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Optimal Active Pointwise Control of Distributed Parameter systems with time delays

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Abstract

A class of time–delayed optimal control systems governed by hyperbolic partial differential equations is considered. Problems of this type are of significant practical interest, for example, in controlling flexible or very large space structures. A control mechanism is proposed to suppress the undesirable vibrations in the structures by means of the simultaneous application of spatially distributed pointwise open–closed loop controllers. Two convex performance indices are introduced and minimized with respect to the open–loop control functions and the closed–loop control parameters. It is shown that the optimal pointwise open–loop control is related to the adjoint variable by a maximum principle. The maximum principle converts the solution of the original problem into the solution of a system of coupled initial–boundary–terminal–value problems with both delayed and advanced terms. The closed–loop control parameters are numerically determined from the minimization of the energy of the system subject to a constraint on the amount of closed–loop control force that can be applied. The proposed theory is demonstrated by applying it to a string with fixed ends subject to time–delayed feedback and open-loop controllers.

1 Introduction

Maximum principles for various distributed optimal control of dynamic systems governed by partial differential equations have been studied extensively in the literature [2, 3, 7, 11, 14, 16]. Numerical examples of applications associated with control of distributed parameter systems are given in references [1, 12, 13]. In many applications, however, the behavior of the state may depend upon its past history. This phenomenon can be induced by the presence of

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time delays in the mathematical models of the motion. The presence of time delay may affect the stability of a dynamic system which would be stable in the absence of time delays [4, 15]. As such, time–delay effects need to be taken into account in the feedback control studies of dynamic systems. Recently, optimal control of distributed parameter systems exhibiting time delays appearing in the state equations or boundary conditions has been considerable interest [5, 6, 8, 9, 10, 15, 17, 18, 19]. However, a general maximum principle which is easily adaptable for control of time-delayed dynamic systems involving the minimization of the energy of vibrating structures does not seem to have been formulated. This study presents such a maximum principle for the case when the governing differential–difference equation involves spatial derivatives of at most four and when the performance index involves spatial derivatives of at most two. In cases where the performance index is a quadratic functional, the proposed maximum principle can lead to an explicit expression for the optimal control function.

The present study deals with a class of control problems for a damped distributed parameter systems. An effective control mechanism is proposed to suppress the excessive vibrations of the system due to the initial disturbances. This suggested mechanism involves the simultaneous application of a finite number of distributed pointwise time–delayed feedback and open–loop controllers extended over the entire spatial region occupied by the system. A theory is first presented to determine the optimal pointwise open–loop controllers for an initial–boundary value problem governed by a linear partial differential–difference equation in which the index is convex. A Hamiltonian functional is introduced and it is shown that the admissible controllers that maximize this Hamiltonian are indeed the optimal controllers. The Hamiltonian involves an adjoint variable with advanced terms related to the optimal controllers by the maximization of this Hamiltonian. The connection between the maximization of the Hamiltonian and the minimum of the index of performance is the maximum principle. The adjoint variable that satisfies a boundary value problem with advanced terms is related to the optimal state variable by terminal conditions at the final time. The theory presented here is capable of providing a method of solution for the open–loop control of these one–dimensional structures. Finally, the gain feedback parameters are numerically determined from the solution of the index of performance including not only the state variable but its first, second derivatives with respect to space and its first derivative with respect to time.

As an example of the applicability of the theory, an undamped beam with fixed ends is considered. The index of performance is made up of the potential and kinetic energies as well as a penalty function involving the pointwise open–loop controllers.

Numerical simulations indicate the transient vibrations are significantly damped in the presence of the the open-closed loop force.

2 Problem Formulation and Statement

Consider a distributed parameter dynamical system described by the following partial differential–difference equation

$$\begin{aligned}\mathcal{L}[w] &= \partial_t[m(x,t)\,\partial_t w(x,t)] + D_x[\partial_t w(x,t)] + L_x[w(x,t)] + \tilde{w}(x,t) = f(x,t) \\ &\qquad (x,t) \in\ D = \Omega \times (0,t_f)\end{aligned} \tag{1}$$

where $m(x,t)$ is the distributed mass, t is the time coordinate, $t_f > 0$ is the terminal time, x is the spatial coordinate defined on $\Omega = (0,l) \subset R^1$, $l > 0$. The function $w(x,t)$ characterizes the state of the system, D_x and L_x are linear time–invariant differential operators of the form

$$D_x[\ \ ] = \sum_{j=0}^{3} d_j(x)\,\partial_x^j, \quad L_x[\ \ ] = \sum_{j=0}^{4} l_j(x)\,\partial_x^j \tag{2}$$

where $d_j(x)$ and $l_j(x)$ are scalar functions to be specified later. The notation ∂_z denotes differentiation with respect to z. In equation (1), $\tilde{w}(x,t)$ is the feedback control distribution of the form

$$\tilde{w}(x,t) = \begin{cases} 0 & \text{for } t \in\ T_0 = [0,\tau], \\ \sum_{j\in J} c_j\ w_t(x_j^s, t-\tau)\delta(x - x_j^s) & \text{for } t \in\ T_\tau = [\tau, t_f] \\ 0 & \text{for } t \in\ T_{t_f} = [t_f, \infty) \end{cases} \tag{3}$$

for all $x \in (0,l)$ and $f(x,t)$ is the open–loop control distribution of the form

$$f(x,t) = \sum_{i\in I} f_i(t)\delta(x - x_i^a), \qquad \text{for all } (x,t) \in D \tag{4}$$

in which $\vec{f}(t) = [f_1(t), f_2(t), \cdots, f_n(t)]^T \in U_{ad}$,with $f_i(t)$ is the i–th amplitude of the actuator located at the $x_i^a \in (0,l)$, and the feedback parameter $c_j > 0$, for $j \in J = \{0,1,2,\cdots,m\}$, $\delta(x - x_i^r)$ are Dirac delta distributions, $r = a$ or s, $i \in I = \{0,1,2,\cdots,n\}$ and where

$$U_{ad} = \left\{\vec{g}(t) | \vec{g}(t) \in L^2[0,t_f]^n,\ |g_i(t)| \le \alpha_i < \infty,\ 1 \le i \le n\right\}$$

Assume the boundary conditions associated with equation (1) are homogeneous of the form

$$B_k\left[w(x,t;\vec{c},\vec{f})\right]_{\partial D}=0, \quad k=1,2,3,4 \quad t\in[0,t_f] \tag{5}$$

where $\partial\Omega$ is the boundary of Ω, $\vec{c}=[c_1,c_2,\ldots,c_m]^T$, and B_k are linear differential operators of the form

$$B_k[\ \ ]=\sum_{j=0}^{3} b_j^k(x,t)\,\partial_x^j, \quad k=1,2,3,4. \tag{6}$$

with $b_j^k(x,t)$, $j=0,1,2,\ldots$ being scalar functions, and the initial conditions

$$w(x,0;\vec{c},\vec{f})=\varphi(x),\ \partial_t w(x,0;\vec{c},\vec{f})=\psi(x),\ x\in\overline{\Omega} \tag{7}$$

in which $\varphi(x)\in H_0^1(\overline{\Omega})$, $\psi(x)\in L^2(\overline{\Omega})$.

The objective of the control for the process described by equation (1) is to minimize the index of performance given by

$$\begin{aligned} J(\vec{c},\vec{f}) &= \int_{\Omega} g_1[x,w(x,t_f,\vec{c},\vec{f}),w_x,w_{xx}]\,dx+\int_{\Omega} g_2[x,w_t(x,t_f;\vec{c},\vec{f})]\,dx \\ &\quad +\int_{T} g_0[t;\vec{c},\vec{f}]\,dt \end{aligned} \tag{8}$$

where the last term is a penalty term, and w_x denotes differentiation with respect to x. In applications, the first two terms may represent quantities related to the energy of the physical systems. The properties of the functions g_1, g_2 and g_0 will be specified in the next section.

Our optimal control problems may now be stated as:

- **Problem 1.** Find $\vec{f_0}(t)\in U_{ad}$ such that

$$J(\vec{c},\vec{f_0})\le J(\vec{c},\vec{f}) \tag{9}$$

  for all $\vec{f}\in U_{ad}$ and for some fixed $\vec{c}\in V_{ad}=\{\vec{c}\mid c_i>0,\ 0\le i\le m\}$, where $J(\vec{c},\vec{f})$ is given by equation (8).

- **Problem 2.** Determine optimal parameter $\vec{c_o}$ such that

$$E(\vec{c}_0,\vec{f_0})\le E(\vec{c},\vec{f_0}) \quad \text{for all } \vec{c}\in V_{ad} \tag{10}$$

  where

$$E(\vec{c},\vec{f_0})=\int_{\Omega} g_1[x,w(x,t_f;\vec{c},\vec{f_0}),w_x,w_{xx}]dx+\int_{\Omega} g_2[x,w_t(x,t_f;\vec{c},\vec{f_0})]dx$$

We now make the following fundamental assumptions throughout the paper.

(A1) The functions $d_j(x)$ and $l_j(x)$ are defined, bounded and measurable in $[0, l]$, and the function $m(x,t)$ is also defined, bounded and measurable in D.

(A2) $\dfrac{\partial^j w}{\partial x^j}, \dfrac{\partial^{j+1} w}{\partial x^j\, \partial t}, \dfrac{\partial^2 w}{\partial t^2} \in L^2(\overline{D}),\ j = 0, 1, \cdots, 4.$

(A3) There exists an optimal control $\overrightarrow{f_0} \in U_{ad}$ of **problem 1.**

(A4) There exists an optimal gain $\overrightarrow{c_0}$ of **problem 2.**

3 Some Preparatory Results

In this section, we introduce the adjoint operator of $\mathcal{L}[w]$ defined in equation (1). We also give some lemmas and definitions which will play crucial roles in the proof of the maximum principle.

Let $\mathcal{L}^*[v]$ be the adjoint operator given by

$$\mathcal{L}^*[v] = \partial_t(m(x,t)\,\partial_t v) - D_x^*[\partial_t v] + L_x^*[v] - \tilde{v}(x,t) \tag{11}$$

where

$$\tilde{v}(x,t) = \begin{cases} \sum_{i \in I} c_j v_t(x_i^s, t+\tau)\delta(x - x_i^s) & t \in [0, t_f - \tau] \\ 0 & t \in [t_f - \tau, t_f] \end{cases} \tag{12}$$

in which

$$D_x^*[v] = \sum_{j=0}^{4} (-1)^j\, \partial_x^j (d_j(x) v) \tag{13}$$

$$L_x^*[v] = \sum_{j=0}^{4} (-1)^j\, \partial_x^j (l_j(x) v) \tag{14}$$

The relations between D_x and D_x^*, L_x and L_x^* are given in the next lemma.

Lemma 1. *For smooth functions $p(x,t)$ and $q(x,t)$, the following relations are satisfied*

$$q(x,t) D_x[p(x,t)] - p(x,t) D_x^*[q(x,t)] = \partial_x \beta_D[p,q] \tag{15}$$

$$q(x,t) L_x[p(x,t)] - p(x,t) L_x^*[q(x,t)] = \partial_x \beta_L[p,q] \tag{16}$$

where

$$\beta_R[p,q] = \sum_{j=1}^{4} \left(\sum_{i=0}^{j-1} (-1)^i \frac{\partial^i}{\partial x^i} (r_j(x) q) \frac{\partial^{j-i-1}}{\partial x^{j-i-1}} (p) \right) \tag{17}$$

and $\beta_R[p,q]$ *is either* $\beta_D[p,q]$ *with* $r_j(x) = d_j(x)$ *or* $\beta_L[p,q]$ *with* $r_j(x) = \ell_j(x)$, $j = 1,2,3,4$.

Proof: The proof of Lemma 1 can be easily be obtained by differentiating β_D and β_L.

From **Lemma 1** follows:

Corollary:

$$q(x,t)D_x[p_t(x,t)] + p(x,t)D_x^*[q_t(x,t)] = \partial_t(qD_x[p]) - \partial_x\beta_D[p,q_t] \tag{18}$$

Proof: We note that

$$\begin{aligned} qD_x[p] + pD_x^*[q_t] &= qD_x[p] + q_tD_x[p] - q_tD_x[p] + pD_x^*[q_t] \\ &= \partial_t(qD_x[p]) - \partial_x\beta_D[p,q_t] \end{aligned} \tag{19}$$

where the second expression on the right follows from **Lemma 1** with q replaced by q_t in equation (15)

Definition 1. *The set of operators* $\{\mathcal{L}^*, B_1^*, B_2^*, B_3^*, B_4^*\}$ *is said to be adjoint to the set of operators* $\{\mathcal{L}, B_1, B_2, B_3, B_4\}$ *provided when*

$$B_k^*[q(x,t)] = 0 \quad \text{on } \partial\Omega, \quad k = 1,2,3,4 \tag{20}$$

then

$$\beta_D[p,q]|_{\partial\Omega} = 0 \text{ and } \beta_L[p,q]|_{\partial\Omega} = 0$$

for all $p \in C^4[\Omega]$ *that belong to the manifold generated by those* p *satisfying*

$$B_k[p(x,t)]|_{\partial\Omega} = 0, \quad k = 1,2,3,4 \tag{21}$$

It is noted that the differential operators B_k^* are similar to B_k, but different in form and, in general, with different coefficients.

Definition 2. *The function* $v(x,t;\vec{c},\vec{f})$ *is said to be a solution of the homogeneous adjoint problem described by the set of operators* $\{\mathcal{L}^*, B_1^*, B_2^*, B_3^*, B_4^*\}$ *provided this set is adjoint to the set of operators* $\{\mathcal{L}, B_1, B_2, B_3, B_4\}$ *and* v *satisfies*

$$\mathcal{L}^*[v] = 0 \tag{22}$$

$$B_k^*\left[v(x,t;\vec{c},\vec{f}\right]_{\partial\Omega} = 0, \quad k = 1,2,3,4 \tag{23}$$

Definition 3. *A real-valued* $g[x;\vec{\theta}]$ *defined on* $\overline{\Omega} \times R^3$ *is said to be convex in* $\vec{\theta} \in R^3$ *if*

$$g[x;\lambda\vec{\theta_1} + (1-\lambda)\vec{\theta_2}] \le \lambda g[x,\vec{\theta_1}] + (1-\lambda)g[x;\vec{\theta_2}]$$

for all $\vec{\theta}_1, \vec{\theta}_2 \in R^3$ *and all* $\lambda, 0 < \lambda < 1$. *If strict inequality holds whenever* $\vec{\theta}_1 \neq \vec{\theta}_2$, g *is said to be strictly convex.*

Let $w(x,t;\vec{c},\vec{f})$ and $w^0(x,t;\vec{c},\vec{f}) \in L^2(\overline{\Omega}\times[0,t_f])$ be the state functions of the **problem 1** corresponding to the controls of $f(x,t;\vec{c})$ and $f_0(x,t;\vec{c})$, respectively. We note that the notation $w(x,t;\vec{c},\vec{f})$ [or $w^0(x,t;\vec{c},\vec{f})$] has been introduced to emphasize the dependence of the state solution w[or w^0] of (1) on the parameter control $\vec{c}$ and the control function $\vec{f}$ (or $\vec{f_0}$) which also depends on $\vec{c}$. For the simplicity of notation, the dependence of the state and control functions will be omitted.

Also, let

$$\Delta w(x,t) = w(x,t) - w^0(x,t) \tag{24}$$

$$\Delta f(x,t) = f(x,t) - f_0(x,t) \tag{25}$$

and in view of (4), (25) becomes

$$\Delta f(x,t) = \sum_{i=1}^{n} \Delta f_i(t)\delta(x - x_i^a) \tag{26}$$

where

$$\Delta f_i(t) = f_i(t) - f_{i0}(t)$$

Then the state function $\Delta w(x,t)$ satisfies the following initial–boundary value problem

$$\mathcal{L}[\Delta w] = \Delta f \quad \text{in } \Omega \times [0,t_f] \tag{27}$$

$$B_k[\Delta w] = 0 \quad \text{on } \partial\Omega,\ k = 1,2,3,4 \tag{28}$$

$$\Delta w(x,0) = 0,\ \Delta w_t(x,0) = 0 \tag{29}$$

The next five lemmas are needed for the proof of the maximum principle.

Lemma 2. *Let w be the solution of the state system (27) – (29) and v be the solution of the adjoint system (22) – (23). Then*

$$I_1[v_t, w_t] = \int_{\overline{\Omega}}\int_{\overline{T}} [\tilde{w}v_t - \tilde{v}w_t]dt\, dx = 0 \tag{30}$$

where $\tilde{w}$ and $\tilde{v}$ are given by (3) and (12), respectively.

Proof: It follows from (3) and (12) that

$$I_1[v,w] \quad = \quad \int_{\Omega}\int_{T_\tau} \left[v_t(x,t)\sum_{j\in J} c_j w_t(x_j^s, t-\tau)\delta(x - x_j^s)\right] dtdx$$

$$
\begin{aligned}
&- \int_{\overline{\Omega}} \int_{0}^{t_f-\tau} \left[w_t(x,t) \sum_{j\in J} c_j v_t(x_j^s, t+\tau)\delta(x-x_j^s) \right] dt\, dx \\
&= \int_{T_\tau} \sum_{j\in J} c_j w_t(x_j^s, t-\tau)\, v_t(x_j^s, t)]dt - \int_{\overline{T}} \sum_{j\in J} c_j v_t(x_j^s, t+\tau) w_t(x_j^s, t)dt \qquad (31)
\end{aligned}
$$

where use has been made of **Fubini's Theorem** to change the order of integration and shifting property for Dirac Delta function. Replacing t by $t+\tau$ in the second integral term on the right side of equation (31) then (30) is obtained.

Lemma 3. *Let $\Delta w, v \in C^4(\overline{\Omega}) \times C^2(\overline{T})$ be solutions of the state system (27) – (29) and the adjoint system (22) – (23). Then*

$$
\begin{aligned}
I_2[v, \Delta f] &= \int_{\overline{\Omega}} \int_{\overline{T}} v(x,t)\Delta f(x,t)dtdx \\
&= \int_{\overline{\Omega}} \{v(m\, \Delta w_t + D_x[v_t]\, \Delta w) - \Delta w\; m\partial_t v\}|_{t=t_f}\, dx \qquad (32)
\end{aligned}
$$

where Δf is given by (26)

Proof: It follows from (27) that

$$
I_2[v, \Delta f] = \int_{\overline{\Omega}} \int_{\overline{T}} v(x,t)\mathcal{L}[\Delta w]dtdx. \qquad (33)
$$

Note that

$$
\begin{aligned}
\overline{I}_2[v, \Delta w] &= v\mathcal{L}[\Delta w] - \Delta w \mathcal{L}^*[v] \\
&= v\partial_t(m\Delta w_t) - \Delta w \partial_t(m v_t) + v D_x[\Delta w_t] + \Delta w D_x^*[v_t] \\
&\quad + v L_x[\Delta w] - \Delta w L^*[v] + v\Delta\tilde{w} - \Delta w \tilde{v} \qquad (34)
\end{aligned}
$$

By taking $p = \Delta w$ and $q = v$ in equations (15), (16) and (18), it follows

$$
\begin{aligned}
\overline{I}_2[v, \Delta w] &= \partial_t[v m\Delta w_t - \Delta w m v_t] + \partial_t(v\; D_x^*[\Delta w]) \\
&\quad - \partial_x \beta_D[\Delta w, v_t] + \partial_x \beta_L[\Delta w, v] \qquad (35)
\end{aligned}
$$

In view of equation (33) and (35), (33) becomes

$$
\begin{aligned}
I_2[v, \Delta f] &= \int_{\overline{\Omega}} \int_{\overline{T}} \{\partial_t[v\, m\, \Delta w_t - \Delta w\, m v_t] + \partial_t(v\; D_x^*[\Delta w]) \\
&\quad - \partial_x \beta_D[\Delta w, v_t] + \partial_x \beta_L[\Delta w, v] + \Delta w \mathcal{L}^*[v]\}dtdx \qquad (36)
\end{aligned}
$$

Using **definitions 1** and **2** and **Fubini's Theorem** to change the order of integration, we have

$$
I_2[v, \Delta f] = \int_{\overline{\Omega}} [v m\Delta w_t - \Delta w m v_t + v\; D_x^*[\Delta w]]|_{t=0}^{t=t_f}\, dx \qquad (37)
$$

Since all terms on the right side of equation (37) vanish, the result (32) follows.

Lemma 4: *Let $g[x; w, w_x, w_{xx}]|_{t=t_f} \in C^2[\overline{\Omega} \times R^3]$ be a convex in w, w_x, and w_{xx}, then*

$$\begin{aligned} I_3[g, \Delta w] &= \int_{\overline{\Omega}} \left[\sum_{i=1}^{2} (-1)^i \partial_x^i (\partial_{w_i} g) \right] \Delta w \, dx \\ &= \int_{\overline{\Omega}} \left[\Delta g - R_2(x; z, z^0, \overline{z}) \right] dx \end{aligned} \tag{38}$$

where

$$w_i = \partial_x^i w, i = 0, 1, 2 \quad \Delta g \equiv g(x; z) - g(x; z^0),$$

$$z = (w, w_x, w_{xx})$$

and $\overline{z}$ denotes an intermediate point on the line joining z to z^0 and $\int_0^1 R_2 dx \geq 0$, with the following condition is satisfied

$$\{[-\partial_{wx} g + \partial_x(\partial_{w_{xx}} g)\Delta w - \partial_{w_{xx}} g \Delta w_x\}|_{\partial\Omega} = 0 \tag{39}$$

Proof: The integration by parts of $I_3[g, \Delta w]$ yields the following:

$$\begin{aligned} I_3[g, \Delta w] &= \{-\partial_{w_x} g \Delta w + \partial_x(\partial_{w_{xx}} g)\Delta w - \partial_{w_{xx}} g \Delta w_x\}|_{\partial\Omega} \\ &+ \int_{\overline{\Omega}} \left[\sum_{i=0}^{2} (-1)^i \partial_x^i (\partial_{w_i} g) \Delta(\partial^i w) \right] dx \end{aligned} \tag{40}$$

The boundary term in equation (40) vanishes in view of equations (39). From the Taylor's formula, it follows that

$$g = g^0 + \sum_{i=0}^{2} \partial_{w_i} g^0 \Delta(\partial^i w) + R_2(x; \overline{z}) \tag{41}$$

where $g^0 = g[x; z^0], z^0 = (w^0, w_x^0, w_{xx}^0)$, and $w_i = \partial_x^i w, i = 0, 1, 2$. Since g is a convex function implies that $R_2 \geq 0$ and the result (38) follows from equations (40) and (41).

Lemma 5. *Let $h[x, w_t] \in C^2(\overline{\Omega} \times R^1)$ be a convex function in w_x, then*

$$\begin{aligned} I_4[h, \Delta w_t] &= \int_{\overline{\Omega}} \partial_{w_t} h \Delta w_t \, dx \\ &= \int_{\overline{\Omega}} [\Delta h - R_3(x; \overline{w}_t)] dx \end{aligned} \tag{42}$$

where $\Delta w_t = w_t - w_t^0$ nd $\Delta h = h[x; w_t] - h[x; w_t^0]$ with $\int_{\overline{\Omega}} R_3(x, \overline{w}_t) dx \geq 0$ and $\overline{w}_t$ is an immediate point on the line joining w_t to w_t^0.

Proof: The equation (42) follows immediately from the use of Taylor's formula and the convexity of h.

4 Maximum Principle

In this section, we state and prove the main result of this paper, the maximum principle for the **problem 1**. The proof is a direct generalization of the proof given by Sloss et al [14].

Introduce the Hamiltonian functional for the **problem 1**.

$$H[t; v, \vec{c}, \vec{f}] = \sum_{j \in J} v(x_j^a, t)\, f_j(t) - g_0[t; \vec{c}, \vec{f}] \tag{43}$$

where $\vec{f}(t) = [f_1(t), f_2(t), \ldots, f_n(t)]^T$ and v is the solution of the boundary value problem (22) – (23). We require the following assumptions regarding the **problem 1**.

(A5) $g_1[x; w, w_x, w_{xx}]$ is a convex function in w, w_x, and w_{xx}.

(A6) $g_2[x; w_t]$ is a convex function in w_t.

(A7) $\{[-\partial_{w_x} g_1 + \partial_x(\partial_{w_{xx}} g_1)\Delta w + \partial_{w_{xx}} g_1 w_x\}|_{\partial\Omega} = 0$ where Δw denotes the solution of the initial–boundary value problem (27) – (29).

Theorem 1. *If $\vec{f_0}(t) \in F_{ad}$ is a solution of the optimal* **Problem 1** *and satisfies the maximum principle*

$$\max_{\vec{f} \in U_{ad}} H[t; v, \vec{c}, \vec{f}] = H[t; v, \vec{c}, \vec{f_0}] \tag{44}$$

where H is defined by (43) and v satisfies (22) – (23). Let w be a solution of the problem (1) – (7) and the conditions **(A5)** – **(A7)** *are satisfied. Also, let v and w be related by the following terminal conditions.*

$$m(x,t)\, \partial_t v(x,t) - D_x[v(x,t)] = \sum_{i=0}^{2} (-1)^i\, \partial_x^i(\partial_{w_i} g_1) \tag{45}$$

$$m(x,t)\, v(x,t) = -\partial_{w_t} g_2[x; w_t] \tag{46}$$

at $t = t_f$ and where $w_i = \partial^i w, i = 0, 1, 2$. Then

$$J[\vec{f}] \geq J[\vec{f_0}] \quad \textit{for all } \vec{f} \in U_{ad} \tag{47}$$

Proof: From **Lemma 3**, it follows that

$$I_2[v, \Delta f] = \int_\Omega \int_T v(x,t)\Delta f(x,t) dt dx = \int_\Omega \{v(m\Delta w_t + D_x[v_t]\Delta w) - \Delta w m \partial_t v\}|_{t=t_f}\, dx \tag{48}$$

where Δf satisfies (27).

Inserting the terminal conditions (45) and (46) into (48), we obtain

$$I_2[v,\Delta f] = \int_\Omega \left\{ \left[\sum_{i=0}^{2} (-1)^{i+1} \partial_x^i (\partial_{w_i} g_1) \right] \Delta w - \partial_{w_t} g_2 \Delta w_t \right\} \Bigg|_{t=t_f} dx. \tag{49}$$

From the definition of $J(\vec{c},\vec{f})$ given by (8), it follows that

$$\Delta J = J(\vec{c},\vec{f}) - J(\vec{c},\vec{f_0}) = \int_\Omega \{\Delta g_1 + \Delta g_2\}|_{t=t_f} dx + \int_T \Delta g_0 dt \tag{50}$$

where

$$\begin{aligned} \Delta g_1 &= g_1[x; w, w_x, w_{xx}] - g_1[x; w^0, w_x^0, w_{xx}^0], \\ \Delta g_2 &= g_2[x; w_t] - g_2[x; w_t^0] \\ \Delta g_0 &= g_0[t; \vec{c}, \vec{f}(t)] - g_0[t; \vec{c}, \vec{f_0}(t)] \end{aligned}$$

Use **Lemmas 4** and **5** with $g = g_1$ and $h = g_2$ gives

$$\begin{aligned} \Delta J &= \int_\Omega \left\{ \left[\sum_{i=0}^{2} (-1)^{i} \partial_x^i (\partial_{w_i} g_1) \right] \Delta w + \partial_{w_t} g_2 \Delta w_t \right\} \Bigg|_{t=t_f} dx \\ &\quad + \int_T \Delta g_0 dt + \int_\Omega (R_2 + R_3) dx \end{aligned} \tag{51}$$

Inserting (49) into (51) and using the fact that R_2 and R_3 are nonnegative functions, we have

$$\Delta J \geq \int_T \Delta g_0 dt - \int_\Omega \int_T v(x,t) \Delta f(x,t) dt dx \tag{52}$$

Applying the shifting property for Dirac delta functions in the second integral of the right hand side of the inequality (52), we have

$$\begin{aligned} \Delta J &\geq \int_T \Delta g_0 dt - \int_T \sum_{j\in J} v(x_j^a, t) \Delta f_j(t) dt \\ &\geq -\int_T \left\{ \sum_{j\in J} v(x_j^a, t) \Delta f_j(t) - g_0[t; \vec{f}(t)] \right. \\ &\qquad \left. - \sum_{j\in J} v(x_j^a, t) \Delta f_{j0}(t) - g_0[t, \vec{f_0}(t)] \right\} dt \\ &\geq -\int_T \left\{ H[t; v, \vec{c}, \vec{f}] - H[t; v, \vec{c}, \vec{f_0}] \right\} dt \geq 0 \end{aligned} \tag{53}$$

in which the equation (44) is used. Hence $J(\vec{c},\vec{f_0}) \leq J(\vec{c},\vec{f})$ and $\vec{f_0}$ is an optimal control and the proof of **Theorem 1** is complete.

Note 1: The uniqueness of the optimal control $\vec{f_0}$ is a direct consequence of the strict convexity of the cost function (8) and convexity of the feasible domain $\overline{\Omega} \times [0, t_f]$.

Note 2: In view of the maximum principal (44), the optimization **Problem 2** is reduced to one with the feedback parameter $\vec{c}$, which can be solved for the optimal value $\vec{c_0}$ by using optimization procedure.

5 Application

As an example of the applications of the maximum principle, consider the control of time–delayed beam with fixed ends. The equation of motion in dimensionless form is given by

$$\mathcal{L}[w] = \partial_t^2 w(x,t) + \partial_x^4 w(x,t) + c\, w_t(x, t-\tau)\delta(x - x^s) = f(t)\delta(x - x^a) \quad 0 < x < 1,\ 0 < t < t_f \tag{54}$$

The prescribed boundary condition is

$$w(0,t) = w(1,t) = 0, \quad \forall\, t \in [0, t_f] \tag{55}$$

and the initial conditions are given by (7).

The problem is to find an optimal open-loop function $f_0(t; c) \in U_{ad}$ and the feedback parameter c_0 such that

$$\min_{f \in U_{ad}} J(c, f) = \min_{f \in U_{ad}} \left(E(c, f) + \mu \int_0^{t_f} f^2(t; c) dt \right) \tag{56}$$

$$\min_{c>0} E(c, f_0) \tag{57}$$

where $\mu > 0$ is the weighting factor and $E(f, c)$ is the total energy given by

$$E(c, f) = \frac{1}{2} \int_0^1 \left[w_x^2(x, t_f) + w_t^2(x, t_f) \right] dx \tag{58}$$

Conditions **(A4)** and **(A7)** are clearly satisfied and to check condition **(A7)**, note that:

$$-2 w_x \Delta w \big|_{x=0}^{x=1} = 0 \tag{59}$$

since $\Delta w(1, t) = \Delta w(0, t) = 0$ for all $t \in [0, t_f]$.

We note that the operator $L_x = \partial_x^4$ is self–adjoint and admits eigenfunctions $\varphi_n(x) = \sin(n\pi x)$ and eigenvalues $\lambda_n = (n\pi)^4, n = 1, 2, 3, \ldots$. By **Theorem 1**, the optimal open–loop controller is of the form

$$f_0(t) = \frac{1}{2\mu} v(x^s, t) \tag{60}$$

where the adjoint function $v(x,t)$ satisfies the adjoint system,

$$\partial_t^2 v(x,t) = \begin{cases} \partial_t^2 v(x,t) + \partial_x^4 v(x,t) - cv(x^s, t+1)\delta(x-x^s) = 0 & (x,t) \in (0,1)\times[0,\tau] \\ \partial_t^2 v(x,t) + \partial_x^4 v(x,t) = 0, & (x,t) \in (0,1)\times[\tau, t_f]. \end{cases} \tag{61}$$

$$v(0,t) = v(1,t), \quad \forall\, t \in [0,t_f] \tag{62}$$

$$\partial_t v(x,t_f) = -2\,\partial_x^2 w(x,t_f), \quad x\in[0,1] \tag{63}$$

$$v(x,t_f) = -2\partial_t w(x,t_f), \quad x\in[0,1] \tag{64}$$

and where the state function satisfies the state system

$$\mathcal{L}[w] = \begin{cases} \partial_t^2 w(x,t) + \partial_x^4 w(x,t) = f(t)\delta(x-x^a), \quad (x,t)\in(0,1)\times[0,\tau] \\ \partial_t^2 w(x,t) + \partial_x^4 w(x,t) + cw_t(x^s, t-\tau)\delta(x-x^s) = f(t)\delta(x-x^a) \\ \qquad (x,t)\in(0,1)\times[\tau,t_f] \end{cases} \tag{65}$$

subject to boundary conditions (55) and initial conditions (7).

The general solution of the adjoint system (61)-(62) is

$$v(x,t;f,c) = \sum_{n=1}^{\infty} \mathrm{v}_n(t;c,f)\phi_n(x) \tag{66}$$

where

$$\mathrm{v}_n(t;f,c) = \begin{cases} a_{1n}\cos(n\pi t) + b_{1n}\sin(n\pi t) + r_{1n}\,[a_{2n}\,k_{1n}(t) + k_{2n}(t)], & t\in[0,\tau] \\ n\pi[-a_{2n}\sin(n\pi t) + b_{2n}\cos(n\pi t)], & t\in[\tau,t_f] \end{cases}$$

in which $k_{1n}(t) = \frac{1}{2}t\sin(n\pi t)$, $k_{2n}(t) = -\frac{1}{2}t\cos(n\pi t) + \frac{1}{2n\pi}\sin(n\pi t)$, and $r_{1n}(t) = -\frac{2c}{n\pi}\sin^2(n\pi x^s)$.

In view of the control (60) and the adjoint solution (66), the general solution of the state equation (54) is obtained in the form of

$$w(x,t;c,f) = \sum_{n=1}^{\infty} \mathrm{w}_n(t;c,f)\;\phi_n(x) \tag{67}$$

where the Fourier coefficients of the function $w(x,t;c,f)$ is given by

$$\mathrm{w}_n(t;c,f) = \int_0^{\tau} w(x,t;c,f)\;\phi_n(x) \tag{68}$$

Using continuity conditions of $v(x,t;c,f)$ at $t=\tau$, we have a system of linear equations

$$\begin{aligned} a_{1n}\cos(n\pi\tau) + b_{1n}\sin(n\pi\tau) &= a_{2n}\,d_{1n}(\tau) + b_{2n}\,d_{2n}(\tau) \\ -a_{1n}\sin(n\pi\tau) + b_{1n}\cos(n\pi\tau) &= a_{2n}\,d_{3n}(\tau) + b_{2n}\,d_{4n}(\tau) \end{aligned} \tag{69}$$

where

$$
\begin{aligned}
d_{1n}(\tau) &= -r_{1n}\, k_{1n}(\tau) + \cos(n\pi\tau) \\
d_{2n}(\tau) &= -r_{1n}\, k_{2n}(\tau) + \sin(n\pi\tau) \\
d_{3n}(\tau) &= -r_{1n}\, k_{3n}(\tau) - \sin(n\pi\tau) \\
d_{4n}(\tau) &= -r_{1n}\, k_{1n}(\tau) + \cos(n\pi\tau) \\
k_{3n}(\tau) &= \frac{1}{2} t \cos(n\pi t) + \frac{1}{2n\pi} \sin(n\pi t)
\end{aligned}
$$

which can be solved for a_{1n} and b_{1n} in terms of a_{2n} and b_{2n}.

Using the continuity conditions for $w(x, t; c, f)$ at $t = \tau$, the state function (68) will be obtained in terms of the constants a_{2n} and b_{2n}. The terminal conditons (63) and (64) will lead to a linear system in a_{2n} and b_{2n}. The solution of the linear system for a_{2n} and b_{2n} will give the optimal state function $w_0(x, t; c, f_0)$ which corresponds to the optimal open-loop control $f_0(t, c)$ given by (60) in conjunction with equation (67). The optimization problem (57) reduces to

$$
\min_{c>0} \; E(c, f_0) = \min_{c>0} \sum_{n=1}^{\infty} E_n(c, f_0) \tag{70}
$$

where

$$
E_n(c, f_0) = \frac{1}{2}\left[\dot{\mathrm{w}}_n^2(t_f; c, f_0) + \lambda_n \mathrm{w}_n^2(t_f; c, f_0)\right] \tag{71}
$$

which can be solved for optimal feedback parameter c_0 by using a one-dimensional search on c.

6 Numerical Simulations

In this part of the study, we consider a beam with a single mode initially active to illustrate the effectiveness of the control method. Introducing the equation (67) into the system (54), then applying the integral transormation $\int_\Omega (.)\, \varphi_l(x)\, dx$, and using the orthogonality property, the modal displacement $\mathrm{w}_n(t)$ and control input $f(t)$ satisfies

$$
\ddot{\mathrm{w}}_n(t) + \lambda_n \mathrm{w}_n(t) + c\,[\phi_n(x^s)]^2\, \dot{\mathrm{w}}_n(t-\tau) + c \sum_{m=1, m\neq n}^{\infty} \phi_n(x^s)\phi_m(x^s)\dot{\mathrm{w}}_m(t-\tau) = \phi_n(x^a) f(t), \quad \text{for } n \geq 1. \tag{72}
$$

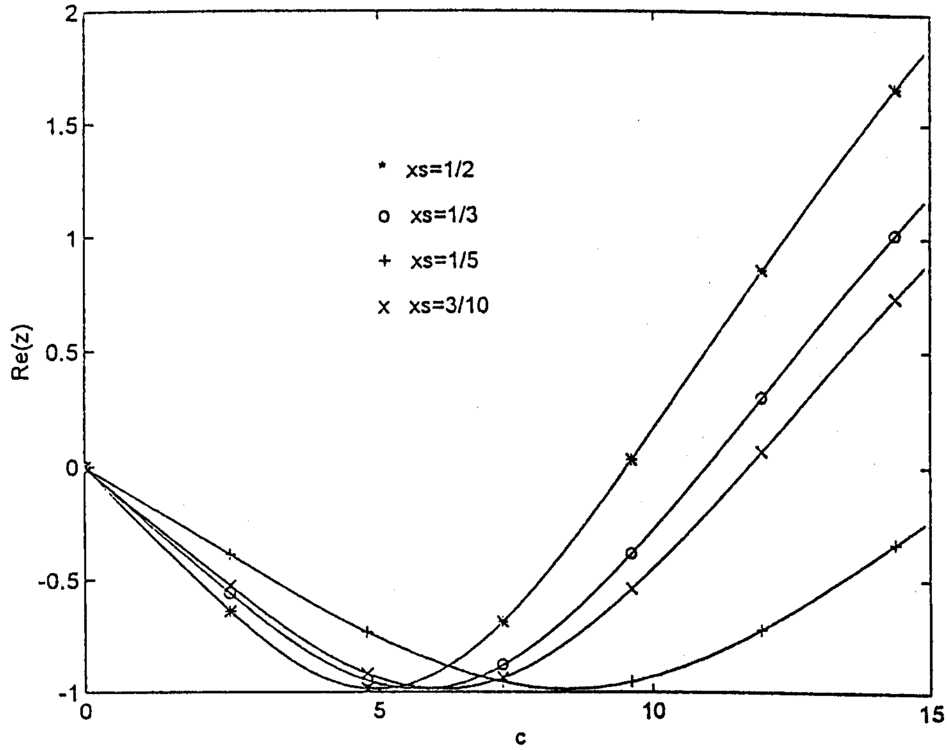

Figure 1: Re(z) plotted versus c for mode 1.

Let $z = z_0 + iz_1$, be a root of the characteristic equation associated with the homogenious system (72)

$$\det\begin{pmatrix} N_1(z) + c\,ze^{-\tau z}\,[\phi_1(x^s)]^2 & c\,ze^{-\tau z}\phi_1(x^s)\phi_2(x^s) & c\,ze^{-\tau z}\phi_1(x^s)\phi_3(x^s) & \cdots \\ c\,ze^{-\tau z}\phi_2(x^s)\phi_1(x^s) & N_2(z) + c\,ze^{-\tau z}\,[\phi_2(x^s)]^2 & c\,ze^{-\tau z}\phi_2(x^s)\phi_3(x^s) & \cdots \\ c\,ze^{-\tau z}\phi_3(x^s)\phi_1(x^s) & c\,ze^{-\tau z}\phi_3(x^s)\phi_2(x^s) & N_3(z) + c\,ze^{-\tau z}\,[\phi_3(x^s)]^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0$$

where $N_n(z) = z^2 + \lambda_n$. In Figure 1, Re$(z) = z_0$ of the first dominant mode is plotted against c, for different values of x^s, with $\tau = 0.5$. From this figure it is shown that the stability region is $0 < c < 9.8$ when $x^s = \frac{1}{2}$.

For the next case simulations we have choosen $\tau = 0.5$, $t_f = 10$, $x^s = 1/2$, and $x^a = 1/2$. The initial conditions are taken to be $\phi(x) = \psi(x) = \sin(\pi x)$.

Figure 2 shows $E_1(c, f_0)$ plotted against c for mode 1. It is observed the energy function is convex whose minimum occurs at $c_0 = 4.8$. Figure 3 shows the first displacement of the mid-point of the beam plotted as a function of time t when both feedback and feedforward controllers are applied together with the case when the feedforward controllers is inactive, with $c = c_0$. It is observed that the addition of a proportional feedback control mechanism reduces the modal displacement of the beam.

Figure 4 shows the controlled first modal displacement of the mid-point of the beam plotted as a function of time t for distinct values of c. This figure shows the effectiveness of

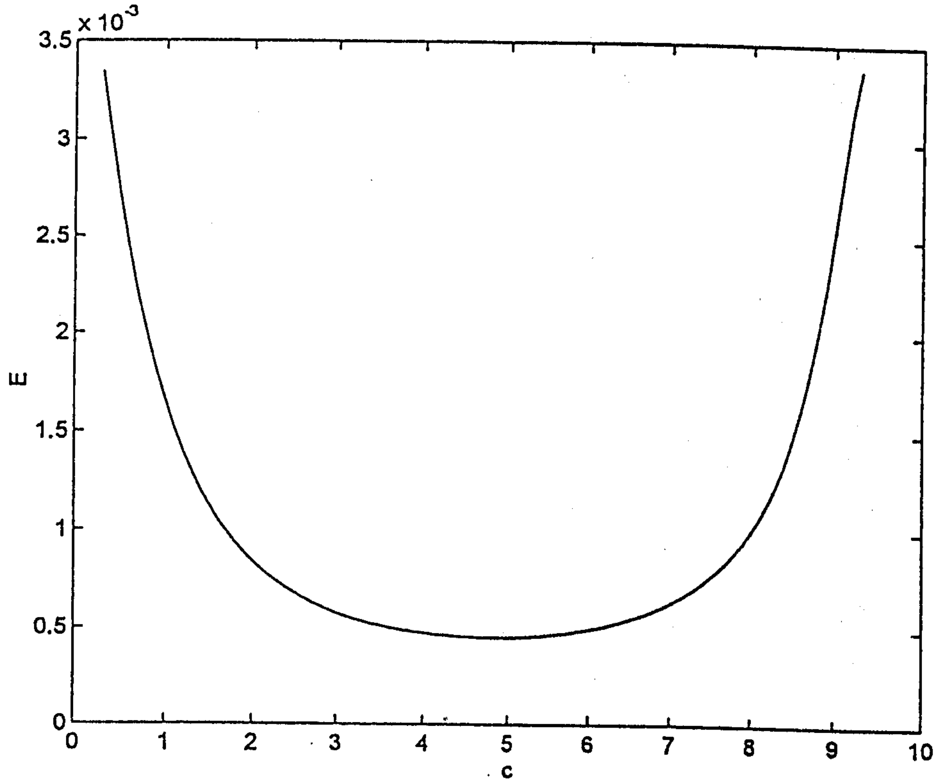

Figure 2: Total energy plotted versus c for mode 1.

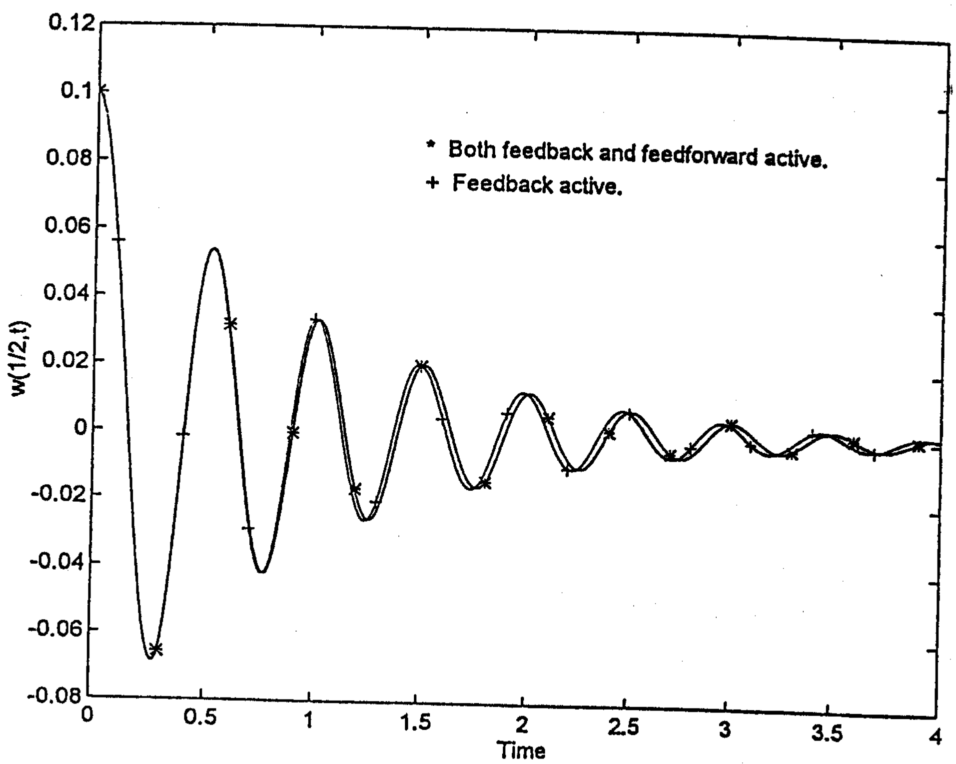

Figure 3: Displacement of the mid-point of the controlled beam as a function of t with $c_0 = 4.8$.

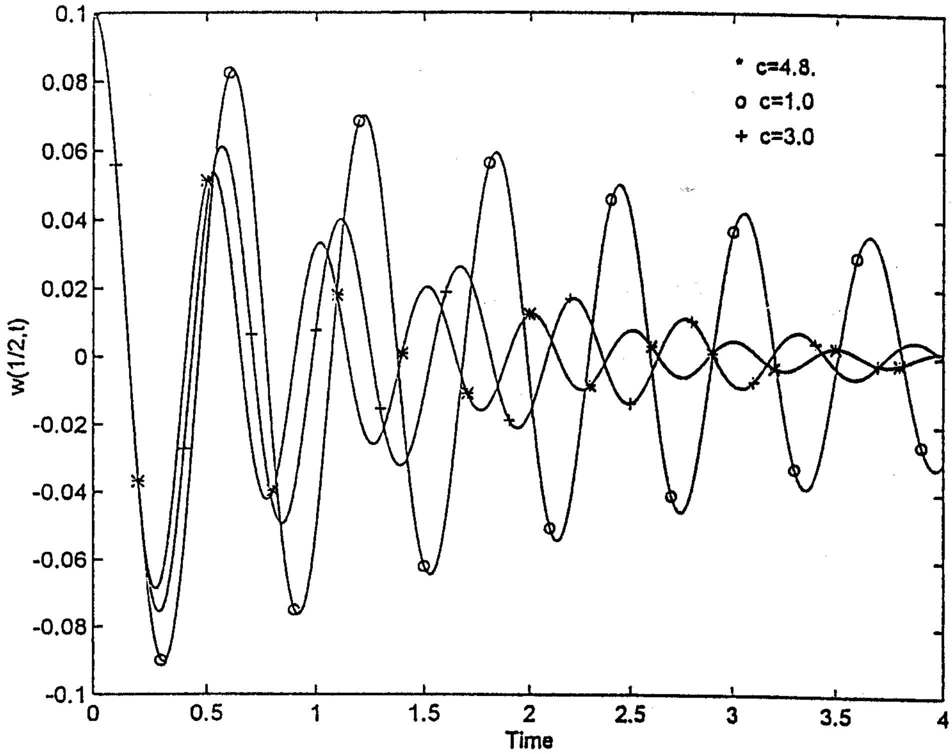

Figure 4: Displacement of the mid-point of the controlled beam as a function of t for different values of c.

the control when $c_0 = 4.8$, in reducing the dynamic displacement of a controlled beam as compared with other values of c. It is also observed that the maximum displacement of the controlled beam dies out gradually when c is within the stability region.

7 Conclusion

The maximum principle for the optimal control of a hyperbolic partial differential equation with deviating argument was presented taking a convex performance index. It was shown that the admissible function which maximizes a given Hamiltonian is an optimal control which minimizes the index of performance. It was also shown that there is at most one optimal control. The problem formulation was made to suit a wide class of applications such as problems in structural control with deviating argument. The basic objective of the structural control problems is to damp out excessive vibrations by pointwise controllers.

The maximum principle given reduces the problem of finding the optimal open–loop control to the problem of solving a coupled system of equations. The solution of the coupled system of equations involve the state equation and the adjoint equation with delayed and advance terms, respectively, subject to boundary conditions for both the state and adjoint variables. Initial conditions are given for the state variable and the terminal condition relate the state and adjoint variables. The closed–loop control parameters are numerically determined from the minimization of the energy of the system subject to a constraint as the amount of closed–loop control function that can be applied. The method is illustrated using the example of a beam with fixed ends taking the cost functional as the sum of potential and kinetic energies. Numerical results indicate that simultaneous use of the open-loop and closed-loop controls leads to a substantial decrease in the magnitude of the beam displacement.

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