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An Application of Melnikov Method

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1 Introduction

Let $F : R^n \to R^n$ be a diffemorphism, $x = 0$ be its hyperbolic fixed point and W^+, W^- be, respectively, the immersed stable and unstable manifolds of the hyperbolic fixed point $x = 0$ characterized by

$$W^{\pm} = \{x \in R^n : \lim_{k \to \pm\infty} F^k(x) = 0\}.$$

If W^+ and W^- intersect transversely at $x_0 \neq 0$, the orbit of $x_0, \{F^k(x_0) : k \in Z\}$, is doubly asymptotic to the hyperbolic fixed point $x = 0$. Such a point x_0 is called a transversal homoclinic point. It is well known that the existence of a transversal homoclinic point is responsible for extremely complicated dynamics of F [3]. In [4] it is shown that near a transversal homoclinic point there exists a horseshoe map and hence the above complexity of the dynamic of F becomes transparent. In [2] it was proved that a diffemorphism F_μ on R^2, depending on a parameter μ, which has a dissipative hyperbolic fixed point and whose stable and unstable manifolds have a nondegenerate tangential intersection at $\mu = \mu_0$ must have infinite number of attracting periodic orbits for μ near μ_0.

Consider the differential equation

$$\frac{dx}{dt} = f(x), \quad x \in R^n, \tag{1.1}$$

and its perturbation

$$\frac{dx}{dt} = f(x) + \epsilon g(t, x, \mu), \tag{1.2}$$

where g is assumed to be periodic in t, and ϵ and μ are parameters with $0 < \epsilon \ll 1$. The diffeomorphism F mentioned above appears as a Poincaré map generated by the differential equation (1.1). To apply the results of [2] and [3] to equation (1.2), we assume that the unperturbed equation has a hyperbolic singular point at $x = 0$ and a homoclinic orbit τ corresponding to $x = 0$. Thus τ, being a homoclinic orbit, is contained in both the stable and unstable manifolds of the hyperbolic singular point $x = 0$. This homoclinic orbit is also a homoclinic orbit of any Poincaré map generated by the equation (1.1). To examine how this homoclinic orbit is perturbed for the equation (1.2) and to detect transversal and tangential homoclinic points we measure the distance between the perturbed stable and unstable manifolds $W^+_{\epsilon,\mu}$ and $W^-_{\epsilon,\mu}$ of the Poincaré map generated by the differential equation (1.2). To this end the Melnikov method in higher dimension developed in [7] can be applied.

The purpose of this note is to make a comment on the use of the Melnikov method which makes it possible to detect not only transversal but also tangential homoclinic points and to give an example of a homoclinic tangency in higher dimension. The rank condition on the derivative of Melnikov vector is usually used to detect a transversal homoclinic point; however, transversal intersection is still possible even if this rank condition is not satisfied. We derive a slightly weaker condition than the rank condition the violation of which implies the homoclinic tangency under some restriction on the system. In section 2 of this note we prove a theorem on homoclinic tangency. In section 3 we give an example of a system in R^4 that creates a tangential homoclinic point and illustrate the use of the theorem proved in section 2.

2 The Homoclinic Tangency by Melnikov Method

We assume that the differential equation (1.1) has a hyperbolic singular point at $x = 0$ and has the n_0–dimensional stable and unstable manifolds W^+, W^- to $x = 0$. Further we assume that $W^+ = W^- = \Gamma$ and that the manifold Γ is parametrized as $\Gamma = \{\gamma(t, \beta) : \ t \in R, \ \beta \in S\}$, where S is an $(n_0 - 1)$–dimensional connected manifold. The n_0–dimensional manifold Γ is called a homoclinic manifold. Since for a fixed $\beta, \gamma(t, \beta), \ \ t \in R$, is an orbit of the differential equation (1.1), (t, β) is frequently identified with a point on Γ. We will denote the stable and unstable manifolds of the Poincaré map $F_{\epsilon,\mu}$ generated by the differential equation (1.2) by $W^+_{\epsilon,\mu}$ and $W^-_{\epsilon,\mu}$, respectively.

Let $\Phi(t, s)$ be the transition matrix of the linear variational equation

$$\frac{dz}{dt} = Df(\gamma(t, \beta))z \tag{2.1}$$

of the system (1.1) along $\gamma(t, \beta)$. Then the expressions of $W^+_{\epsilon,\mu}$ and $W^-_{\epsilon,\mu}$ as graphs on the (t, β)–space are given by the following (see [7] for details)

$$\begin{aligned} F^s(\alpha, \beta, \epsilon) &= (\alpha, \beta) + (I - P(\alpha, \beta)) \left[\int_\infty^\alpha \Phi(\alpha, t) g(t - \alpha, \gamma(t, \beta)) dt + h.o.t\,(\epsilon) \right] \\ &\equiv (\alpha, \beta) + m^s(\alpha, \beta, \epsilon), \end{aligned} \tag{2.2}$$

and

$$
\begin{aligned}
F^u(\alpha,\beta,\epsilon) &= (\alpha,\beta) + (I - P(\alpha,\beta))\left[\int_{\infty}^{\alpha} \Phi(\alpha,t)g(t-\alpha,\gamma(t,\beta))dt + h.o.t\,(\epsilon)\right] \\
&\equiv (\alpha,\beta) + m^u(\alpha,\beta,\epsilon) \qquad (2.3)
\end{aligned}
$$

where in the above equations μ was suppressed and $P(\alpha,\beta) : T_{(\alpha,\beta)}R^n \rightarrow T_{(\alpha,\beta)}\Gamma$ is the orthogonal projection with respect to the standard euclidean structure on $T_{(\alpha,\beta)}R^n$.

If $W^+_{\epsilon,\mu}$ and $W^-_{\epsilon,\mu}$ have a point of intersection, then for some (α,β) we must have $F^s(\alpha,\beta,\epsilon) = F^u(\alpha,\beta,\epsilon)$. At this point the tangent spaces to $W^+_{\epsilon,\mu}$ and $W^-_{\epsilon,\mu}$ are spanned by the column vectors of each of the following matrices.

$$
D_{(\alpha,\beta)}F^s = \begin{bmatrix} 1 & 0 \\ 0 & I \\ \dfrac{\partial m^s}{\partial \alpha} & \dfrac{\partial m^s}{\partial \beta} \end{bmatrix} \qquad (2.4)
$$

and

$$
D_{(\alpha,\beta)}F^u = \begin{bmatrix} 1 & 0 \\ 0 & I \\ \dfrac{\partial m^u}{\partial \alpha} & \dfrac{\partial m^u}{\partial \beta} \end{bmatrix} \qquad (2.5)
$$

If one of $\dfrac{\partial}{\partial\alpha}(m^u - m^s)$ and $\dfrac{\partial}{\partial\beta}(m^u - m^s)$ is zero, then the intersection of $W^+_{\epsilon,\mu}$ and $W^-_{\epsilon,\mu}$ is tangential. Hence at this point $q, T_qW^+_{\epsilon,\mu}$ and $T_qW^-_{\epsilon,\mu}$ do not span the whole space T_qR^n. To coordinatize $(m^u - m^s)$, we use the bounded solutions φ_i^* on $-\infty < t < \infty$ of the adjoint system of (2.1)

$$
\frac{d\varphi}{dt} + [Df(\gamma(t,\beta))]^*\varphi = 0.
$$

There are n_0 such solutions where Dim $T^{\perp}_{(\alpha,\beta)} = n_0$. The separation vector, $M(\alpha,\beta,\epsilon)$, which measures the distance between $W^+_{\epsilon,\mu}$ and $W^-_{\epsilon,\mu}$ in the normal direction $T^{\perp}_{(\alpha,\beta)}\Gamma$, for the system (1.2) is defined to be the n_0–vector with components $M_i(\alpha,\beta,\epsilon)$, where

$$M_i(\alpha,\beta,\epsilon)=\varphi_i^*(\alpha,\beta,\epsilon)(m^u(\alpha,\beta,\epsilon)-m^s(\alpha,\beta,\epsilon))=\int_{-\infty}^{\infty}\varphi_i^*(t,\beta)g(t-\alpha,\gamma(t,\beta))dt+h.o.t.(\epsilon)$$

The first order approximation in ϵ of $M(\alpha,\beta,\epsilon)$ is the Melnikov vector for the system (1.2). We also define

$$M(\alpha,\beta)=\int_{-\infty}^{\infty}\Phi^*(t,\beta)g(t-\alpha,\gamma(t,\beta))dt,$$

where $\Phi^*=(\varphi_1^*,\dots,\varphi_{n_0}^*)$.

Using these notations we state and prove the following theorem.

Theorem 2.1. Let $\mu\in R^{n_0}$. Suppose that there exist values $p=(\alpha_0,\beta_0,\mu_0)$ at which the following conditions are satisfied.

(i) $M(p)=0$,

(ii) at least one of the column vectors in the matrix $\left[\frac{\partial M}{\partial\alpha}(p),\frac{\partial M}{\partial\beta}(p)\right]$ is zero,

(iii) rank $\frac{\partial}{\partial\mu}\begin{bmatrix} M(p)\\ \frac{\partial M}{\partial\rho}(p)\end{bmatrix}=2n_0$, where $\frac{\partial M}{\partial\rho}$ is a column vector in the matrix in (ii) such that $\frac{\partial M}{\partial\rho}(p)=0$.

Then for a sufficiently small ϵ the stable manifold $W^+_{\epsilon,\mu}$ and the unstable manifold $W^-_{\epsilon,\mu}$ of the Poincaré map generated by system (1.2) has a tangential homoclinic point at some values (α,β,μ) near p.

Proof. It is clear from matrices (2.4) and (2.5) that the separation vector $M(\alpha,\beta,\epsilon,\mu)$ and its directional derivatives completely determine the intersection of $W^+_{\epsilon,\mu}$ and $W^-_{\epsilon,\mu}$. More precisely if $M=0$ and $\frac{\partial M}{\partial\rho}=0$ at some values $(\alpha,\beta,\epsilon,\mu)$, then the intersection is tangential. Since $M(\alpha, \beta, 0, \mu) = M(\alpha, \beta, \mu)$, the condition (iii) implies, by the implicit mapping theorem, that there exist values $p' = (\alpha', \beta', \epsilon', \mu')$ in a neighborhood of $(\alpha_0, \beta_0, 0, \mu_0)$ such that $M(p') = 0$ and $\frac{\partial M}{\partial \rho}(p') = 0$. Thus the intersection is tangential. □

Remark 2.2. The simplest situation occurs when dim $\Gamma = 1$. In this case $M(\alpha, \mu)$ is a scalar function and so the conditions (i), (ii), (iii) and $\partial^2 M(p)/\partial\alpha^2 \neq 0$ imply the quadratic tangency. The next simplest situation is that only $\partial M/\partial\alpha$ among $[\partial M/\partial\alpha \ \ \partial M/\partial\beta]$ is zero in condition (ii) and $\partial^2 M_i/\partial\alpha^2 \neq 0, i = 1, \ldots, n_0$. An example satisfying these conditions will be given in the next section.

Remark 2.3. It seems that the detailed analysis of critical points of a general Melnikov vector can be done using the singularity theory. However, to our knowledge, very little has been done in this direction. Also the relation between the characterization of critical points of the Melnikov vector and the dynamics of the original system is still not clear. This is because the Newhouse phenomena (and the homoclinic tangency in general) in higher dimension is much less clear than in two-dimensional case.

3 Example

We consider the two degrees of freedom Hamiltonian system.

$$\dot{x} = J\Delta H(x), x = (q_1, q_2, p_1, p_2) \in R^4 \tag{3.1}$$

with the Hamiltonian

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + U(q_1, q_2).$$

The potential $U(q_1, q_2)$ is assumed to be

$$U(q_1, q_2) = -\frac{1}{2}(q_1^2 + q_2^2) + \frac{1}{2}(q_1^2 + q_1^2)^2 = \frac{1}{2}r^2(r^2 - 1)$$

where $q_1 = r\cos\beta$ and $q_2 = r\sin\beta$. Simple calculations show that system (3.1) has a hyperbolic singular point at $x = 0$ and it is easy, from the last expression of $U(q_1, q_2)$, to see that system (3.1) has a homoclinic manifold Γ. In fact Γ is given by

$$\begin{aligned} \Gamma &= \{\gamma(t, \beta) : -\infty < t < \infty,\ \ 0 \le \beta < 2\pi\} \\ &= \{r(t)\cos\beta, r(t)\sin\beta, \dot{r}\cos\beta, \dot{r}(t)\sin\beta) : -\infty < t < \infty, 0 \le \beta < 2\pi\} \end{aligned}$$

where $r(t) = \operatorname{sech} t$.

Now notice that system (3.1) has an integral $F(x) \equiv q_1p_2 - q_2p_1$, and it is easy to show that the differentials $dH(x)$ and $dF(x)$ are linearly independent on Γ. In fact the Poisson bracket $\{H, F\} = 0$ and hence system (3.1) is a completely integrable system. Now we consider the perturbation of system (3.1).

$$\dot{x} = J\Delta H(x) + \epsilon g(t, x, \mu) \tag{3.2}$$

where $0 < \epsilon \ll 1, \mu = (\mu_1, \mu_2, \mu_3, \mu_4) \in R^4$ and $g = (g_1, g_2, g_3, g_4)$ is given by the following expressions

$$\begin{aligned} g_1(t, x, \mu) &= \left[\frac{2(\mu_4 - 1)}{1 + \omega^2}(x_1^2 + 3x_3^2)\right]\cos\omega t, \\ g_2(t, x, \mu) &= \left[-3\mu_1x_1 - \mu_2x_2 + \frac{2\mu_3}{1 + \omega^2}(3x_1^2 + x_3^2) + \frac{4\mu_4}{1 + \omega^2}x_1x_3\right]\cos\omega t, \\ g_3(t, x, \mu) &= \left[\frac{2(\mu_4 - 1)}{1 + \omega^2}x_1x_3\right]\cos\omega t, \\ g_4(t, x, \mu) &= \left[-\mu_1x_3 - \mu_2x_4 + \frac{4\mu_3}{1 + \omega^2}x_1x_3 + \frac{2\mu_4}{1 + \omega^2}(x_1^2 + 3x_3^2)\right]\cos\omega t. \end{aligned}$$

This choice of the perturbation g is a modification of the one in Gruendler [1]. More physically relevant models are expected to be given.

Now for a completely integrable Hamiltonian system $\phi_1(t,\beta) = dH(\gamma(t,\beta))$ and $\phi_2(t,\beta) = dF(\gamma(t,\beta))$ are bounded solutions of the adjoint system $\dot{\phi}+[JD^2H(\gamma(t,\beta)]^*\phi = 0$. Hence the Melnikov vector for system (4.2) takes the following form

$$M(\alpha,\beta,\mu) = \begin{bmatrix} M_1(\alpha,\beta,\mu) \\ M_2(\alpha,\beta,\mu) \end{bmatrix} = \begin{bmatrix} \int_{-\infty}^{\infty} dH(\gamma(t,\beta))g(t-\alpha,\gamma(t,\beta),\mu)dt \\ \int_{-\infty}^{\infty} dF(\gamma(t,\beta))g(t-\alpha,\gamma(t,\beta),\mu)dt \end{bmatrix}.$$

Several routine calculations give the following $M(\alpha,\beta,\mu)$.

$$\begin{aligned} M_1(\alpha,\beta,\mu) &= -\frac{2}{3}\mu_2 - c\omega(\mu_3\cos\beta + \mu_4\sin\beta)\sin\omega\alpha + 2(\mu_4-1)[d_1(\cos^3\beta - \cos\beta\sin^2\beta) \\ &\quad +d_2(-2\cos^3\beta + 2\cos\beta\sin^2\beta)]\cos\omega\alpha \\ M_2(\alpha,\beta,\mu) &= 2\mu_1\sin 2\beta + c(-\mu_3\sin\beta + \mu_4\cos\beta)\cos\omega\alpha \\ &\quad -\frac{1}{3}c\omega(\mu_4-1)(\sin^2\beta + \sin^2\beta)\sin\omega x \end{aligned}$$

where $c = \operatorname{sech}\left[\frac{\pi\omega}{2}\right]$, $d_1 = \frac{1}{1+\omega^2}\int_{-\infty}^{\infty}\operatorname{sech}^3 t\cos\omega t\, dt$ and $d_2 = \frac{1}{1+\omega^2}\int_{-\infty}^{\infty}\operatorname{sech}^5 t\cos\omega t\, dt$.

Now at $\omega\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$ for example,

$$M = \begin{bmatrix} -\frac{2}{3}\mu_2 - \frac{1}{2\sqrt{2}}c\omega\mu_3 - \frac{1}{2\sqrt{2}}(\sqrt{3}c\omega + d_1 - 2d_2)\mu_4 - \frac{1}{2\sqrt{2}}(d_1 - 2d_2) \\ \sqrt{3}\mu_1 - \frac{\sqrt{3}}{2\sqrt{2}}c\mu_3 + \frac{1}{6\sqrt{2}}c(3-(\sqrt{3}+3)\omega)\mu_4\frac{\sqrt{3}+3}{6\sqrt{2}}c\omega \end{bmatrix},$$

$$\frac{\partial M}{\partial\alpha} = \begin{bmatrix} -\frac{1}{2\sqrt{2}}c\omega^2\mu_3 - \frac{1}{2\sqrt{2}}c\omega(\sqrt{3}c\omega - d_1 + 2d_2)\mu_4 - \frac{1}{2\sqrt{2}}\omega(d_1 - 2d_2) \\ \frac{\sqrt{3}}{2\sqrt{2}}c\omega\mu_3 - \frac{1}{6\sqrt{2}}c\omega(3+(\sqrt{3}+3)\omega)\mu_4 + \frac{\sqrt{3}+3}{6\sqrt{2}}c\omega^2 \end{bmatrix},$$

$$\frac{\partial M}{\partial \beta} = \begin{bmatrix} \frac{\sqrt{3}}{2\sqrt{2}}c\omega\mu_3 - \frac{1}{2\sqrt{2}}(c\omega + \sqrt{3}(d_1 - 2d_2))\mu_4 + \frac{\sqrt{3}}{2\sqrt{2}}(d_1 - 2d_2) \\ -2\mu_1 - \frac{1}{2\sqrt{2}}c\mu_3 - \frac{1}{6\sqrt{2}}c(3\sqrt{3} + (1 + 2\sqrt{3})\omega)\mu_4 + \frac{1+2\sqrt{3}}{6\sqrt{2}}c\omega \end{bmatrix},$$

$$\frac{\partial}{\partial \mu}\begin{bmatrix} M \\ \partial M/\partial \alpha \end{bmatrix} = \begin{bmatrix} 0 & -2/3 & -c\omega/2\sqrt{2} & -(\sqrt{3}c\omega + d_1 - 2d_2)/2\sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3}c/2\sqrt{2} & c(3 - (\sqrt{3} + 3)\omega)/6\sqrt{2} \\ 0 & 0 & -c\omega^2/2\sqrt{2} & -\omega(\sqrt{3}c\omega - d_1 + 2d_2)/2\sqrt{2} \\ 0 & 0 & \sqrt{3}c\omega/2\sqrt{2} & -c\omega(3 + (\sqrt{3} + 3)\omega)/6\sqrt{2} \end{bmatrix},$$

$$\frac{\partial^2 M}{\partial \alpha^2} = \begin{bmatrix} \frac{1}{2\sqrt{2}}c\omega^3\mu_3 + \frac{1}{2\sqrt{2}}\omega^2(\sqrt{3}c\omega + d_1 - 2d_2)\mu_4 - \frac{1}{2\sqrt{2}}\omega^2(d_1 - 2d_2) \\ \frac{\sqrt{3}}{2\sqrt{2}}c\omega^2\mu_3 - \frac{1}{6\sqrt{2}}c\omega^2(3 - (\sqrt{3} + 3)\omega)\mu_4 - \frac{\sqrt{3}+3}{6\sqrt{2}}c\omega^3 \end{bmatrix}.$$

From these expressions it can be shown that there exists $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*)$ such that $\mu_i \neq 0, i, = 1, 2, 3$ and $\mu_4 - 1 \neq 0$ at which $M = \dfrac{\partial M}{\partial \alpha} = 0$, $\dfrac{\partial M}{\partial \beta} \neq 0$, $\operatorname{rank} \dfrac{\partial}{\partial \mu}\left[M, \dfrac{\partial M}{\partial \alpha}\right] = 4$ and $\dfrac{\partial^2 M_1}{\partial \alpha^2} \neq 0 \neq \dfrac{\partial^2 M_2}{\partial \alpha^2}$.

Therefore by theorem 2.1 the quadratic homoclinic tangency occurs.

4 Conclusion

In this paper we studied the Melnikov method for the homoclinic tangency. In [5] it is reported that a Hénon-like map exists near a tangential homoclinic point for a two-dimensional C^∞ diffeomorphism. In higher dimensions the corresponding situation is less well understood. The homoclinic tangency in higher dimensions can occur in many different situations and there the coexistence of a horeshoe-like map and a Hénon-like map might be possible. Theoretical study of the dynamics in this direction is still very infant at present.

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