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On 2–normal Operators

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Abstract

In this paper, we introduce the class of 2–normal operators and give some characterizations of 2–normal operators. We study the relations between the class of 2–normal operators and some other classes of operators. We also give some conditions under which a 2–normal operator becomes normal.

1 Let $L(H)$ be the algebra of all bounded linear operators acting on a Hilbert space H and let $T = A + iB$ be the cartesian decomposition of an operator T in $L(H)$. If A and B commute, then T is normal. Correspondingly, E. Kamei, [5], called an operator $T = A + iB$ skew–normal if $AB = -BA$. Since $T^2 = (A^2 - B^2) + i(AB + BA)$, the definition of skew–normality gives us the following two facts:

(i) T is skew–normal if and only if T^2 is hermitian;

(ii) T is skew–normal and normal if and only if $AB = 0$.

A general two-nilpotent operator T (i.e. $T^2 = 0$) is a skew–normal operator which is not normal.

If $T = A + iB$ is an operator in $L(H)$, then we say that T is 2–normal if and only if $AB^2 = B^2A$ and $A^2B = BA^2$. The set of all 2–normal operators is denoted by $[2N]$.

Our first proposition is a characterization of 2–normal operators.

Proposition 1.1 *If $T \in L(H)$ then $T \in [2N]$ if and only if $T^2T^* = T^*T^2$.*

Proof. Let $T = A + iB$ be the cartesian decomposition of T, then by direct computation we have

$$T^2T^* = (A^3 - B^2A + AB^2 + BAB) + i(ABA + BA^2 - A^2B + B^3) \quad \text{(i)}$$

$$T^*T^2 = (A^3 - AB^2 + BAB + B^2A) + i(A^2B + ABA - BA^2 + B^3). \quad \text{(ii)}$$

Suppose that $T \in [2N]$ then $AB^2 = B^2A$ and $A^2B = BA^2$. Substituting in (i) and (ii) we get the right hand sides of (i) and (ii) equal. Thus $T^2T^* = T^*T^2$. Now if $T^2T^* = T^*T^2$, then we have

$$A^3 - B^2A + AB^2 + BAB = A^3 - AB^2 + BAB + B^2A \quad \text{(iii)}$$

and

$$ABA + BA^2 - A^2B + B^3 = A^2B + ABA - BA^2 + B^3. \quad \text{(iv)}$$

From (iii) we get $-B^2A + AB^2 = -AB^2 + B^2A$ which implies that

$$AB^2 = B^2A, \quad \text{(v)}$$

and from (iv) we get $BA^2 - A^2B = A^2B - BA^2$ which implies that

$$A^2B = BA^2. \quad \text{(vi)}$$

From (v) and (vi) we conclude that $T \in [2N]$.

In the following proposition we prove some results about 2–normal operators.

Proposition 1.2 *Let $T \in L(H)$ such that $T \in [2N]$ then*

(a) $T^* \in [2N]$

(b) $\alpha T \in [2N]$ *for all complex numbers* α

(c) If T is invertible then $T^{-1} \in [2N]$

Proof.

(a) direct from Proposition 1.1

(b) direct from Proposition 1.1

(c) Let T be invertible then

$$\begin{aligned}(T^{-1})^2(T^{-1})^* &= (T^*T^2)^{-1} \\ &= (T^2T^*)^{-1} \\ &= (T^{-1})^*(T^2)^{-1}.\end{aligned}$$

Thus $T^{-1} \in [2N]$.

Proposition 1.3 *If $T, S \in L(H)$ such that $T \in [2N]$ and S is unitarily equivalent to T then $S \in [2N]$.*

Proof. Since S is unitarily equivalent to T, there is a unitary operator U such that $S = U^*TU$ which implies that $S^* = U^*T^*U$. Now by direct computation we have

$$S^2S^* = U^*T^2T^*U, \tag{i}$$

and

$$S^*S^2 = U^*T^*T^2U. \quad \text{(ii)}$$

Since $T^2T^* = T^*T^2$, the right hand sides of (i) and (ii) above are equal and hence $S^2S^* = S^*S^2$. Thus $S \in [2N]$.

Unitary equivalence, in Proposition 1.3, cannot be replaced by similarity as the following example shows.

Example 1.1 Consider the two operators $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ acting on the two–dimensional space R^2. Then it can be easily shown that T is 2–normal. However S is not 2-normal because we can prove that $SS^* \neq S^*S$. Now let $X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then $X^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and direct computation shows that $S = X^{-1}TX$.

Proposition 1.4 *Let $T \in L(H)$ be a 2–normal operator and let M be a closed subspace of H that reduces T. Then T/M (the restriction of T to M) is 2–normal.*

Proof.

$$\begin{aligned} (T/M)^2(T/M)^* &= (T^2/M)(T^*/M) \\ &= (T^2T^*/M) \\ &= (T^*T^2/M) \\ &= (T/M)^*(T/M)^2. \end{aligned}$$

Thus $T/M \in [2N]$.

Proposition 1.5 *Let $T = A + iB$ be the cartesian decomposition of $T \in L(H)$. Then T is 2–normal if and only if AB is normal.*

Proof. Suppose $T \in [2N]$, then we have

$$AB^2 = B^2A, \tag{i}$$

$$A^2B = BA^2. \tag{ii}$$

Multiplying (1) on the left by A we get

$$A^2B^2 = AB^2A. \tag{iii}$$

Multiplying (2) on the left by B we get

$$BA^2B = B^2A^2. \tag{iv}$$

Since $B^2A^2 = B(BA^2) = BA^2B = A^2B^2$, then we conclude by using (3) and (4) that $AB^2A = BA^2B$ which implies that $ABB^*A^* = B^*A^*AB$. Thus $(AB)(AB)^* = (AB)^*(AB)$. Hence AB is normal.

Now suppose that AB is normal, then BA is normal. Since $(AB)A = A(BA)$, then, by the well-known Fuglede–Putnam theorem, we have $(AB)^*A = A(BA)^*$ which implies that $BA^2 = A^2B$. Similarly, we obtain $AB^2 = B^2A$. Hence, T is 2–normal.

In the following proposition we show that the class of 2–normal operators coincides with the class of operators studied in [6].

Proposition 1.6 *If $T \in L(H)$ then $T \in [2N]$ if and only if T^2 is normal.*

Proof. Let T^2 be normal. Since $T^2T = TT^2$, then by Fuglede theorem $T^{2^*}T = TT^{2*}$ which implies that $T^*T^2 = T^2T^*$. Thus $T \in [2N]$.

Now suppose that $T \in [2N]$ then $T^2T^* = T^*T^2$. Multiplying on the right by T^* we get $T^2T^{*^2} = T^*T^2T^* = T^{*^2}T^2$. thus $T^2T^{2^*} = T^{2^*}T^2$ which implies that T^2 is normal.

Corollary 1.1 *If $T \in L(H)$ then $T \in [2N]$ if and only if $\|T^2x\| = \|T^{*^2}x\|$ for all $x \in H$.*

Proof. The proof follows immediately from Proposition 1.6 and from ([1], Theorem 1, p. 154).

Proposition 1.7 *The direct sum and the tensor product of two 2–normal operators are 2–normal.*

Proof. Let $x = x_1 \oplus x_2 \in H \oplus H$ and let S, T be two 2–normal operators in $L(H)$ then

$$\begin{aligned}
(T \oplus S)^2(T \oplus S)^*x &= (T \oplus S)^2(T \oplus S)^*(x_1 \oplus x_2) \\
&= (T \oplus S)^2(T^* \oplus S^*)(x_1 \oplus x_2) \\
&= (T \oplus S)^2(T^*x_1 \oplus S^*x_2) \\
&= T^2T^*x_1 \oplus S^2S^*x_2 \\
&= T^*T^2x_1 \oplus S^*S^2x_2 \\
&= (T \oplus S)^*(T \oplus S)^2(x_1 \oplus x_2) \\
&= (T \oplus S)^*(T \oplus S)^2x.
\end{aligned}$$

Thus $(T \oplus S)^2(T \oplus S)^* = (T \oplus S)^*(T \oplus S)^2$ which means that $T \oplus S \in [2N]$. Also

$$(T \otimes S)^2(T \otimes S)^*x = (T^2 \otimes S^2)(T^* \otimes S^*)(x_1 \otimes x_2)$$

$$= (T^2 \otimes S^2)(T^* x_1 \otimes S^* x_2)$$

$$= T^2 T^* x_1 \otimes S^2 S^* x_2$$

$$= T^* T^2 x_1 \otimes S^* S^2 x_2$$

$$= (T \otimes S)^* (T \otimes S)^2 x.$$

Thus $T \otimes S \in [2N]$.

2 In section two of this paper we investigate the relation between the class $[2N]$ and some other classes of operators.

Proposition 2.1 *If $T \in L(H)$ and T is normal or skew–normal then $T \in [2N]$.*

Proof. If T is normal or skew–normal then T^2 is normal and thus, by Proposition 1.6, $T \in [2N]$.

Definition 2.1 Let $T \in L(H)$, then T is called quasinormal (respectively binormal, θ–operator) if $TT^*T = T^*T^2$ (respectively T^*T commutes with TT^*, T^*T commutes with $T + T^*$). The following inclusion relations are proper

normal $\subset$ quasinormal $\subset$ binormal; quasinormal $\subset$ θ–operator.

Proposition 2.2 *If $T \in L(H)$is 2–normal then T is binormal.*

Proof. Since $T \in [2N]$, $T^2T^* = T^*T^2$ which implies that $TT^{*^2} = T^{*^2}T$. Multiplying on the right by T we get $TT^{*^2}T = T^{*^2}T^2 = T^*(T^*T^2) = T^*(T^2T^*) = T^*T^2T^*$. Thus T is binormal.

Proposition 2.3 *If $T \in L(H)$ such that T is 2–normal and quasinormal then T is normal.*

Proof. Since T is quasinormal, we have

$$T^*T^2 = TT^*T \tag{1}$$

which implies that

$$T^{*2}T = T^*TT^*. \tag{2}$$

Multiplying (2) above on the left by T we get

$$TT^{*2}T = TT^*TT^*. \tag{3}$$

Now $T \in [2N]$ implies that

$$T^2T^* = T^*T^2. \tag{4}$$

From (1) and (4) we have

$$T^2T^* = TT^*T. \tag{5}$$

Multiplying (5) on the left by T^* we get

$$T^*T^2T^* = T^*TT^*T. \tag{6}$$

Now

$$(TT^* - T^*T)^2 = TT^*TT^* - TT^{*2}T - T^*T^2T^* + T^*TT^*T. \tag{7}$$

Substituting from (3) and (6) in (7) we get $(TT^* - T^*T)^2 = 0$. Since TT^* and T^*T are self–adjoint, $TT^* - T^*T$ is self–adjoint. Thus $TT^* - T^*T = 0$ which means that T is normal.

Remark 2.1 Since there are nonnormal quasinormal operators and nonnormal 2–normal operators, we conclude from Proposition 2.3 that the class $[2N]$ and the class of all quasinormal operators are independent. We also conclude that the class of all binormal operators is not contained in the class $[2N]$.

Thus the converse of Proposition 2.2 is not true.

Lemma 2.1 *If $T \in L(H)$ is binormal and a θ-operator then T is quasinormal.*

Proof. ([3], p. 459).

Lemma 2.2 *If $T \in L(H)$ such that T is compact and $T \in \theta$, then T is normal.*

Proof. ([2], p. 55).

In the following we show that Lemma 2.2 remains true if we replace compactness by 2–normality.

Proposition 2.4 *If $T \in L(H)$ is a θ-operator and a 2–normal operator then T is normal.*

Proof. Since $T \in [2N]$, then, by Proposition 2.2, T is binormal. Thus, by Lemma 2.1, T is quasinormal. Now the result follows from Proposition 2.3.

Remark 2.2 Combining Proposition 2.4 and the fact that a normal operator is both 2–normal and a θ–operator, we conclude that an operator $T \in L(H)$ is normal if and only if T is both a 2–normal and a θ–operator.

3 In the third and last section of this paper we give some conditions on a 2–normal operator to become normal.

Proposition 3.1 *Let $T \in L(H)$ be 2–normal and isometric, then T is unitary.*

Proof. Since $T \in [2N]$, $T^2T^* = T^*T^2$. Since T is isometric, $T^*T = I$. Thus the last equation becomes $T^2T^* = T$ which, when multiplied on the left by T^*, becomes $T^*TTT^* = T^*T$. Thus $TT^* = T^*T = I$ which means that T is unitary.

Proposition 3.2 *Let $T \in L(H)$ be 2–normal and let $T = VP$ be the polar decomposition of T. If $VP = PV$ then T is normal.*

Proof. Since $VP = PV$ then, by ([4], Problem 137), T is quasinormal. Since $T \in [2N]$ then, by Proposition 2.3, T is normal.

Proposition 3.3 *Let $T \in L(H)$ be 2–normal and let $T + \alpha I$ be 2–normal for some complex $\alpha \neq 0$, then T is normal.*

Proof. Since T, $T + \alpha I$ are 2–normal operators, therefore by Prop. 1.6 T^2 and $(T + \alpha I)^2$ are normal operators. Thus $T^2 + \alpha^2 I$ is normal and commuting with $(T + \alpha I)^2$. Hence $(T + \alpha I)^2 - (T^2 + \alpha^2 I)$ is normal, which implies that $(2\alpha)^{-1} [(T + \alpha I)^2 - (T^2 + \alpha^2 I)] = T$ is normal.

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