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Abstract

This note is concerned with the construction of solutions of a class of two-point weakly singular nonlinear boundary value problems. An iterative method for constructing the solutions is presented. The method is amenable to numerical implementation.

KEY WORDS: Singular boundary value problems, weakly singular boundary value problems, linear self-adjoint operators, Arzelà-Ascoli theorem.

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1 Introduction

We consider the two-point nonlinear boundary value problem

$$\begin{cases} \hat{\ell}(y) \equiv -(p(x)y')' = \hat{f}(x,y), & 0 < x < 1 \\ y(0) = y(1) = 0, \end{cases} \tag{1.1}$$

with $p(x) \geq 0$, $p^{-1} \in L^1(0,1)$, $p(0) = 0$, and $\hat{f}(x,y)$ is a nonlinear forcing term. Such a boundary value problem frequently occurs in practice and is usually termed weakly singular. Weakly singular boundary value problems have been the subject of several recent papers. The reader is referred to [1], [2], [3], [5], [6] and [7] and the references therein for extensive accounts on both the analytical and numerical aspects of these problems.

This paper is concerned with the construction of the solution of (1.1) and examining its behavior at the singular point $x = 0$. The proofs of the existence and uniqueness of solution are based on the contraction mapping theorem and, therefore, can be made constructive. Assuming $\hat{f}(x,y)$ is bounded we prove that the solution of (1.1) can be obtained as the limit of a sequence of functions constructed by the solution of some initial value problems.

We organize the rest of this paper as follows. In Section 2 we show that the boundary conditions in (1.1) determine a unique self-adjoint extension of the minimal operator corresponding to the linear differential operator $\hat{\ell}$. In Section 3 we consider the existence and uniqueness of solution of the nonlinear problem (1.1) and its construction. In Section 4 we apply our results to the case when $p(x) = x^\alpha, \quad 0 < \alpha < 1$.

2 Self-Adjoint Extension of the Minimal Operator of $\hat{\ell}$

Consider the transformation

$$t(x) = 1 + \int_x^1 \frac{1}{p(s)} ds. \tag{2.1}$$

Upon using (2.1), (1.1) becomes

$$\begin{cases} \ell(y) \equiv -\dfrac{1}{\omega(t)} \dfrac{d^2 y}{dt^2} = f(t, y(t)), \quad 1 < t < b, \\ y(1) = y(b) = 0, \end{cases} \tag{2.2}$$

where $\omega(t) = p(x(t))$, $f(t, y(t)) = \hat{f}(x(t), y(x(t))$ and $b = 1 + \int_0^1 \frac{1}{p(s)} ds$.

Let L^2_ω be the Hilbert space consisting of all complex-valued measurable functions y for which

$$\int_1^b |y(t)|^2 \omega(t) dt < \infty,$$

with inner product defined by

$$\langle y, z \rangle = \int_1^b y(t)\overline{z(t)}\omega(t) dt, \quad y, z \in L^2_\omega.$$

It is well known that (2.1) gives rise to a a unitary transformation from $L^2(0,1)$ onto L^2_ω which leaves invariant the classification properties of (1.1).

Define the linear operator L on L^2_ω by

$$\begin{aligned} D(L) &= \left\{ y \in L^2_\omega : \ell(y) \in L^2_\omega, \ y(1) = y(b) = 0 \right\}. \\ L(y) &= \ell(y) \end{aligned}$$

Then $L_0 \subset L \subset L_M$, where L_0 and L_M are minimal and maximal operators associated with the differential operator ℓ, respectively. Since $D(L_0)$ is dense in L^2_ω, it follows that both $D(L)$ and $D(L_M)$ are dense in L^2_ω.

We examine some properties of L in the following lemma.

Lemma 2.1. *$L : D(L) \to L^2_\omega$ is a one-to-one onto operator and is a self-adjoint extension of L_0.*

Proof: First we note that for any $y \in D(L)$, there exist $g \in L^2_\omega$ and constants a and c such that

$$y'(t) = a - \int_1^t g(s)\omega(s)ds,$$

and

$$y(t) = at + c - \int_1^t (t-s)g(s)\omega(s)ds,$$

which imply that

(i) $y'(1)$ and $y'(b)$ are both finite,

(ii) L is onto.

Now using (i), it can be easily shown that L is symmetric. Using the same arguments as those used in Lemma 3.1 of [4], it follows that L is self-adjoint. Since L is obviously one-to-one, (ii) completes the proof of the lemma.

3 The Nonlinear Problem

Assuming that the nonlinearity $f(t, y)$ satisfies

$$f(t, y(t)) \in L^2_\omega, \quad y \in L^2_\omega, \tag{3.1}$$

we can write the boundary value problem (2.2) as

$$L(y) = f(t, y), \quad y \in D(L). \tag{3.2}$$

Using the arguments used in Section 4 of [4] for the singular case, we can prove the following theorem.

Theorem 3.1. *Equation (3.2) has a unique solution provided that the closure of the spectrum of L does intersect* $\left[\inf \frac{\partial f}{\partial y}, \sup \frac{\partial f}{\partial y}\right]$.

Theorem 3.2. *Assume that the nonlinearity $f(t,y)$ satisfies the conditions of Theorem 3.1 and is bounded. Then the unique solution of (3.2) is the limit of the sequence $\{y_n\}$ defined by*

$$y_n(t) = z_n(t) - \frac{(t-1)}{(b-1)} z_n(b), \quad n \geq 1, \tag{3.3}$$

where $z_n(t)$ is the unique solution of the initial value problem

$$\begin{cases} -z_n'' = \omega f(t, y_{n-1}), & 1 < t < b \\ z_n(1) = z_n'(1) = 0, \end{cases} \tag{3.4}$$

for $n \geq 2$, $z_1(t)$ is the unique solution of the initial value problem

$$\begin{cases} -z_n'' = \omega f(t, z_1), & 1 < t < b \\ z_n(1) = z_1'(1) = 0, \end{cases}$$

with respect to the maximum norm.

Proof: The proof is an application of Arzelà-Ascoli Theorem. Consider a subsequence of $\{y_n\}$, which we denote by $\{y_n\}$ again. We note that the unique solution of (3.4) can be written as

$$z_n(t) = -\int_1^t \int_1^s \omega(\xi) f(\xi, y_{n-1}(\xi)) d\xi ds.$$

Since $f(t,y)$ is bounded, it follows that the sequence $\{z_n\}$ is bounded and uniformly equicontinuous on $[1,b]$. It follows from (3.3) that $\{y_n\}$ has in turn a subsequence which converges to some $y \in C[1,b]$ satisfying the integral equation

$$y(t) = -\int_1^t \int_1^s \omega(\xi) f(\xi, y(\xi)) d\xi ds + \frac{(t-1)}{b-1} \int_1^b \int_1^s \omega(\xi) f(\xi, y(\xi)) d\xi ds,$$

which in turn proves that $y(t)$ is the unique solution of (3.2). This proves that the sequence $\{y_n\}$ converges to the unique solution of (3.2).

Remarks

1. The results obtained above can be applied to the boundary value problems

$$\begin{cases} -(p(x)y')' = \hat{f}(x,y), & 0 < x < 1 \\ y(0) = A, \quad y(1) = B, \end{cases} \tag{3.5}$$

   with inhomogeneous boundary conditions.

2. It is found that the unique solution $y(x)$ of (3.5) behaves like $c \cdot \int_0^x \frac{1}{p(s)} ds$, near the singularity, where c is a finite constant given by $\lim_{x \to 0^+} p(x)y'(x)$.

4 Example

In this section, we consider the class of boundary value problems

$$\begin{cases} -(x^\alpha y')' = \hat{f}(x,y), & 0 < x < 1 \\ y(0) = A, \quad y(1) = B, \end{cases} \tag{4.1}$$

where $0 < \alpha < 1$ and A and B are real constants. We apply the results obtained in the previous sections to examine the existence and uniqueness of solution of (4.1) and its behavior near the singular point $x = 0$ and to construct a sequence $\{y_n\}$ of solutions of certain initial value problems which converges to the solution of (4.1) with respect to the maximum norm.

It is well known that the spectrum of L in this case consists of the positive zeros of the equation

$$J_r(\sqrt{\lambda}) = 0, \tag{4.2}$$

where $r = \dfrac{\alpha - 1}{2 - \alpha}$ and $J_r(x)$ is the Bessel function. It follows that (4.1) has a unique solution if $\sup \dfrac{\partial f}{\partial y}$ is less than the first zero of (4.2) or $\left[\inf \dfrac{\partial f}{\partial y}, \sup \dfrac{\partial f}{\partial y}\right]$ lies between two consecutive zeros of that equation and that the unique solution $y(x)$ of (4.1) behaves like $cx^{1-\alpha}$, where c is a constant determined by $c = (1-\alpha) \lim_{x \to 0} x^\alpha y'(x)$. Finally, if $\hat{f}(x, y)$ is bounded, then the sequence $\{y_n\}$ defined by

$$y_n(x) = z_n(x) + [B - z_n(1)]\, x^{1-\alpha}, \qquad n \geq 1$$

where $z_n(x)$ is the unique solution of

$$\begin{cases} -(x^\alpha z_n')' = \hat{f}(x, y_{n-1}), & 0 < x < 1, \\ z_n(0) = A, \ \ z_n'(0) = 0, \end{cases}$$

for $n \geq 2$, and $z_1(x)$ is the unique solution of

$$\begin{cases} -(x^\alpha z_1')' = \hat{f}(x, z_1), & 0 < x < 1, \\ z_1(0) = A, \ \ z_1'(0) = 0, \end{cases}$$

converges to the unique solution of (4.1) in the maximum norm. These results are the basis for the numerical treatment of (4.1) in [5].

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