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# Operator-valued Multiplication Operators on Weighted Function Spaces

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**Operator–valued Multiplication Operators on Weighted Function Spaces**

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**Abstract**

Let $X$ be a completely regular Hausdorff space, $E$ a Hausdorff topological vector space, $V$ a Nachbin family of weights on $X$, and $CV_b(X,E)$ the weighted space of continuous $E$–valued functions on $X$. Let $B(E)$ be the vector space of all continuous linear mappings from $E$ into itself, endowed with the topology of uniform convergence on bounded sets. If $\psi : X \to B(E)$ is a continuous mapping and $f \in CV_b(X,E)$, let $M_\psi(f) = \psi f$, where $(\psi f)(x) = \psi(x)(f(x))$ $(x \in X)$. In this paper we give a necessary and sufficient condition for $M_\psi$ to be the multiplication operator (i.e. continuous self–mapping) on $CV_b(X,E)$, where $E$ is a general space or a locally bounded space. These results extend recent work of Singh and Manhas to a non–locally convex setting and that of the authors where $\psi$ has been considered to be a complex or $E$–valued map.

## 1. Introduction

The fundamental work on weighted space of continuous scalar–valued functions has been done mainly by Nachbin [5, 6] in 1960's. Since then it has been studied extensively for a variety of problems such as weighted approximation, characterization of the dual space, approximation property etc. for both scalar– and vector–valued functions (see, for instance, [1, 2, 5, 6, 7, 11]). Recently Singh and Summers [10] have studied the notion of composition operators on $CV_b(X,E)$. Later, Singh and Manhas [8, 9] made an analogous study of multiplication operators on $CV_b(X,E)$, assuming $E$ to be a locally convex space or a normed space. This paper is a continuation of our earlier paper [3] in which we have studied, in the non–locally convex framework, multiplication operators on weighted function spaces which are induced by scalar– and vector–valued mappings.

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The purpose of this paper is to characterize those multiplication operators which are induced by operator–valued mappings. These results extend, in particular, some results of Singh and Manhas [9] for $E$ a locally convex space or a normed space to the case of $E$ a general space or a locally bounded space, respectively.

## 2. Preliminaries

Throughout this paper we shall assume, unless stated otherwise, that $X$ is a completely regular Hausdorff space and $E$ is a non–trivial Hausdorff topological vector space with a base $\mathcal{W}$ of balanced "shrinkable" neighborhoods of 0. (A neighborhood $G$ of 0 in $E$ is called shrinkable [4] if $r\overline{G} \subseteq \text{int } G$ for $0 \leq r < 1$. By ([4], Theorems 5 and 6), every Hausdorff topological vector space has a base of shrinkable neighborhoods of 0 and also the Minkowsksi functional $\rho_G$ of any such neighborhood $G$ is continuous.

Let $S^+(X)$ denote the set of all non–negative upper–semicontinuous functions on $X$, and let $S_0^+(X)\,(S_c^+(X))$ be the subset of $S^+(X)$ consisting of those functions which vanish at infinity (have compact support). A Nachbin family on $X$ is a subset $V$ of $S^+(X)$ such that, for any $x \in X$, there is some $v \in V$ with $v(x) > 0$ and given $u, v \in V$, there exist $w \in V$ and $t > 0$ so that $u, v \leq tw$ (pointwise); the elements of $V$ are called weights. Let $C(X,E)\,(C_b(X,E))$ be the vector space of all continuous (and bounded) $E$–valued functions on $X$, and let $CV_b(X,E)$ denote the subspace of $C(X,E)$ consisting of those $f$ such that $vf$ is bounded for each $v \in V$. When $E = C$ (or $R$), these spaces are denoted by $C(X)$, $C_b(X)$, and $CV_b(X)$. If $\phi \in C(X)$ and $a \in E$, then $\phi \otimes a$ is a function in $C(X,E)$ defined by $(\phi \otimes a)(x) = \phi(x)a \quad (x \in X)$.

Given any Nachbin family $V$ on $X$, the weighted topology $w_v$ on $CV_b(X,E)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$N(v,G) = \{f \in CV_b(X,E) : \ (vf)(X) \subseteq G\},$$

where $v \in V$ and $G \in \mathcal{W}$; $CV_b(X,E)$ endowed with $w_v$ is called a weighted space. We mention that if $V = S_0^+(X)$, then $CV_b(X,E) = CV_0(X,E) = C_b(X,E)$ and $w_v = \beta$, the strict topology, and we write it as $(C_b(X,E),\beta)$; if $V = S_c^+(X)$, then $CV_b(X,E) = CV_0(X,E) = C(X,E)$ and $w_v = k$, the compact–open topology, and we write it as $(C(X,E),k)$. For more information on weighted spaces, we refer to [1–3, 5–11].

Let $B(E)$ denote the vector space of all continuous linear mappings from $E$ into itself, endowed with the topology of uniform convergence on bounded sets, i.e. the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$U(A,G) = \{T \in B(E) : \ T(A) \subseteq G\},$$

where $A$ is a bounded subset of $E$ and $W \in \mathcal{W}$. Given a mapping $\psi : X \to B(E)$ and $f \in CV_b(X,E)$, we let $M_\psi(f) = \psi f$, where $(\psi f)(x) = \psi(x)(f(x)) \ \ (x \in X)$. If $M_\psi$ maps $CV_b(X,E)$ into itself and is continuous, then it is called a multiplication operator on $CV_b(X,E)$ induced by $\psi$.

## 3. Main Results

In this section, we first give a necessary and sufficient condition for $M_\psi$ to be the multiplication operator on the weighted space $CV_b(X,E)$. We then use this characterization to show that, in case $E$ is a locally bounded space, then $M_\psi$ is a multiplication operator if $\psi$ is a bounded mapping or $V = S_c^+(X)$. Our main result is

**Theorem 3.1.** *For any continuous mapping $\psi : X \to B(E)$, the following are equivalent:*

*(a) For every $v \in V$ and $G \in \mathcal{W}$, there exist $u \in V$ and $H \in \mathcal{W}$ such that*

$$v(x)\rho_G(\psi(x)(y)) \leq u(x)\rho_H(y) \text{ for all } x \in X \text{ and } y \in E.$$

*(b) $M_\psi$ is a multiplication operator on $CV_b(X, E)$.*

*Proof.* We may assume that $\mathcal{W}$ consists of closed, balanced, and shrinkable sets.

(a) $\Rightarrow$ (b). We first show that $M_\psi$ maps $CV_b(X, E)$ into itself. Let $f \in CV_b(X, E)$, and let $\{x_\alpha : \alpha \in I\}$ be a net in $X$ with $x_\alpha \to x \in X$. Let $G \in \mathcal{W}$, and let $G_1 \in \mathcal{W}$ with $G_1 + G_1 \subseteq G$. Now $\{f(x_\alpha)\}$, being convergent, is bounded and so, by continuity of $\psi$, there exists an $\alpha_1 \in I$ such that

$$\psi(x_\beta)(f(x_\alpha)) - \psi(x)(f(x_\alpha)) \in G_1 \tag{1}$$

for all $\alpha \in I$ and all $\beta \geq \alpha_1$. Since $\psi(x) : E \to E$ is a continuous linear operator, there exists an $H \in \mathcal{W}$ such that $\psi(x)(y) \in G_1$ for all $y \in H$. Choose an $\alpha_2 \in I$ such that $f(x_\alpha) - f(x) \in H$ for all $\alpha \geq \alpha_2$; consequently,

$$\psi(x)(f(x_\alpha) - f(x)) \in G_1 \tag{2}$$

for all $\alpha \geq \alpha_2$. Choose $\alpha_0 \in I$ with $\alpha_0 \geq \alpha_1, \alpha_2$. Then, by (1) and (2)

$$\psi(x_\beta)(f(x_\beta)) - \psi(x)(f(x)) = \psi(x_\beta)(f(x_\beta)) - \psi(x)(f(x_\beta)) + \psi(x)(f(x_\beta)) -$$
$$\psi(x)(f(x)) \in G_1 + G_1 \subseteq G$$

for all $\beta \geq \alpha_0$. Hence $\psi f \in C(X, E)$. To show that $\psi f \in CV_b(X, E)$, let $v \in V$ and $G \in \mathcal{W}$. By hypothesis, there exist $u \in V$ and $H \in \mathcal{W}$ such that

$$v(x)\rho_G(\psi(x)(y)) \leq \rho_H(u(x)(y)) \tag{3}$$

for all $x \in X$ and $y \in E$. Choose $\lambda > 0$ such that $u(x)f(x) \in \lambda H$ for all $x \in X$. Then, it follows from (3) that $v(x)\psi(f(x)) \in \lambda G$ for all $x \in X$. This proves that $\psi f \in CV_b(X, E)$.

We next establish the continuity of $M_\psi$. Let $\{f_\alpha : \alpha \in I\}$ be a net in $CV_b(X, E)$ with $f_\alpha \to 0$, and let $v \in V$ and $G \in \mathcal{W}$. Choose $u \in V$ and $H \in \mathcal{W}$ as above and which satisfy (3). There exists an $\alpha_0 \in I$ such that $f_\alpha \in N(u, H)$ for all $\alpha \geq \alpha_0$. Then, for any $x \in X$ and $\alpha \geq \alpha_0$,

$$v(x)\rho_G(\psi(x)(f_\alpha(x))) \leq u(x)\rho_H(f_\alpha(x)) \leq 1$$

or equivalently $v(x)\psi(x)(f_\alpha(x)) \in G$. So $M_\psi$ is continuous at 0 and hence, by linearity, it is continuous on $CV_b(X, E)$.

(b) $\Rightarrow$ (a). Let $v \in V$ and $G \in \mathcal{W}$. By hypothesis, there exist $u \in V$ and $H \in \mathcal{W}$ such that

$$M_\psi(N(u, H)) \subseteq N(v, G). \tag{4}$$

We claim that

$$v(x)\rho_G(\psi(x)(y)) \leq 2u(x)\rho_H(y)$$

for all $x \in X$ and $y \in E$. Let $x_0 \in X$ and $y_0 \in E$. Then we consider four cases:

(I) $u(x_0)\rho_H(y_0) \neq 0$.

(II) $u(x_0) = 0,\ \rho_H(y_0) \neq 0$.

(III) $u(x_0) \neq 0,\ \rho_H(y_0) = 0$.

(IV) $u(x_0) = 0,\ \rho_H(y_0) = 0$.

**Case I.** Suppose $u(x_0)\rho_H(y_0) \neq 0$, and let $\epsilon = u(x_0)\rho_H(y_0)$. Then the set $D = \{x \in X : u(x)\rho_H(y_0) < 2\epsilon\}$ is an open neighborhood of $x_0$. By ([6], p. 69), there exists an $h \in CV_b(X)$ such that $0 \leq h \leq 1$, $h(x_0) = 1$, and $h(X \backslash D) = 0$. Define $g = (h \otimes y_0)/2\epsilon$. Then it is easily seen that $g \in N(u, H)$. Hence, by (4), $\psi g \in N(v, G)$. This implies that, for any $x \in X$,

$$v(x)h(x)\rho_G(\psi(x)(y_0)) \leq 2\,u(x_0)\rho_H(y_0).$$

In particular, by taking $x = x_0$, we have

$$v(x_0)\rho_G(\psi(x_0)(y_0)) \leq 2\,u(x_0)\rho_H(y_0).$$

**Case II.** Suppose $u(x_0) = 0$ but $v(x_0)\rho_G(\psi(x_0)(y_0)) > 0$. Put $\epsilon = v(x)\rho_G(\psi(x_0)(y_0))/2$. Then, by the arguments used in Case I with $D = \{x \in X : u(x)\rho_H(y_0) < \epsilon\}$ and $g = (h \otimes y_0)/\epsilon$, we easily obtain

$$v(x_0)\rho_G(\psi(x_0)(y_0)) \leq 2\,v(x_0)\rho_G(\psi(x_0)(y_0))$$

which is impossible unless $v(x_0)\rho_G(\psi(x_0)(y_0)) = 0$. The proofs for Cases III and IV, being similar to Case II, are omitted. This completes the proof.

We now deduce that, under some additional conditions on $\psi$, $M_\psi$ is automatically a multiplication operator.

**Corollary 3.2.** *Let $\psi : X \rightarrow B(E)$ be a constant mapping. Then $M_\psi$ is a multiplication operator on $CV_b(X, E)$.*

*Proof.* We need to verify that condition (a) of Theorem 3.1 holds in this case. Let $v \in V$ and $G \in \mathcal{W}$. Choose $T \in B(E)$ such that $\psi(x) = T$ for all $x \in X$. Choose a closed and shrinkable $H \in \mathcal{W}$ with $H \subseteq G$. Since $T$ being continuous and linear, is bounded, there exists $m > 0$ such that $T(E) \subseteq mH$ i.e. $\rho_H(\psi(x)(y)) \leq m$ for all $x \in X$ and $y \in E$. Let $u = mv$. Then, for any $x \in X$ and $y \in E$.

$$\begin{aligned} v(x)\rho_G(\psi(x)(y)) &\leq v(x)\rho_H(\psi(x)(y)) \\ &\leq u(x)\rho_H(y), \end{aligned}$$

as required.

**Corollary 3.3.** *Suppose $E$ is a locally bounded space and $\psi : X \to B(E)$ is a continuous and bounded mapping. Then $M_\psi$ is a multiplication operator on $CV_b(X, E)$.*

*Proof.* Let $v \in V$ and $G \in \mathcal{W}$. We may assume that $G$ is a bounded neighborhood of 0. Choose a closed and shrinkable $H \in \mathcal{W}$ with $H \subseteq G$. Since $\psi(X)$ is bounded in $B(E)$, there exists $m > 0$ such that $\psi(X) \subseteq mU(H, H)$; i.e.,

$$\rho_H(\psi(x)(y)) \leq m \text{ for all } x \in X \text{ and } y \in H.$$

Let $u = mv$. Then, for any $x \in X$ and $y \in E$,

$$\begin{aligned} v(x)\rho_G(\psi(x)(y)) &\leq v(x)\rho_H(\psi(x)(y)) \\ &\leq u(x)\rho_H(y). \end{aligned}$$

This completes the proof.

Finally, we apply Theorem 3.1 to the cases $V = S_c^+(X)$ and $V = S_0^+(X)$. The following result with $E$ being a locally bounded space extends ([9], Proposition 2.3) where $E$ has been taken to be a normed space.

**Theorem 3.4.** *Suppose $E$ is a locally bounded space and $\psi : X \to B(E)$ is a* *continuous mapping. Then $M_\psi$ is a multiplication operator on $(C(X,E),k)$.*

*Proof.* We need to verify that condition (a) of Theorem 3.1 holds for $V = S_c^+(X)$. Let $v \in V$ and $G \in \mathcal{W}$. Choose a compact $K \subseteq X$ such that $v(x) = 0$ for all $x \in X \backslash K$. We may assume that $G$ is a bounded neighborhood of 0. Choose a closed and shrinkable $H \in \mathcal{W}$ with $H \subseteq G$. Since $\psi : X \to B(E)$ is continuous, $\psi(K)$ is compact in $B(E)$ and so there exists $m > 0$ such that $\psi(K) \subseteq mu(H,H)$; i.e.

$$\rho_H(\psi(x)(y)) \leq m \text{ for } x \in K \text{ and } y \in H.$$

Let $t = \sup\{v(x) : x \in K\}$, and put $u = mt\chi_K$. Then $u \in V$. Let $x \in X$ and $y \in E$. If $x \in K$, then

$$\begin{aligned} v(x)\rho_G(\psi(x)(y)) &\leq t\rho_H(\psi(x)(y)) \\ &\leq u(x)\rho_H(y). \end{aligned}$$

If $x \in X\backslash K$, the above holds trivially (since $v(x) = 0$). This completes the proof.

The following example shows that the above result need not hold when $(C(XE),k)$ is replaced by $(C_b(X,E),\beta)$.

**Example 3.5.** Let $X = \mathbb{R}^+$, $E = \mathbb{C}$, and $V = S_0^+(X)$. Let $\psi : X \to B(E) = \mathbb{C}$ be given by $\psi(x) = x^2$ $(x \in X)$, and let $v \in V$ be given by $v(x) = \frac{1}{x}$ $(x \in X)$. Then $v(x)|\psi(x)| = x$ for all $x \in X$. Since each $u \in V$ is a bounded function, $v|\psi| \leq u$ for all $u \in V$. Hence, by Theorem 3.1, $M_\psi$ is not a multiplication operator on $(C_b(X),\beta)$.

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