King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 193

December 1995

Semi-open and Regularly Closed Set in Separable and Locally Separable Metric Spaces

Raja Mohammad Latif, Stephen Willard

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

SEMI-OPEN AND REGULARLY CLOSED SETS IN SEPARABLE AND LOCALLY SEPARABLE METRIC SPACES

RAJA MOHAMMAD LATIF
Department of Mathematical Sciences
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia

STEPHEN WILLARD
Department of Mathematics
University of Alberta
Edmonton, T6G 2G1 Canada

ABSTRACT

In this paper we extend the results of [LATIF & WILLARD; 1995] to separable and locally separable metric spaces. This paper deals with the finite and countable intersections of semi-open sets and regularly closed sets in separable metric spaces. We prove that a set in a separable metric space is the intersection of two semi-open sets if and only if it is the union of an open set and a nowhere dense set. We also show that a set in a separable metric space is the intersection of finitely many regularly closed sets if and only if it is the union of a regularly closed set and a closed nowhere dense set. We also prove that a countable intersection of semi-open sets is a closed set minus a set of the first category. It is also shown that a set in a locally separable metric space is the intersection of two semi-open sets if and only if it is the union of an open set and a nowhere dense set.

AMS Subject Classification Numbers:

54B, 54D, 54F, 54H

KEY WORDS:

Topological space, Metric space, Semi-open set, Semi-closed set, Open set, Closet set, Interior, Nowhere dense set, Union, Intersection, Countable, Regularly closed set, Compact space, Complete space, Separable, Locally Separable, First category.

INTRODUCTION

In 1963, N. Levine introduced the concept of semi-open set by defining a subset A of a topological space X to be semi-open if there exists an open set U in X such that A contains U and A is contained in the closure of U in X. He proved that a set A in a topological space X is semi-open if and only if A is contained in the closure of the interior of A in X. SO(X) will denote the class of all semi-open sets in a topological space X. We note that every open set in a topological space X is a semi-open set but a semi-open set may not be an open set in X. N. Levine also proved that the union of a collection of semi-open sets in a topological space is always semi-open. However the intersection of even two semi-open sets may not be a semi-open set. It is clear that a nowhere dense set in a space X is always not semi-open in X and the complement of a nowhere dense set in X is always semi-open in X. In particular for any semi-open set S in a space X, the difference of the closure of S and S is not semi-open in X.

MAIN RESULTS

A. SEMI-OPEN SETS AND REGULARLY CLOSED SETS IN SEPARABLE METRIC SPACES

In this main section of this paper, we give a characterization of the intersection of semi-open sets and regularly closed sets in separable metric spaces.

Our first result shows that a nowhere dense set in a separable metric space can be represented in terms of two semi-open sets.

THEOREM.1. If X is a separable metric space and N is a nowhere dense subset of X, then there exist two semi-open sets S_1 and S_2 in X such that $N = S_1 \cap S_2$.

PROOF: It is well-known by Urysohn's Metrization Theorem that a separable metrizable space X can be embedded as a subspace of the Hilbert cube I^{χ_0}. So there exists $\mu : X \to I^{\chi_0}$ an embedding. Let $K = Cl_{I^{\chi_0}}[\mu(X)]$. Then K is a compact metric space, $\mu(X)$ is dense in K and $\mu : X \to K$ is an embedding. Now to verify that $\mu(N)$ is nowhere dense in K, let V be a nonempty open subset of K. Then $\mu^{-1}(V)$ being a nonempty open subset of K, we can fix $x \in \mu^{-1}(V) \cap (X - Cl_X(N))$ as N is nowhere dense in X. Then $\mu(x) \in V$ and $x \notin Cl_X(N)$. But as $\mu : X \to \mu(X)$ is a homeomorphism, so $x \notin Cl_X(N)$ implies that $\mu(x) \notin Cl_{\mu(X)}[\mu(N)]$. This shows that $\mu(N)$ is nowhere dense in $\mu(X)$. Then it follows easily that $\mu(N)$ is nowhere dense in K as $\mu(X)$ is dense in K. Hence by Theorem 4 of [LATIF & WILLARD; 1995], there exist disjoint open sets G_1 and G_2 in K such that

$$Cl_K[\mu(N)] \subseteq Cl_K(G_i) - G_i, \text{ for } i = 1, 2.$$

Let $H_i = G_i \cap \mu(X)$, for $i = 1, 2$.

Then H_i is open in $\mu(X)$ for each $i = 1, 2$, and H_1 and H_2 are disjoint. Also $\mu(N) \subseteq Cl_K(H_i) - H_i$, for $i = 1, 2$. For if $y \in \mu(N)$, then every K-neighborhood of y meets G_i whence every $\mu(X)$-neighborhood of y meets H_i, for $i = 1, 2$. Now set $T_i = H_i \cup \mu(N)$, for $i = 1, 2$. Then clearly T_1 and T_2 are semi-open in $\mu(X)$ and $\mu(N) = T_1 \cap T_2$.

Now let $S_1 = \mu^{-1}(T_1)$ and $S_2 = \mu^{-1}(T_2)$. Then as $\mu : X \to \mu(X)$ is a homeomorphism, so it follows immediately that S_1 and S_2 are semi-open in X and $N = S_1 \cap S_2$.

COROLLARY.2. If X is a separable metric space and $A = G \cup N$ where G is open and N is nowhere dense, then $A = S_1 \cap S_2$ where S_1 and S_2 are semi-open.

PROOF: By Theorem 1, we can write $N = T_1 \cap T_2$ where T_1 and T_2 are semi-open. Then

$$G \cup N = G \cup (T_1 \cap T_2) = (G \cup T_1) \cap (G \cup T_2).$$

Clearly $S_1 = G \cup T_1$ and $S_2 = G \cup T_2$ are semi-open and $A = S_1 \cap S_2$.

Now we will show that a closed set in a separable metric space X can be represented in terms of regularly closed sets in X. To see this we need the following lemmas.

LEMMA.3. If X is dense in a space K and A is open in K, then

$$X \cap Cl_K(A) = Cl_X(X \cap A).$$

PROOF: Since $X \cap Cl_K(A)$ is closed in X and contains $X \cap A$, so $Cl_X(X \cap A) \subseteq X \cap Cl_K(A)$. On the other hand, suppose $x \in X \cap Cl_K(A)$. Then $x \in X$ and every K-neighborhood of x meets A. Let U be an open neighborhood of x in X. Then $U = V \cap X$, where V is open in K. Now V is a K-neighborhood of x and hence meets A, whence $V \cap A$ is a nonempty open set in K, so $(V \cap A) \cap X \neq \Phi$, and therefore $(V \cap X) \cap A \neq \Phi$ whence $U \cap (X \cap A) \neq \Phi$. Thus $x \in Cl_X(X \cap A)$.

The next lemma describes the regularly closed property possessed by the intersection of a regularly closed set and a dense subspace of a space.

LEMMA.4. Let X be dense in a space K. If R is regularly closed in K, then $R \cap X$ is regularly closed in X.

PROOF: We have $R = Cl_K[Int_K(R)]$. But then using lemma 3, we have

$$\begin{aligned} X \cap R &= X \cap Cl_X[Int_K(R)] \\ &= Cl_X\,[X \cap (Int_K(R))]. \end{aligned}$$

Hence $X \cap R$ is the closure of an open set in X, whence $X \cap R$ is regularly closed.

LEMMA.5. Every closed set A in a space X is the union of a regularly closed set and a nowhere dense closed set.

PROOF: It is enough to note that

$$A = [Cl_X\,(Int\,(A))] \cup [A - Int\,(A)].$$

Now let us state and prove our central result.

THEOREM.6. Every closed set in a separable metric space is the intersection of two regularly closed sets.

PROOF: Let X be a separable metric space and let A be a closed subset of X. Embed X as a dense subset of a compact metric space K. Then $A = X \cap Cl_K(A)$. By Lemma 5, $Cl_K(A) = R \cup N$ where R is a regularly closed set and N is a closed nowhere dense set in K. Then by Lemma 7 of [LATIF & WILLARD; 1995], $N = F_1 \cap F_2$ where F_1 and F_2 are regularly closed in K. Thus

$$\begin{aligned} Cl_K(A) &= R \cup (F_1 \cap F_2) \\ &= (R \cup F_1) \cap (R \cup F_2) \\ &= T_1 \cap T_2 \end{aligned}$$

where $T_1 = R \cup F_1$ and $T_2 = R \cup F_2$. Also clearly T_1 and T_2 are regularly closed in K. Hence

$$\begin{aligned} A &= X \cap (T_1 \cap T_2) \\ &= (X \cap T_1) \cap (X \cap T_2) \\ &= E_1 \cap E_2. \end{aligned}$$

By Lemma 4, $E_1 = X \cap T_1$ and $E_2 = X \cap T_2$ are regularly closed in X.

Now our next result is that in a separable metric space X, the difference of a closed set and a set of the first category can be characterized in terms of a countable intersection of semi-open sets in X.

For this purpose, we will make use of the following lemma.

LEMMA.7. If R is regularly closed and T is nowhere dense in a space X, then $R - T$ is semi-open.

PROOF: Let $G = \mathrm{Int}(R - T)$. Then $G \subseteq (R - T)$. We claim that $(R - T) \subseteq \mathrm{Cl}[\mathrm{Int}(R - T)] = \mathrm{Cl}(G)$. To justify our claim, let $x \in (R - T)$ and let V be an open set containing x. It is sufficient to show that $V \cap \mathrm{Int}(R - T) \neq \Phi$. Now $R = \mathrm{Cl}[\mathrm{Int}(R)]$ and $x \in R$, so V meets Int(R). Hence, since T is nowhere dense, V meets $[\mathrm{Int}(R) - \mathrm{Cl}(T)]$. But $[\mathrm{Int}(R) - \mathrm{Cl}(T)]$ is open and is contained in $(R - T)$, so $[\mathrm{Int}(R) - \mathrm{Cl}(T)] \subseteq \mathrm{Int}(R - T)$. Hence V meets $\mathrm{Int}(R - T)$ as required.

Now let us state and prove the result we have been aiming for.

THEOREM.8. Let X be a separable metric space. Let $A = F - T$ where F is closed and T is of first category. Then $A = \cap_{n=1}^{\infty} S_n$ where each S_n is semi-open.

PROOF: By Theorem 6, we can write

$$F = R_1 \cap R_2$$

where R_1 and R_2 are regularly closed. Since T is of first category, we can write $T = \cap_{n=1}^{\infty} T_n$ where each T_n is nowhere dense. Then

$$\begin{aligned} A &= F - T \\ &= (R_1 \cap R_2) - (\cup_{n=1}^{\infty} T_n) \\ &= \cap_{n=1}^{\infty} [(R_1 \cap R_2) - T_n] \\ &= \cap_{n=1}^{\infty} [(R_1 - T_n) \cap (R_2 - T_n)]. \end{aligned}$$

By Lemma 7, for all $n \in N$, $R_1 - T_n$ and $R_2 - T_n$ are semi-open.

The following proposition shows that for each given infinite cardinal number τ, there exists a compact T_2-space Y containing a dense discrete subspace X of cardinality τ such that each non-isolated point in Y is not the intersection of two semi-open sets in Y.

PROPOSITION.9. Let X be an infinite discrete space. Let βX be the Stone-Cech compactification of X. Let $p \in \beta X - X$. Then $\{p\}$ is nowhere dense but $\{p\}$ is not the intersection of two semi-open sets in βX.

PROOF: We note that $\{p\}$ is closed in βX and p is non-isolated because βX is compact T_2 and X is dense in βX. Thus $\{p\}$ is nowhere dense in βX.
In order to prove that $\{p\}$ is not the intersection of two semi-open sets, it is enough to show that for each pair of disjoint open sets, say G and H in βX, $Cl_{\beta X}(G) \cap Cl_{\beta X}(H) = \phi$. For this let $U = G \cap X$ and $V = H \cap X$. Then since X is dense in βX, so we have $Cl_{\beta X}(U) = Cl_{\beta X}(G)$ and $Cl_{\beta X}(V) = Cl_{\beta X}(H)$.
But then U and V are disjoint zero-sets in X, so $Cl_{\beta X}(U) \cap Cl_{\beta X}(V) = \phi$.
Hence $Cl_{\beta X}(G) \cap Cl_{\beta X}(H) = \phi$ as required.

In Particular the preceding proposition shows that a non-isolated point in a separable compact T_2-space may not be an intersection of two semi-open sets.

Now we will show that a non-isolated point in a T_4-space with a discrete subspace may not be an intersection of countably many semi-open sets.

Let X be a Tychonoff space. Then the set $\{t \in \beta X : X \text{ is c-embedded in } X \cup \{t\}\}$ denoted by υX (Pronounced: upsilon X) is called the real compactification of X. We call X realcompact if $\upsilon X = X$.

We call a cardinal m measureable if a set X of cardinal m admits a $\{0,1\}$-valued measure μ such that $\mu(X) = 1$, and $\mu(\{x\}) = 0$ for every $x \in X$.

THEOREM.10. A discrete space is realcompact if and only if its cardinal is nonmeasureable.

PROOF: 12.2 of [GILLMAN & JERISON; 1960].

PROPOSITION.11. Let X be a discrete space with its cardinal nonmeasureable. By Theorem 10, fix $p \in \upsilon X - X$. Then $\{p\}$ is not an intersection of countably many semi-open sets in υX.

PROOF: Contrawise if possible, then suppose that there exists a sequence $\{S_n: n \in N\}$ of semi-open sets such that $\{p\} = \bigcap_{n=1}^{\infty} S_n$. Then there exists a sequence $\{G_n : n \in N\}$ of open sets such that $G_n \subseteq S_n \subseteq Cl_{vX}(G_n)$, for each $n \in N$. This implies that $\bigcap_{n=1}^{\infty} G_n \subseteq \bigcap_{n=1}^{\infty} S_n = \{p\} \subseteq \bigcap_{n=1}^{\infty} Cl_{vX}(G_n)$. Since p being a point in $vX - X$, is not a G_δ in vX. Thus $\bigcap_{n=1}^{\infty} G_n = \phi$.

Now for each $n \in N$, put $H_n = G_n \cap X$. Since X is dense in vX and each G_n is open in vX. So it follows that

$$Cl_{vX}(H_n) = Cl_{vX}(G_n), \text{ for each } n \in N.$$

Then since each H_n is a zero-set in X and $\bigcap_{n=1}^{\infty} H_n = \phi$, so it implies that $\bigcap_{n=1}^{\infty} Cl_{vX}(H_n) = \phi$. Hence $\bigcap_{n=1}^{\infty} Cl_{vX}(G_n) = \phi$. But then $\{p\} = \Phi$, which is impossible. Hence we conclude that $\{p\}$ is not the intersection of countably many semi-open sets in vX.

Our next aim is to give the characterization of a separable closed and nowhere dense subspace of a metrizable space in terms of semi-open sets. To achieve our aim, we will need the following well-known results. Their proofs may be seen in some standard book on General Topology. Firstly let us recall totally bounded spaces. A subset D of a metric space X with the metric ρ will be called an ε-net in X if for every point $x \in X$ there exists an $x^* \in D$ such that $\rho(x, x^*) < \varepsilon$. A metric space (X, ρ) is said to be totally bounded if for each $\varepsilon > 0$ there exists a finite ε-net $\{x_1, x_2, \ldots, x_k\}$ in X. A metrizable space X will be called metrizable in a totally bounded manner provided that there exists a metric ρ on X such that (X, ρ) is totally bounded.

THEOREM.12. In order that a metrizable space X be metrizable in a totally bounded manner it is necessary and sufficient that X be separable.

THEOREM.13. If X is any metrizable space, A is a closed subset of X, and ρ is a compatible metric on A, then ρ can be extended to a compatible metric on X.

THEOREM.14. Every metric space X can be isometrically embedded as a dense subset of a complete space.

THEOREM.15. A set A is nowhere dense in the space X if and only if every nonempty open set contains a nonempty open set which is disjoint from A.

We also need the following Lemma.

LEMMA.16. Let (X^*, ρ^*) be a complete metric space. Let $A \subseteq X^*$ such that A is totally bounded in X^*. Then $Cl_{X^*}(A) = \overline{A}$ is compact.

PROOF: Let $\varepsilon > 0$. Since A is totally bounded. There exists a finite subset D of A such that for each $a \in A$, there exists $d_a \in D$ such that $\rho^*(a, d_a) < \varepsilon/2$. Let $x \in \overline{A}$. Then $A \cap O_{\varepsilon/2}(x) \neq \phi$. Fix any $a_x \in A \cap O_{\varepsilon/2}(x)$. Then $a_x \in A$ and $a_x \in O_{\varepsilon/2}(x)$. i.e., $a_x \in A$ and $\rho^*(a_x, x) < \varepsilon/2$. But also for $a_x \in A$, we have $d_{a_x} \in D$ such that $\rho^*(a_x, d_{a_x}) < \varepsilon/2$. Thus $\rho^*(x, d_{a_x}) \leq \rho^*(x, a_x) + \rho^*(a_x, d_{a_x}) < \varepsilon$. Hence for each $x \in \overline{A}$, there exists $d_x^* = d_{a_x} \in D$ such that $\rho^*(x, d_x^*) < \varepsilon$ and D is a finite subset of $\overline{A}$. Hence $\overline{A}$ is totally bounded in X^*.
Now also $\overline{A}$ being a closed subspace of a complete space X^*, $\overline{A}$ is complete. Thus $\overline{A}$ being complete and totally bounded, $\overline{A}$ is compact.

Now we are ready to prove our main theorem.

THEOREM.17. Let X be a metrizable space. Let A be a separable, closed and nowhere dense subset of X. Then there exist two semi-open sets R_1 and R_2 in X such that $A = R_1 \cap R_2$.

PROOF: By using Theorem 12, let ρ be a compatible metric on A such that (A, ρ) is totally bounded.
Then by applying Theorem 13, we can extend ρ to a compatible metric on X.
Now by making use of Theorem 14, there exists a compatible metric space (X^*, ρ^*) and $f : X \to X^*$ such that f is an isometry of X onto $f(X)$ and $f(X)$ is dense in X^*.
Now to see that $f(A)$ is totally bounded in X^*, let $\varepsilon > 0$. Since A is totally bounded, there exists a finite subset D of A such that for each $a \in A$, there exists $d_a \in D$ such that $\rho(a, d_a) < \varepsilon$. So for any $a \in A$, $f(d_a) \in f(D)$ and $\rho^*(f(a), f(d_a)) = \rho(a, d_a) < \varepsilon$. Also $f(D)$ is a finite subset of $f(A)$. Thus $f(A)$ is totally bounded in X^*.
Put $Z = Cl_{X^*}[f(A)]$. Then by Lemma 16, Z is compact.
Now to show that $f(A)$ is nowhere dense in $f(X)$, let G be a nonempty open subset of $f(X)$. Then clearly $f^{-1}(G)$ is a nonempty open subset of X.
Since A is nowhere dense in X. So by Theorem 15, $f(A)$ is nowhere dense in $f(X)$.

Now to verify that $f(A)$ is nowhere dense in X^*, let G^* be a nonempty open subset of X^*. Then as $f(X)$ is dense in X^*. $G^{**} = G^* \cap f(X)$ is a nonempty open subset of $f(X)$. Hence by Theorem 15, there exists a nonempty open subset H^{**} of $f(X)$ such that $H^{**} \subseteq G^{**}$ and $H^{**} \cap f(A) = \Phi$.

Let H^* be a nonempty open subset of X^* such that

$$H^{**} = H^* \cap f(X) \text{ and } H^* \cap f(A) = \Phi.$$

Then $K^* = H^* \cap G^*$ is a nonempty open subset of X^* such that

$K^* \subseteq G^*$ and $K^* \cap f(A) = \Phi$.

Hence by Theorem 15, $f(A)$ is nowhere dense in X^*.

Thus Z is a closed compact nowhere dense subset of X^*.

So $Z = S_1 \cap S_2$ where S_1 and S_2 are semi-open sets in X^*.

Let U_1 and U_2 be open sets in X^* such that

$$U_i \subseteq S_i \subseteq Cl_{X^*}(U_i), \quad \text{for } i = 1, 2.$$

Put $W_i = f(X) \cap U_i$, $T_i = f(X) \cap S_i$, for $i = 1, 2$.

Then since $f(X)$ is dense in X^*, so $Cl_{X^*}(W_i) = Cl_{X^*}(U_i)$, for $i = 1, 2$.

This implies that $Cl_{f(X)}(W_i) = Cl_{f(X)}(U_i)$, for $i = 1, 2$.

Now $U_i \cap f(X) \subseteq S_i \cap f(X) \subseteq [Cl_{X^*}(U_i)] \cap f(X) = Cl_{f(X)}(U_i)$, for $i = 1, 2$.

Thus we have $W_i \subseteq T_i \subseteq Cl_{f(X)}(W_i)$, for $i = 1, 2$.

Hence T_1 and T_2 are semi-open in $f(X)$.

Also

$$\begin{aligned} Cl_{f(X)}[f(A)] &= f(X) \cap Cl_{X^*}[f(A)] \\ &= f(X) \cap Z \\ &= f(X) \cap (S_1 \cap S_2) \\ &= [f(X) \cap S_1] \cap [f(X) \cap S_2] \\ &= T_1 \cap T_2. \end{aligned}$$

Since $f(A)$ is closed in $f(X)$. So $f(A) = T_1 \cap T_2$.

Let $R_1 = f^{-1}(T_1)$, $R_2 = f^{-1}(T_2)$.

Since $f : X \to f(X)$ is a homeomorphism, we have $A = R_1 \cap R_2$ and also it follows easily that R_1 and R_2 are semi-open in X.

THEOREM. 18. Let (X, ρ) be a metric space. Let A be a closed separable subset of X. Then there exist $T_1, T_2 \in SO(X)$ such that $A = T_1 \cap T_2$.

PROOF: Note that $A = Int(A) \cup (A - Int(A))$. Clearly $(A - Int(A))$ is a separable, closed and nowhere dense subset of X. So by Theorem 17, there exist $S_1, S_2 \in SO(X)$ such that

$$A - \mathrm{Int}(A) = S_1 \cap S_2.$$

Hence $A = (\mathrm{Int}(A)) \cup (S_1 \cap S_2)$

$= (\mathrm{Int}(A) \cup S_1) \cap (\mathrm{Int}(A) \cup S_2)$.

Let $T_i = \mathrm{Int}(A) \cup S_i$, for $i = 1, 2$. Then $A = T_1 \cap T_2$. Also T_1 and T_2 are semi-open sets each being a union of an open set and a semi-open set.

DEFINITION.19. A set A in a topological space X is said to be semi-closed if and only if there exists a closed set F such that $\mathrm{Int}(F) \subseteq A \subseteq F$.

THEOREM.20. A set A in a space X is semi-closed if and only if $\mathrm{Int}(\overline{A}) \subseteq A$.

PROOF: This follows easily.

Now we give a characterization of a semi-closed set in a separable metric space.

THEOREM.21. Let (X, ρ) be a separable metric space. Let A be a semi-closed subset of X. Then there exist two semi-open sets S_1 and S_2 such that $A = S_1 \cap S_2$.

PROOF: By Theorem 20, $\mathrm{Int}(\overline{A}) \subseteq A$.

Hence $A = (\mathrm{Int}(\overline{A})) \cup (A - \mathrm{Int}(\overline{A}))$.

Take $B = A - (\mathrm{Int}(\overline{A}))$.

Then clearly $\mathrm{Int}(\overline{B}) \subseteq \mathrm{Int}(\overline{A})$.

Since $B \cap (\mathrm{Int}(\overline{A})) = \Phi$. So $B \cap (\mathrm{Int}(\overline{B})) = \Phi$.

This implies that $\overline{B} \cap (\mathrm{Int}(\overline{B})) = \Phi$.

Hence $\mathrm{Int}(\overline{B}) = \Phi$.

Thus B is nowhere dense in X.

Now by Theorem 1, there exist $T_1, T_2 \in SO(X)$ such that

$$B = A - \mathrm{Int}(\overline{A}) = T_1 \cap T_2.$$

It follows that $A = (\mathrm{Int}(\overline{A})) \cup (T_1 \cap T_2)$

$= (\mathrm{Int}(\overline{A}) \cup T_1) \cap (\mathrm{Int}(\overline{A}) \cup T_2)$.

Let $S_i = \mathrm{Int}(\overline{A}) \cup T_i$, for $i = 1, 2$.

Then S_1 and S_2 are semi-open.

Also $A = S_1 \cap S_2$.

B. SEMI-OPEN SETS IN LOCALLY SEPARABLE METRIC SPACES

In this section, we extend some results of the previous section to locally separable metric spaces.

THEOREM.22. For a metric space to be locally separable it is necessary and sufficient that it be a disjoint union of open separable subsets.

PROOF: Theorem of [SIERPINSKI; 1933].

THEOREM.23. Let (X, ρ) be a locally separable metric space. Let A be a nowhere dense subset of X. Then there exist $S, T \in SO(X)$ such that $A = S \cap T$.

PROOF: If X is separable, then we already know that $A = S \cap T$ for some $S, T \in SO(X)$. If X is non-separable, then by Theorem 22, there exists $\{X_\beta : \beta \in I\}$ a pairwise disjoint class of separable open sets such that $X = \cup_{\beta \in I} X_\beta$.
For each $\beta \in I$, let $A_\beta = A \cap X_\beta$. Then each A_β is nowhere dense in X_β.
So by Theorem 1, there exist $S_\beta, T_\beta \in SO(X_\beta)$ such that $A_\beta = S_\beta \cap T_\beta$.
Then for $\beta \in I$, there exist open sets G_β and H_β in X_β such that
$G_\beta \subseteq S_\beta \subseteq Cl_{X_\beta}(G_\beta)$ and $H_\beta \subseteq T_\beta \subseteq Cl_{X_\beta}(H_\beta)$.
Clearly G_β and H_β are open in X, for all $\beta \in I$.
We claim that $Cl_{X_\beta}(G_\beta) \subseteq Cl_X(G_\beta)$, for each $\beta \in I$.
For that let $x \in Cl_{X_\beta}(G_\beta)$. Let U be any open subset of X such that $x \in U$.
Then $x \in U \cap X_\beta$ and $U \cap X_\beta$ is open in X_β.
So $(U \cap X_\beta) \cap G_\beta \neq \phi$. This implies that $U \cap G_\beta \neq \phi$.
Hence $x \in Cl_X(G_\beta)$. Thus $Cl_{X_\beta}(G_\beta) \subseteq Cl_X(G_\beta)$. Similarly, $Cl_{X_\beta}(H_\beta) \subseteq Cl_X(H_\beta)$.
We conclude that S_β and T_β are semi-open in X, for all $\beta \in I$.
Put $S = \cup_{\beta \in I} S_\beta$, $T = \cup_{\beta \in I} T_\beta$.
Then clearly S and T are semi-open in X. Further, $S_\beta \cap T_\partial = \Phi$, for all $\beta, \partial \in I$ such that $\beta \neq \partial$. Hence $S_\beta \cap T = S_\beta \cap T_\beta = A_\beta$, for each $\beta \in I$.
Thus $\cup_{\beta \in I}(S_\beta \cap T) = \cup_{\beta \in I} A_\beta$. So $(\cup_{\beta \in I} S_\beta) \cap T = \cup_{\beta \in I}(A \cap X_\beta)$.
Therefore $S \cap T = A \cap (\cup_{\beta \in I} X_\beta) = A \cap X = A$,
i.e., $A = S \cap T$.

COROLLARY.24. Let (X, ρ) be a locally compact metric space. Let A be a nowhere dense subset of X. Then there exist $S, T \in SO(X)$ such that $A = S \cap T$.

PROOF: Note that every locally compact metric space is locally separable. The desired result now follows from Theorem 23.

Our next Theorem characterizes the union of an open set and a nowhere dense set in terms of two semi-open sets in a locally separable metric space.

THEOREM.25. Let (X, ρ) be a locally separable metric space. Let $A \subseteq X$. Then A is the intersection of finitely many semi-open sets in X if and only if A is the union of an open set and a nowhere dense set in X.

PROOF: $\Rightarrow$: Follows from Theorem 1 of [LATIF & WILLARD; 1995].

$\Leftarrow$: Let $A = G \cup N$ where G is an open set in X and N is a nowhere dense set in X. By Theorem 23, there exist two semi-open sets S_1 and S_2 in X such that $N = S_1 \cap S_2$.

Thus $A = G \cup (S_1 \cap S_2)$

$= (G \cup S_1) \cap (G \cup S_2)$.

Let $S = G \cup S_1$, $T = G \cup S_2$.

Then S and T are semi-open in X.

Also $A = S \cap T$.

COROLLARY.26. Let (X, ρ) be a locally compact metric space. Let $A \subseteq X$. Then A is the intersection of finitely many semi-open sets in X if and only if A is the union of an open set and a nowhere dense set in X.

PROOF: We note that every locally compact metric space is locally separable. Now the result follows immediately from Theorem 25.

ACKNOWLEDGMENT

The authors are indebted to the University of Alberta, Edmonton, Canada, for providing all necessary research facilities during the completion of this paper. The first author also gratefully acknowledges the support provided by the King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia. Thanks are also due to the Canadian International Development Agency for providing financial aid.

REFERENCES

1. BOURBAKI, N., General Topology, Part 2 (transl.), Addison-Wesley, Reading (1966).

2. ENGELKING, R., Outline of General Topology, Wiley Interscience (1968).

3. GILLMAN, L. and JERISON, M., Rings of Continuous Functions, Van Nostrand, Princeton (1960).

4. LATIF, RAJA M. and WILLARD, S., "Semi-open and regularly closed sets in compact metric spaces", Technical Report # 192, Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia (1995).

5. LEVINE, N., " Semi-open sets and semi-continuity in topological spaces", Amer. Math. Monthly. 70, 36-41 (1963).

6. SIERPINSKI, W., "Sur les espaces metriques localement separables", Fund. Math. 21, 107-113 (1933).

7. SIERPINSKI, W., General Topology, 2nd Ed. (transl.), University of Toronto Press, Toronto (1956).

8. STEEN, L. A. Jr., Counter-examples in Topology, Holt, Rinehart and Winston, 1970.

9. WILANSKY, A., Topology for Analysis, Ginn. (1970).

10. WILLARD, S., General Topology, Addison-Wesley (1970).

RAJA MOHAMMAD LATIF
Department of Mathematical Sciences
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia

STEPHEN WILLARD
Department of Mathematics
University of Alberta
Edmonton, T6G 2G1 Canada