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ON TIME-DELAYED FEEDBACK STABILIZABILITY OF STRUCTURALLY DAMPED SYSTEMS

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Abstract

The stability of a class of multi-dimensional damped structures using active time-delayed displacement feedback control is discussed. The conditions under which the system remains stable are derived. An analysis of the effectiveness of using displacement control under these conditions is studied. Specific numerical results are given for a structurally damped beam for various parameters when the time-delayed displacement control is applied.

1 Introduction

Time delay optimal control problems have long been recognized as important models for real-life phenomena. Various mathematical models associated with control of distributed parameter systems with simple time delays appearing in the state equations or boundary conditions have been studied in the literature cf.[11 - 15, 19 - 20].

In recent publications on structural control, it has been shown that time delays in a control mechanism may cause instability in the system e.g. [7, 8]. This matter has been experimentally investigated in [5]. The stability of systems described by delay-differential and more general functional differential equations were investigated by several authors see e.g. [3, 4, 6, 10, 9].

On the other, feedback stabilization of time-lag systems for distributed parameters is minimally studied in the literature. The effect of the delays in boundary-feedback stabilization schemes for wave equations is studied in [7]. Time-delayed feedback control of beams subject to displacement constraints is studied in [13]. The asymptotic stability for delays of controlled and uncontrolled linear functional differential equations is established [8]. It was

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also indicated in this paper [8] that boundary feedback stabilization of hyperbolic partial differential equations may not be robust with respect to small delays in the feedback.

Two examples, the one-dimensional wave equation and an Euler-Bernoulli beam equation, demonstrate this phenomenon. In reference [1] showed the time delay effect on distributed parameter structures that are controlled using direct velocity feedback. It is shown how the stability of the structure could be lost due to time delay. Active velocity feedback control of flexible damped structures is also studied [16] where there is a time delay in applying the control. A sufficient condition, independent of the time delay, for the system to remain asymptotically stable is formulated. When this condition is violated, it is shown constructively that for any control constant there is a time delay for which the structure is unstable. However the effect of time delay on the stability of distributed parameter systems, in which active displacement control is used, has been adequately investigated in [17].

In the present paper, we study the stability and control of damped distributed parameter systems with time delayed feedback which are governed by partial differential equation. In particular, this study is concerned with the effect of time delayed displacement feedback control in minimizing a performance index for a class of self-adjoint damped flexible systems. The linear distributed parameter system with delayed control action is transformed into a modal time-delayed lumped parameter system using modal space technique [17]. The linear lumped parameter system is transformed into a system without delays. Under this transformation, the system is a set of ordinary differential control equations. We demonstrate that problems of stabilization and controllability can be dealt by addressing the reduced system. Numerical investigation of a simply supported beam with structural damping is carried out.

2 The Dynamic Model

Consider the motion of a structure whose deviation from rest at the point $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \Omega \subset \mathcal{R}^d, \Omega$ a bounded domain, and at time t is given by $w(x,t)$. Moreover, assume that the motion of the structure is governed by the non-dimensional partial differential equation:

$$\mathcal{L}[w(x,t)] \equiv w_{tt} + 2M[w_t] + L[w] = u(x, t-\tau) \quad x \in \Omega,\, t > 0 \tag{2.1}$$

subject to homogeneous boundary conditions of the form:

$$B[w]|_{\partial\Omega} = [0]_{m\times 1} \tag{2.2}$$

and prescribed initial conditions of the form:

$$w(x,0) = \alpha(x), \;\; w_t(x,0) = \beta(x) \tag{2.3}$$

with $\alpha(x) \in L_2(\Omega)$ and $\beta(x) \in L_2(\Omega)$ where $L_2(\Omega)$ denotes the Hilbert space of all real valued square-integrable functions on Ω.

M and L are time-invariant scalar linear partial differential operators of order m and $2m$ respectively, B is a time-invariant vector linear partial differential operator of order m, and the highest order derivatives in B are less than the highest derivative in L. It is assumed that B can vary from one subregion of the boundary $\partial\Omega$ to another so long as there are at most a finite number of subregions and the boundaries of the subregions are smooth enough.

Assume that $\{M, B\}$ and $\{L, B\}$ are self-adjoint and there is a complete set of orthogonal eigenfunctions $\{\varphi_n(x)\}_{n\geq 1}$ of M and L such that

$$\begin{aligned} M[\varphi_n] &= \mu_n\varphi_n, \quad 0 < \mu_1 < \mu_2 < \mu_3 < \dots && (2.4) \\ L[\varphi_n] &= \lambda_n\varphi_n, \quad 0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots && (2.5) \end{aligned}$$

and

$$B[\varphi_n]|_{\partial\Omega} = 0 \tag{2.6}$$

where μ_n and λ_n are the eigenvalues of the operators M and L respectively.

3 Time-Delayed Control Problem

We wish to apply displacement-feedback control to the distributed-parameter system described in (2.1)-(2.3). The control is assumed to act on the structure with a time delay $\tau > 0$. The differential equation governing the vibrations of controlled structure is given by

$$\mathcal{L}[w(x,t)] = u(x, t-\tau), \qquad (x,t) \in \Omega \times (0, t_f) \tag{3.1}$$

where $u(x,t)$ is assumed to be of the form

$$u(x,t) = \ A_\tau \overline{u}(x,t) \tag{3.2}$$

where

$$\overline{u}(x,t) = -c \ \ [w(x,t) + F_\tau \overline{u}(x,s)] \tag{3.3}$$

which represents the closed–loop control and $c > 0$ is the gain control parameter. The transformations A_τ and F_τ are linear operators that depend on τ, which are to be given later. Note that there exists a unique solution to (2.1)-(2.3) with (3.1)-(3.3) for each $c > 0$, with the assumption of $u(x,t) = 0$ for $-\tau \leq t \leq 0$ and $0 \leq x \leq 1$.

The effectiveness of the control may be measured by introducing the following performance criterion

$$E(c;t_f) = \frac{1}{2}\int_\Omega \left\{w_t^2(x,t_f) + w(x,t_f)L\left[w\left(x,t_f\right)\right]\right\} d\Omega \tag{3.4}$$

for some finite time t_f. Equation (3.4) indeed provides a measure of some appropriate physical quantity such as the total energy.

Then the problem is to find an optimal gain control $c^* > 0$ such that

$$E(c^*;t_f) = \min_{c>0} \ E(c;t_f)) \tag{3.5}$$

with $w(x,t)$ subject to (2.1)-(2.3) and (3.1)-(3.3), and such that $w(x,t) \to 0$ asymptotically.

4 Stability

We wish to study the stability of the feedback control system given by (2.1)-(2.3) and (3.1)-(3.3). The constant c is the feedback gain of the controller. Let us define the Fourier series expansion:

$$
\begin{aligned}
w(x,t) &= \sum_{n=1}^{\infty} \mathbf{w}_n(t)\, \varphi_n(x) && (4.1)\\
u(x,t) &= \sum_{n=1}^{\infty} \mathbf{u}_n(t)\, \varphi_n(x) && (4.2)
\end{aligned}
$$

Using equations (3.2)-(3.3), and introducing equations (4.1)-(4.2) into the system (3.2), then applying the integral transormation $\int_\Omega (.)\, \varphi_l(x)\, dx$, and using the orthogonality property, the modal displacement $\mathbf{w}_n(t)$ and control input $\mathbf{u}_n(t)$ satisfies

$$\ddot{\mathbf{w}}_n(t) + 2\mu_n \dot{\mathbf{w}}_n(t) + \lambda_n \mathbf{w}_n(t) = \mathbf{u}_n(t-\tau), \quad \text{for } n \geq 1. \tag{4.3}$$

The modal system (4.3) is completely controllable for finite values of n, and completely stabilizable for $n \geq 1$. For the modal equation (4.3) with delay in the control, let $\mathbf{v}_n(t)$ be defined as in [2]

$$\mathbf{v}_n(t) = e^{-\mu_n \tau}\mathbf{w}_n(t) + \int_{t-\tau}^{t} e^{-\mu_n \tau} f_n(t-\tau-s)\mathbf{u}_n(s)ds, \qquad \text{for } n \geq 1. \tag{4.4}$$

with

$$f_n(t) = \frac{e^{p_n t} - e^{q_n t}}{p_n - q_n} = e^{-\mu_n t}\frac{\sin(\Delta_n t)}{\Delta_n}$$

in which

$$\Delta_n = \sqrt{\lambda_n - \mu_n^2}$$

where p_n and q_n are the roots of the characteristic equation associated with the homogenious of the modal equation (4.3). The last equality is satisfied if $\lambda_n - \mu_n^2 > 0$, which means the system is underdamped. The motivation for control is more attractive for the case when the system is under or critically damped. From now on, we assume that $\lambda_n - \mu_n^2 > 0$. The transformation (4.4) of $\mathbf{v}_n(t)$ is valid for any arbitrary control $\mathbf{u}_n(t-\tau)$.

The transformation (4.4) reduces the system (4.3) to:

$$\ddot{\mathbf{v}}_n(t) + 2\mu_n \dot{\mathbf{v}}_n(t) + \lambda_n \mathbf{v}_n(t) = e^{-\mu_n \tau} f_n(-\tau)\dot{\mathbf{u}}_n(t) + e^{-\mu_n \tau} g_n(-\tau)\,\mathbf{u}_n(t), \quad \text{for } n \geq 1. \tag{4.5}$$

where

$$g_n(t) = \frac{q_n e^{p_n t} - p_n e^{q_n t}}{p_n - q_n} = e^{-\mu_n t}\left[\cos(\Delta_n t) - \frac{\mu_n}{\Delta_n}\sin(\Delta_n t)\right], \text{ for } n \geq 1.$$

Remark: The newly resulting transformed system (4.5) is delay free, where are existing known techniques to stabilize such a system. As it will be seen later, the asymptotic stability of (4.5) implies the asymptotic stability of the system (4.3) which in turns guarantees the stability of the feedback control system given by equations (2.1)-(2.3) and (3.1)-(3.3).

The modal system equation (4.5) can be written as:

$$\ddot{\mathbf{v}}_n(t) + 2\mu_n \dot{\mathbf{v}}_n(t) + \lambda_n \mathbf{v}_n(t) = \frac{-\sin(\Delta_n \tau)}{\Delta_n} \dot{\mathbf{u}}_n(t)$$

$$+ \left[\frac{-\mu_n}{\Delta_n} \sin(\Delta_n \tau) + \cos(\tau \Delta_n)\right] \mathbf{u}_n(t), \quad \text{for } n \geq 1. \tag{4.6}$$

Since the modal equation (4.3) is completely controllable for finite n and completely stabilisable for $n \geq 1$, the new modal equation (4.6) is completely controllable for finite n and completely stabilisable for $n \geq 1$ [2].

Using the modal state $\mathbf{v}_n$ as a feedback variable, then

$$\mathbf{u}_n(t) = -c\,\mathbf{v}_n(t) \tag{4.7}$$

is the feedback control law of the modal equation (4.5). The resulting closed-loop equation becomes:

$$\ddot{\mathbf{v}}_n(t) + \left[2\mu_n - c\frac{\sin(\Delta_n \tau)}{\Delta_n}\right] \dot{\mathbf{v}}_n(t)$$

$$+ \left[\lambda_n + c\frac{-\mu_n}{\Delta_n} \sin(\Delta_n \tau) + c\cos(\Delta_n \tau)\right] \mathbf{v}_n(t) = 0, \quad \text{for } n \geq 1. \tag{4.8}$$

Using the Hurwitz stability criterian, the closed-loop modal equation (4.8) is stable iff there exists a constant gain c such that:

$$\left\{\begin{array}{l} 2\mu_n - c\dfrac{\sin(\Delta_n \tau)}{\Delta_n} > 0 \\ \text{and} \\ \lambda_n + c\left[\dfrac{-\mu_n}{\Delta_n} \sin(\tau \Delta_n) + \cos(\tau \Delta_n)\right] > 0 \end{array}\right. \qquad \text{for } n \geq 1.$$

or

$$\left\{\begin{array}{l} c\,\sin(\Delta_n \tau) < 2\mu_n \Delta_n \\ \text{and} \\ \left[\dfrac{\mu_n}{\Delta_n} \sin(\Delta_n \tau) - \cos(\Delta_n \tau)\right] c < \lambda_n \end{array}\right. \qquad \text{for } n \geq 1. \tag{4.9}$$

The two conditions on the feedback gain c given by inequalties (4.9), are necessary and sufficient for stability of the closed-loop modal equation (4.8). This means that the feedback control (4.7) stabilizes the modal system (4.6).

The next step is to find out whether there exists a feedback gain c so that the conditions of (4.9) are satisfied for every $n \geq 1$. The next theorem establishes the main result.

Theorem: If for $n \geq 1$, $\mu_n > 0$, then for some values of $c > 0$, the feedback control law

$$\mathbf{u}_n(t) = -c\left[e^{-\mu_n \tau} \mathbf{w}_n(t) + \int_{-\tau}^{0} e^{-\mu_n \tau} f_n(s) \mathbf{u}_n(t - \tau - s) ds\right] \tag{4.10}$$

asymptotically stabilizes the modal system (4.3), for $t > 2\tau$.

Proof: First, let us turn our attention to the modal system equation (4.8). From (4.9), the modal system (4.8) is asymptotically stable if the feedback gain c satisfies the inequality:

$$0 \leq c < \min_{n\geq 1}\left\{\frac{2\mu_n\Delta_n}{|\sin(\Delta_n\tau)|}\ ;\ \frac{\lambda_n}{\left|\frac{\mu_n}{\Delta_n}\sin(\Delta_n\tau)-\cos(\Delta_n\tau)\right|}\right\}$$

For $n \geq 1$, $\mu_n > 0$ and $\lambda_n > 0$, then $c > 0$ always exists.

Now using the transformation equation (4.4) together with the feedback control (4.10), one obtains

$$|\mathbf{w}_n(t)| \leq e^{\mu_n\tau}|\mathbf{v}_n(t)| + c\tau \max_{-\tau\leq s\leq 0}||e^{-\mu_n s}\frac{\sin(\Delta_n s)}{\Delta_n}||\ ||\mathbf{v}_n^\tau(t)|| \quad \text{for } n \geq 1. \tag{4.11}$$

where $||\mathbf{v}_n^\tau(t)|| = \max_{t-\tau\leq s\leq t}|\mathbf{v}_n(s)|$.

Suppose that a feedback gain c for which all the roots of the characteristic equation associated with the closed-loop modal system (4.8) have negative real parts, is the feedback control gain to (4.8). Thus the system (4.8) is asymptotically stable (i.e. $\mathbf{v}_n(t) \to 0$ asymptotically).

When n is finite, μ_n is also finite, then $\mathbf{v}_n^\tau(t) \to 0$ asymptotically as $t \to \infty$ (because $\mathbf{v}_n(t) \to 0$ asymptotically as $t \to \infty$). Thus, from (4.11) $\mathbf{w}_n(t) \to 0$ asymptotically as $t \to \infty$.

If $\mu_n \to \infty$ when $n \to \infty$, the modal system equation (4.8) reduces to

$$\ddot{\mathbf{v}}_n(t) + 2\overline{\mu}_n\dot{\mathbf{v}}_n(t) + \overline{\lambda}_n\mathbf{v}_n(t) = 0 \tag{4.12}$$

with $\overline{\mu}_n \to \mu_n$ and $\overline{\lambda}_n \to \lambda_n$. One has $\mathbf{v}_n(t) \to 0$ asymptotically. In fact the modal state variable $\mathbf{v}_n$ converges at the rate of $e^{-\mu_n t}$ (i.e. $|\mathbf{v}_n(t)| \leq Ke^{-\mu_n t}$ for some positive real number K). Hence equation (4.11) becomes:

$$|\mathbf{w}_n(t)| \leq Ke^{-\mu_n(t-\tau)}\left[1 + c\tau\max_{-\tau\leq s\leq 0}\left|\left|e^{-\mu_n s}\frac{\sin(\Delta_n s)}{\Delta_n}\right|\right|\right] \quad \text{for } n \geq 1. \tag{4.13}$$

So, for $t > 2\tau$, $\mathbf{w}_n(t) \to 0$ asymptotically. This complete the proof of the theorem.

The next step is to derive the control law to the system (2.1)-(2.3). If $\overline{u}(x,t) = \sum_{n=1}^{\infty}\overline{\mathbf{u}}_n(t)\,\varphi_n(x)$, using equations (4.1) and (4.2) one obtains

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\mu_n\tau}\,\overline{\mathbf{u}}_n(t)\,\varphi_n(x), \tag{4.14}$$

with

$$\overline{u}(x,t) = -c\left[w(x,t) + \sum_{n=1}^{\infty}\varphi_n(x)\int_{-\tau}^{0} e^{-\mu_n\tau}f_n(s)\,\overline{\mathbf{u}}_n(t-\tau-s)ds\right] \tag{4.15}$$

It is to be noted that the asymptotic stability of the feedback controlled modal system (4.3) and (4.10) implies the asymptotic stability of the system (2.1)-(2.3) and (4.14)-(4.15). So any c that satisfies the inequalities (4.9) asymptotically stabilizes the system (2.1)-(2.3) and (4.14)-(4.15).

	mode 1	mode 2	mode 3	mode 4	mode 5
$\gamma = 0.001,\ \tau = 0.1$	0.2335	——	30.5837	——	——
$\gamma = 0.001,\ \tau = 0.5$	——	4.0125	37.779	——	——
$\gamma = 0.001,\ \tau = 1.0$	——	3.1861	20.7904	67.3422	122.7182
$\gamma = 0.0002,\ \tau = 0.1$	0.0467	——	6.1168	——	——
$\gamma = 0.0002,\ \tau = 0.5$	——	0.8025	7.5556	——	——
$\gamma = 0.0002,\ \tau = 1.0$	——	0.6372	4.1579	13.4675	24.5440

Table 1: Upper bound of the feedback gain c for which stability is guaranteed.

For this control mechanism to be practical, only 'finite' modal displacement terms are to be used for feedback control. The minimum number N^* of modes to be kept is based on the stability criteria. If $\mu_n > 0$, then with the observation from (4.14) and (4.15) that $e^{-\mu_n \tau} \to 0$ and $e^{-\mu_n \tau} f_n(s) \to 0$ as $n \to \infty$, one can conclude that there exist an integer N^* sufficiently large such that the control law given in (4.14) and (4.15) still stabilizes the system (2.1)-(2.3), when the summation in the control law (4.14) and (4.15) are performed on $1 \leq n \leq N$, with $N \geq N^*$.

5 Numerical example

As an example, the results of the paper are applied to a simply supported beam of length l with structural damping in which the controlled equation is given by

$$w_{tt}(x,t) + 2Mw_t(x,t) + Lw(x,t) = u(x, t-\tau)$$

with

$$M = -\gamma \frac{\partial^2}{\partial x^2} \quad \text{and} \quad L = \frac{\partial^4}{\partial x^4}$$

where γ is the structural damping coefficient. The eigenvalues problem (2.4)-(2.5) admits a closed-form solutions: $\lambda_n = (n\pi)^4$, $\mu_n = \gamma(n\pi)^2$, and $\varphi_n(x) = \sin(n\pi x)$, for $n \geq 1$.

The region of stability of the control mechanism as obtained in the **Theorem** is illustrated in **Table 1** for modes 1 to 5, for different values of γ and τ. The notation '——' indicates that there is no upper bound. It is observed from the table that there exists a feedback controller gain $c > 0$ which stabilizes the system. Moreover the gain c, that stabilizes the first or second mode, stabilizes all other remaining higher modes.

In this part of the study it is assumed that $\gamma = 0.001$, and $\tau = 0.5$. The boundary conditions are taken as:

$$B[w]|_{\partial\Omega} \begin{bmatrix} w(0,t) & w_{xx}(0,t) \\ w(1,t) & w_{xx}(1,t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the initial conditions are of the form $\alpha(x) = x(x-1)$ and $\beta(x) = 0$.

As a measure of the total force spent in the control process we consider the integral

$$F = c \int_0^1 \int_\tau^{t_f} w^2(x,t)\, dt\, dx.$$

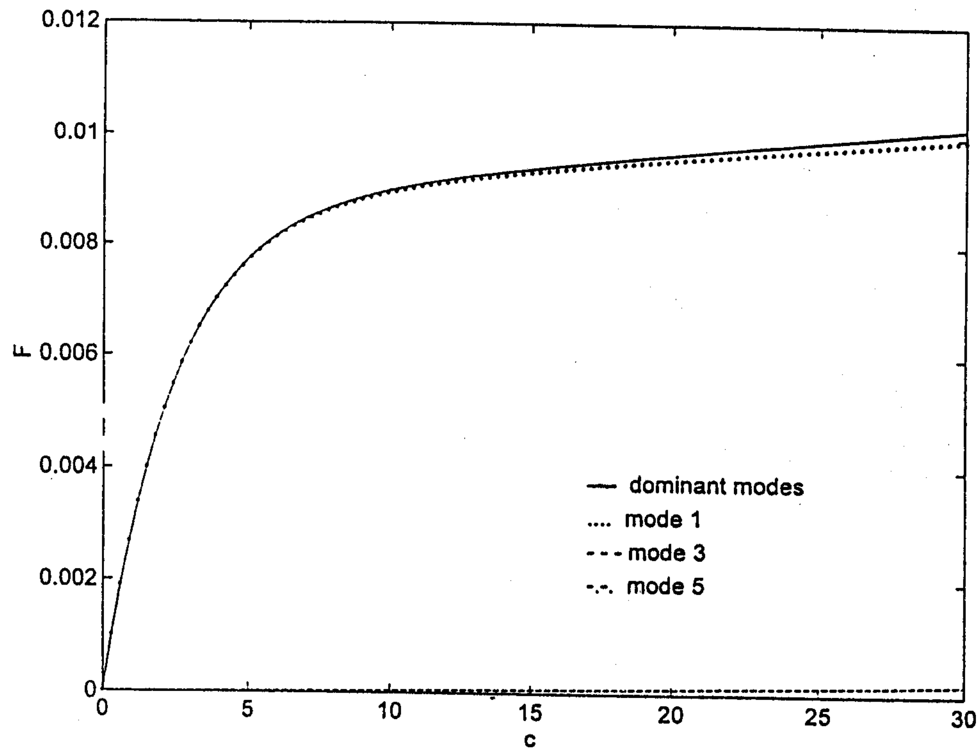

Figure 1: The measure of the total force spent, F, plotted against c for modes 1, 2, 3, and the dominant modes, with $\gamma = 0.001$ and $\tau = 0.5$.

In **Figure 1** the total force F is plotted against the feedback gain c. The total feedback force increases as c increases. It is observed that the feedback force for each mode increases as c increases. In **Figure 2**, the total energy E of the system is plotted against the feedback gain c, where $t_f = 5$. It is observed that the energy function $E(c)$ of the system is convex whose minimum occurs at $c^* = 10.8$.

The real part ζ of the characteristic equation is plotted against c in **Figure 3**. The first modal displacement w_1 plotted against time, for $\gamma = 0.001$ and $\tau = 0.5$.The first modal displacement w_1 plotted against time, for $\gamma = 0.001$ and $\tau = 0.5$. It is observed that the eigenvalues associated with the mode 2 and 3 gets close to the imaginary axis when c increases, while the eigenvalues associated with the mode 1, 4, and 5 go away of the imaginary axis when c increases. This result is confirmed by row 2 of the **Table 1**.

In **Figure 4**, the first modal displacement of the damped beam for mode 1 is plotted as a function of time. The simulation shows that the controller drives the modal state $w_1(t)$ to the origin in relatively short time.

6 Conclusion

Linear systems with delayed control action are transformed into equivalent systems without delays. It is shown that the feedback gain control exists and asymptotically stabilizes the closed-loop modal system.

The effectiveness of the control mechanism under the stability conditions is studied numerically. Specific numerical results plotted in **Figure 4** are obtained for a structurally

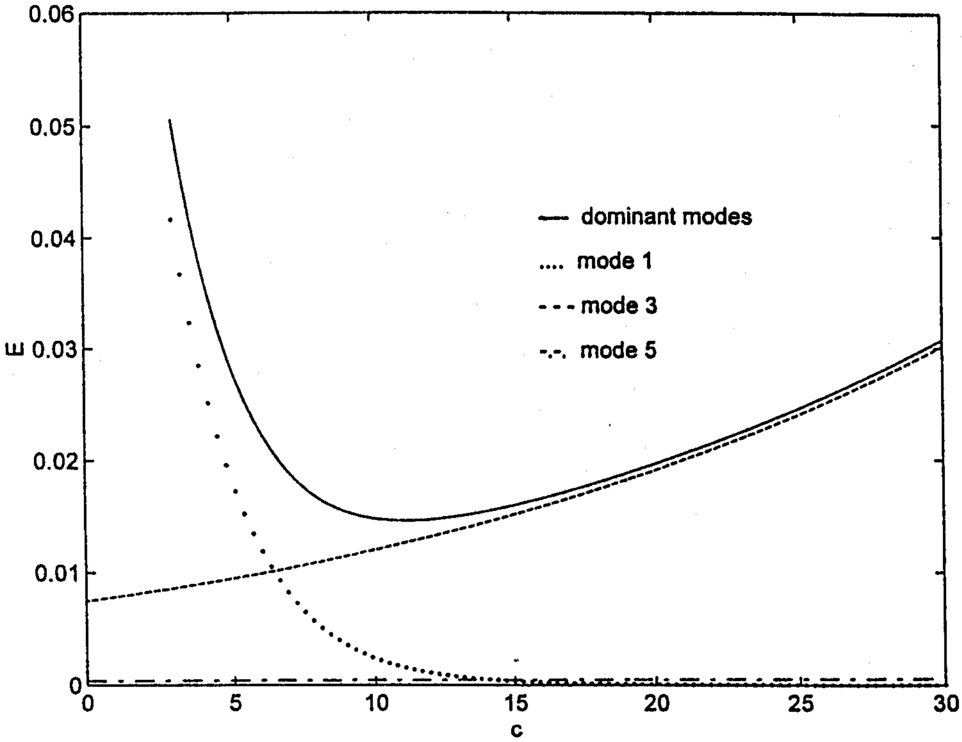

Figure 2: The total energy E plotted against c for modes 1, 3, 5, and the dominant modes, with $\gamma = 0.001$ and $\tau = 0.5$.

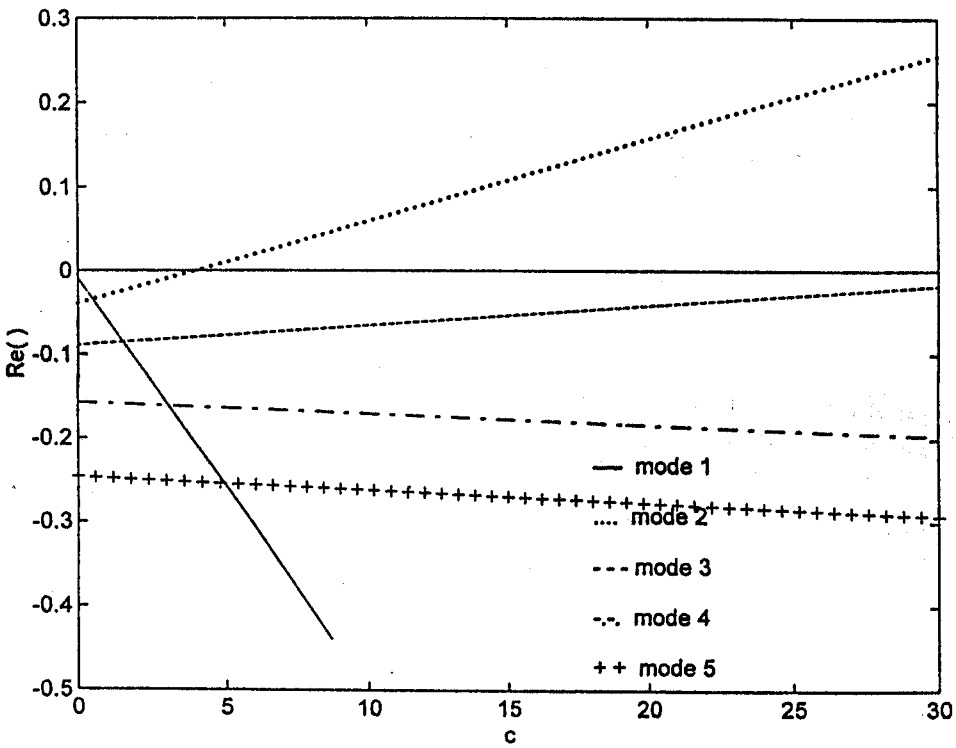

Figure 3: Re(ζ) plotted versus c for modes 1, 2, $\cdots$5, with $\gamma = 0.001$, $\tau = 0.5$.

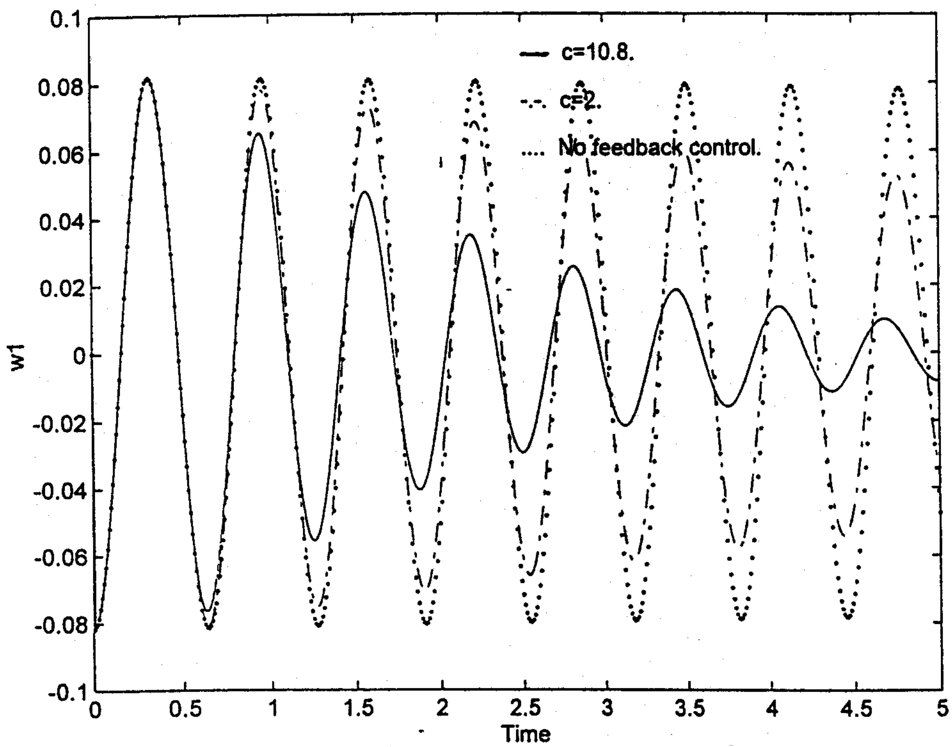

Figure 4: The first modal displacement w_1 plotted against time, for $\gamma = 0.001$ and $\tau = 0.5$.

damped beam to illustrate the phenomenon of control effectiveness.

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