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DOUBLE MULTIPLIERS ON TOPOLOGICAL ALGEBRAS

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Abstract

Let $M_d(A)$ be the double multiplier algebra of a topological algebra A, and let u and s respectively be the uniform and strong operator topologies on $M_d(A)$. It is shown, under some additional hypotheses on A, that (1) $M_d(A)$ is u– and s–complete; (2) A is a u–closed two-sided ideal in $M_d(A)$; (3) A is s–dense in $M_d(A)$.

1. Introduction

The theory of double multipliers (i.e. of double centralizers) was developed for topological algebras by Johnson [6] and further investigated in the case of Banach algebras and C^*–algebras by Busby [2], Fontenot [4], Taylor [11], Tomiuk [12], Arün and Rowlands [1], and others; see the monographs [7, 8] for additional references. If A is a commutative C^*–algebra, that is, $A = C_0(X)$ – the algebra of all complex–valued continuous functions which vanish at infinity on a locally compact Hausdorff space X –, then the algebra of all double multipliers of A is $C_b(X)$ – the algebra of all complex–valued bounded continuous functions on X [13]. The non–commutative generalization of the relationship between $C_0(X)$ and $C(X)$ was found to be useful in the work of Busby [2], Davenport [3], and Lazar and Taylor [8].

In this paper, we study the properties of the uniform operator topology u and strong operator topology s on $M_d(A)$, the algebra of all continuous double multipliers on an arbitrary topological algebra A. In section 2, we define multipliers and double multipliers on an algebra and summarize some basic results for later use. In section 3,

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we introduce the u and s topologies on $M_d(A)$ and establish, under some additional hypotheses on A, that (1) $M_d(A)$ is u– and s–complete; (2) A is a u–closed two–sided ideal in $M_d(A)$; (3) A is s–dense in $M_d(A)$. These results extend the corresponding ones of Busby [2] from Banach algebras to topological algebras.

2. Preliminaries

Let A be a complex Hausdorff topological algebra in which multiplication is associative and separately continuous. An algebra A is said to be *proper* (or *without order*) if $aA = Aa = \{0\}$ implies that $a = 0$. We note that A is proper in each of the following cases: (i) A has an identity; (ii) A is a topological algebra with an approximate identity (e.g. A is a B^*–algebra); (iii) A is a topological algebra with an orthogonal basis [5]. A mapping $T : A \to A$ is called a *multiplier* on A if $aT(b) = T(a)b$ for all $a, b \in A$. A pair (S, T) of mappings $S, T : A \to A$ is called a *double multiplier* (or a *double centralizer*) on A if $aS(b) = T(a)b$ for all $a, b \in A$. Let $M(A)$ denote the set of all continuous multipliers on A, and let $M_d(A)$ denote the set of all double multipliers (S, T) on A with S and T continuous. For convenience, we summarize some basic properties of these multipliers in the following theorems. (The reader is referred to the books of Schaefer [10] and Mallios [9] for the general theory of topological vector spaces and topological algebras.)

Theorem 2.1. *Let A be a proper topological algebra. Then*

(a) If $(S, T) \in M_d(A)$, then $S(ab) = S(a)b$ and $T(ab) = aT(b)$ for all $a, b \in A$.

(b) Each $T \in M(A)$ is linear; if $(S, T) \in M_d(A)$, then S and T are linear.

(c) $M(A)$ is a commutative algebra with composition as multiplication (i.e. $(T_1T_2)(a) = T_1(T_2(a))$ and has the identity $I : A \to A$, $I(a) = a$.

(d) $M_d(A)$ *is an algebra with identity* (I, I) *under the operations*

$$(S,T) + (S_1,T_1) = (S + S_1, T + T_1), \quad \lambda(S,T) = (\lambda S, \lambda T)$$
$$\textit{and } (S,T)(S_1,T_1) = (SS_1, T_1T), \quad \lambda \in \mathbb{C}.$$

(e) If A *is commutative, then* $M_d(A)$ *is commutative and* $M_d(A) \equiv M(A)$*; in fact, if* $(S,T) \in M_d(A)$*, then* $S = T$.

(f) If A *is a Banach algebra, then so is* $M_d(A)$ *with the norm given by* $\|(S,T)\| = \max\{\|S\|, \ \|T\|\}$.

Proof. See [2, 6].

For any $a \in A$, let $L_a, R_a : A \to A$ be given by $L_a(x) = ax$ and $R_a(x) = xa, \quad x \in A$. Clearly $(L_a, R_a) \in M_d(A)$. It is easy to see that, for any $a \in A$ and $(S,T) \in M_d(A)$.

$$\begin{aligned}(L_a, R_a)(S,T) &= (L_{T(a)}, R_{T(a)}) \\ (S,T)(L_a, R_a) &= (L_{S(a)}, R_{S(a)}).\end{aligned}$$

We define a map $\mu : A \to M_d(A)$ by $\mu(a) = (L_a, R_a), \quad a \in A$.

Theorem 2.2. *Let* A *be a Hausdorff topological algebra. Then:*

(a) μ *is linear, algebra homomorphism, and continuous.*

(b) μ *is one–one iff* A *is proper.*

(c) μ *is onto iff* A *has identity.*

(d) If A *is proper, then* $\mu(A)$ *is a two-sided ideal in* $M_d(A)$.

Proof. See [2, 6].

The following result, due to Johnson [6] (see also Wang [13], p. 1132), is concerned with the continuity of S and T, where (S,T) is a double multiplier on A. For completeness, we include its proof here.

Theorem 2.3. *Let A be a proper topological algebra, and let (S,T) be a double multiplier on A. Then:*

(a) S and T have closed graphs.

(b) If A is complete and metrizable, then S and T are continuous.

Proof. (a) Let $\{a_\alpha\}$ be a net in A with $a_\alpha \to a \in A$ and $S(a_\alpha) \to b \in A$. Since multiplication is separately continuous, therefore, for any $x \in A$, $xa_\alpha \to xa$ and $xS(a_\alpha) \to xb$; hence

$$\begin{aligned} xS(a) &= T(x)a = \lim_\alpha T(x)a_a \\ &= \lim_\alpha xS(a_\alpha) = xb. \end{aligned}$$

Since A is proper, we have $S(a) = b$. Hence S has closed graph. Similarly, T has also closed graph.

(b) This follows from (a) and the closed graph theorem ([10], p. 78).

3. Uniform and Strong Operator Topologies on $M_d(A)$

In the sequel, A denotes a proper Hausdorff topological algebra with multiplication jointly continuous. Following Johnson [6], the *uniform operator topology* u (respectively, the *strong operator topology* s) on $M_d(A)$ is defined as the linear topology which has a base of neighborhood of 0 consisting of all sets of the form

$$M(B,W) = \{(S,T) \in M_d(A) : S(B) \subseteq W \text{ and } T(B) \subseteq W\},$$

where B is a bounded (respectively, finite subset) of A and W is a neighborhood of 0 in A. (The topology s is sometimes called the *strict topology* in the literature; see, e.g., [1–4, 7, 8, 11–13].) Clearly, $s \leq u$. It is easy to see that $M_d(A)$ endowed with each of u and s is a topological algebra in which multiplication is separately continuous. In [6], Johnson has observed that if A is a locally convex barrelled quasi–complete metrizable algebra, then $(M_d(A), u)$ and $(M_d(A), s)$ are quasi-complete. In this section we consider the completeness of $(M_d(A), u)$ and $(M_d(A), s)$ without the local convexity assumption on A. We also consider some conditions on A under which A is u–closed and s–dense in $M_d(A)$.

The following result extends ([2], Theorem 2.1 and Proposition 3.1) to topological algebras.

Theorem 3.1. *(a) If A is complete and metrizable, then $(M_d(A), u)$ is complete.*

(b) If A is complete, then A is a u–closed two–sided ideal in $M_d(A)$, under the identification $\mu : a \longrightarrow (L_a, R_a)$.

Proof. (a) Suppose A is complete, and let $\{(S_\alpha, T_\alpha)\}$ be a Cauchy net in $(M_d(A), u)$. Then it easily follows that, for each $a \in A$, $\{S_\alpha, (a)\}$ and $\{T_\alpha(a)\}$ are Cauchy nets in A. Consequently, the mappings $S, T : A \longrightarrow A$, given by $S(a) = \lim_\alpha S_\alpha(a)$ and $T(a) = \lim_\alpha T_\alpha(a)$ $(a \in A)$, are well–defined. Further, for any $a, b \in A$,

$$aS(b) = \lim_\alpha aS_\alpha(b) = \lim_\alpha T_\alpha(a)b = T(a)b;$$

hence, by Theorem 2.3(b), $(S,T) \in M_d(A)$. We now show that $(S_\alpha, T_\alpha) \xrightarrow{u} (S,T)$. Let B be a bounded subset of A and W a closed neighborhood of 0 in A. There exists an index α_0 such that

$$S_\alpha(a) - S_\beta(a) \in W \text{ and } T_\alpha(a) - T_\beta(a) \in W$$

for all $a \in B$ and $\alpha, \beta \geq \alpha_0$. Since W is closed, fixing $\alpha \geq \alpha_0$ and taking $\lim_\beta$, we have

$$S_\alpha(a) - S(a) \in W \text{ and } T_\alpha(a) - T(a) \in W$$

for all $a \in B$. Hence, for any $\alpha \geq \alpha_0$,

$$(S_\alpha, T_\alpha) - (S, T) \in N(B, W).$$

Thus $(M_d(A), u)$ is complete.

(b) Suppose A is complete and metrizable. We have already seen in Theorem 2.2 that $\mu(A)$ is a two-sided ideal in $M_d(A)$. To show that $\mu(A)$ is u-closed in $M_d(A)$, let $(S,T) \in M_d(A)$ with $(S,T) \in \overline{\mu(A)}^{u}$. There exists a net $\{a_\alpha\} \subseteq A$ such that $(L_{a_\alpha}, R_{a_\alpha}) \xrightarrow{u} (S,T)$. Then for any $b \in A$, $a_\alpha b \to S(b)$ and $ba_\alpha \to T(b)$, and so $\{a_\alpha b\}$ and $\{ba_\alpha\}$ are Cauchy nets in A. Since A is proper, it easily follows that $\{a_\alpha\}$ is a Cauchy net in A. Since A is complete, $a_\alpha \to a \in A$. Hence, by continuity of μ,

$$(S,T) = \lim_\alpha \mu(a_\alpha) = \mu(a) = (L_a, R_a),$$

and so $(S,T) \in \mu(A)$. Thus $\mu(A)$ is u-closed in $M_d(A)$.

The next result is a generalization of ([2], Propositions 3.5 and 3.6).

Theorem 3.2. *(a) If A is complete and metrizable, then $(M_d(A), s)$ is complete.*

(b) If A is complete and has a two-sided approximate identity (not necessarily bounded), then A is s-dense in $M_d(A)$.

Proof. (a) Suppose A is complete and metrizable, and let $\{(S_\alpha, T_\alpha)\}$ be a Cauchy net in $(M_d(A), s)$. Then, for any $a \in A$, $\{S_\alpha(a)\}$ and $\{T_\alpha(a)\}$ are Cauchy nets in

A. Consequently, the mappings $S, T : A \to A$ given by $S(a) = \lim_\alpha S_\alpha(a)$ and $T(a) = \lim_\alpha T_\alpha(a)$ $(a \in A)$ are well–defined and linear. Further, for any $a, b \in A$,

$$aS(b) = \lim_\alpha aS_\alpha(b) = \lim_\alpha T_\alpha(a)b = T(a)b,$$

and so by Theorem 2.3(b), $(S,T) \in M_d(A)$. To show that $(S_\alpha, T_\alpha) \xrightarrow{s} (S,T)$, let B be a finite subset of A and W a neighborhood of 0 in A. There exists an index α_0 such that

$$S_\alpha(a) - S(a) \in W \text{ and } T_\alpha(a) - T(a) \in W$$

for all $\alpha \geq \alpha_0$ and all $a \in B$ (since B is finite). Hence $(S_\alpha, T_\alpha) - (S,T) \in N(B,W)$ for all $\alpha \geq \alpha_0$. Thus $(M_d(A), s)$ is complete.

(b) Suppose A is complete, and let $\{e_\lambda : \lambda \in I\}$ be a two–sided approximate identity for A. We need to show that $\mu(A)$ is s–dense in $M_d(A)$. Let $(S,T) \in M_d(A)$, and let B be a finite subset of A and W a neighborhood of 0 in A. We claim that, for some $\lambda \in I$, $\mu(T(e_\lambda)) - (S,T) \in N(B,W)$. Now, by definition, $e_\lambda b \to b$ and $be_\lambda \to b$ for all $b \in A$. Since B is finite, we can choose a $\lambda_0 \in I$ such that

$$e_\lambda S(a) - S(a) \in u \text{ and } T(ae_\lambda) - T(a) \in W$$

for all $a \in B$ and all $\lambda \geq \lambda_0$. Then, for any $a \in B$ and $\lambda \geq \lambda_0$,

$$\begin{aligned} L_{T(e_\lambda)}(a) - S(a) &= T(e_\lambda)a - S(a) \\ &= e_\lambda S(a) - S(a) \in W \end{aligned}$$

and

$$\begin{aligned} L_{T(e_\lambda)}(a) - T(a) &= aT(e_\lambda) - T(a) \\ &= T(ae_\lambda) - T(a) \in W. \end{aligned}$$

Thus $\mu(A)$ is s-dense in $M_d(A)$ and this completes the proof.

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