty} Cl[Int[Cl(S_{4n+1})]]$.

But we note that $Cl(S_{4n+1})$ is a regularly closed set. Hence, $Cl[Int[Cl(G_1)]] = \cup_{n=0}^{\infty} Cl(S_{4n+1}) = Cl(G_1)$. Thus $Cl(G_1)$ is regularly closed. Similarly, $Cl(G_2)$ is regularly closed. Now as in Theorem 4, we note that $A \subseteq Cl(G_1)$ and $A \subseteq Cl(G_2)$. Also if $x \notin A$, then $\rho(x, A) > 0$.

It can be seen that x is a closure point of at most two consecutive members of the sequence $\{S_n : n \in N\}$. Say $x \in Cl(S_k)$ and $x \in Cl(S_{k+1})$. Thus $x \notin Cl(G_1) = \cup_{n=0}^{\infty} Cl(S_{4n+1})$ or $x \notin Cl(G_2) = \cup_{n=0}^{\infty} Cl(S_{4n+3})$.

So $x \notin Cl(G_1) \cap Cl(G_2)$. Consequently, $A = Cl(G_1) \cap Cl(G_2)$.

The next Theorem gives a characterization of the finite intersection of regularly closed sets in a compact metric space.

THEOREM.8. Let (X, ρ) be a compact metric space. Then a subset A of X is the intersection of finitely many regularly closed sets if and only if $A = F \cup B$ where F is regularly closed and B is closed nowhere dense in X.

PROOF: NECESSITY. Suppose $A = K_1 \cap K_2 \cap \ldots \cap K_n$ where each K_i is regularly closed. Then each K_i is semi-open, whence by Theorem 1, $A = G \cup C$ where G is open and C is nowhere dense. Let $F = Cl(G)$, $B = Cl(C)$. Then F is regularly closed and B is closed nowhere dense. Also we note that $A = F \cup B$.

SUFFICIENCY. Suppose $A = F \cup B$ where F is regularly closed and B is closed nowhere dense. By Lemma 7, we can write $B = F_1 \cap F_2$ where F_1 and F_2 are regularly closed. Hence,

$$\begin{aligned} A &= F \cup (F_1 \cap F_2) \\ &= (F \cup F_1) \cap (F \cup F_2) \end{aligned}$$

where $F \cup F_1$ and $F \cup F2$ are regularly closed (as it is well known that the union of two regularly closed sets is a regularly closed set).

The following lemma represents a semi-open set in terms of a regularly closed set and a nowhere dense set.

LEMMA.9. Every semi-open set S in a space X can be written as $F - T$ where F is regularly closed and T is nowhere dense in X.

PROOF: Let G be an open set with $G \subseteq S \subseteq Cl(G)$. Then $S = Cl(G) - [Cl(G) - S]$, and $Cl(G)$ is regularly closed while $Cl(G) - S$ is contained in $Cl(G) - G$ and is thus nowhere dense.

Finally, using the last result, we give the characterization of the countable intersection of semi-open sets in terms of a closed set and a set of first category.

THEOREM.10. Every countable intersection of semi-open sets in a space X, is a closed set minus a set of first category.

PROOF: Let S_n be semi-open in X for each $n \in N$. Then by Lemma 9, $S_n = F_n - T_n$ where F_n is regularly closed and T_n is nowhere dense in X. Thus

$$\begin{aligned}\cap_{n=1}^{\infty} S_n &= \cap_{n=1}^{\infty} (F_n - T_n)\\ &= \cap_{n=1}^{\infty} [F_n \cap (X - T_n)]\\ &= (\cap_{n=1}^{\infty} F_n) \cap [\cap_{n=1}^{\infty} (X - T_n)]\\ &= (\cap_{n=1}^{\infty} F_n) \cap (X - \cup_{n=1}^{\infty} T_n)\\ &= \cap_{n=1}^{\infty} F_n - \cup_{n=1}^{\infty} T_n.\end{aligned}$$

As $\cap_{n=1}^{\infty} F_n$ is closed, and $\cup_{n=1}^{\infty} T_n$ is of first category, the theorem follows.

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