King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 207

June 1996

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Characterization of Small Solutions in Functional Differential Equations

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Abstract

A characterization is given of the initial functions which yield small solutions in a class of linear, autonomous, retarded functional differential equations.

Keywords— Retarded functional differential equation, partial multiplicities, Jordan form, generalized left characteristic matrix equation, transformation, small solution.

1 Introduction

Consider the delay system, $\mathcal{S}_d$, described by the linear autonomous retarded functional differential equation (rfde)

$$\mathcal{S}_d : \quad \dot{x}(t) = \int_{-r}^{0} d\alpha(\theta)x(t+\theta) \tag{1.1}$$

where $t \in (0,\infty)$ denotes time, $r \in (0,\infty)$ denotes the system memory span, the instantaneous state $x(t) \in \mathbb{R}^n$ and $\alpha \in \mathrm{BV}([-r,0];\mathbb{R}^{n\times n})$, the class of $n \times n$ matrix valued functions of bounded variation on $[-r,0]$. The state of $\mathcal{S}_d$ is the segment function, x_t, defined by $x_t(\theta) = x(t+\theta), \quad -r \le \theta \le 0$. In this notation, the initial function of (1.1) can be written as

$$x_0(\theta) = \phi(\theta) \quad -r \le \theta \le 0 \tag{1.2}$$

where $\phi \in C([-r,0];\mathbb{R}^n)$, the class of $\mathbb{R}^n$−valued continuous function on $[-r,0]$. By the solution of $\mathcal{S}_d$ is meant the continuous function $x \in C([-r,\infty);\mathbb{R}^n)$ for which $x_0(\theta) = \phi(\theta)$ and which satisfies (1.1) for $t > 0$.

The linear *non-autonomous* rfde, $\dot{x}(t) = -2te^{1-2t}x(t-1)$, admits the solution $x(t) = e^{-t^2}$ on $[-1,\infty)$. See [1, p. 97]. This solution approaches zero faster than any exponential but is never zero. Such a solution is said to be small. More precisely, corresponding to an initial function $\phi \neq 0$, the solution of an rfde is said to be a small solution if $\lim_{t\to\infty} e^{kt}x(t) = 0 \quad \forall k \in \mathbb{R}$. The question then arises as to whether a linear *autonomous* rfde, $\mathcal{S}_d$, may admit a small solution [1,2]. Further research [1] has shown that if $x(\cdot)$ is a small solution of $\mathcal{S}_d$, then $x(t) = 0$ for $t \geq nr - E(det\Delta(\lambda))$ where

$$\Delta(\lambda) = \lambda I_n - \int_{-r}^{0} e^{\lambda\theta} d\alpha(\theta)$$

is the system characteristic matrix and

$$E(det\Delta(\lambda)) = \overline{\lim}_{|\lambda|\to\infty} \frac{\ln|det\Delta(\lambda)|}{|\lambda|}$$

is the exponential type of $det\Delta(\lambda)$. It also turns out that the eigenfunctions of (1.1) are complete if and only if it has no small solutions. See also [3,4] and references therein for further background on the subject.

Let $T(t) : C([-r,0];\mathbb{R}^n) \to C([-r,0];\mathbb{R}^n)$ defined by $x_t = T(t)\phi$ denote the solution operator of (1.1). Following [1], $\{T(t); t \geq 0\}$ is a $\mathcal{C}_0$ semi-group of bounded linear operators whose infinitesimal generator, $\mathcal{A}$, has a countable point spectrum given by

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} : \ \det\Delta(\lambda) = 0\}.$$

and each eigenvalue has a finite algebraic multiplicity.

In [1], a small solution is characterized through analysis of its exponential type. Our aim is to obtain a spectral characterization based on a transformation of $\mathcal{S}_d$ to an ordinary differential equation. For systems whose eigenvalues of interest are simple (algebraic multiplicity equal to unity), the transformation is documented in [5]. The transformation theory for the general case of multiple eigenvalues (algebraic multiplicity greater than one) has recently been given in [6]. The relevant portions of this theory are summarized in Section II. Section III contains the principal result which gives a characterization of the initial functions that lead to small solutions in $\mathcal{S}_d$ with n linearly independent left eigenvectors and/or generalized left eigenvectors.

2 Background Material

2.1 Jordan Form

Let $\mathbb{C}^+_{-\nu_0} = \{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda \geq -\nu_0, \ \nu_0 \geq 0\}$ denote an arbitrary closed right half of the complex plane bounded by $\lambda = -\nu_0$. It is known (see [1]) that for any $\nu_0 \geq 0$, there are finitely many eigenvalues in $\mathbb{C}^+_{-\nu_0}$. Let $\lambda_j, \ j = 1, 2, \ldots, N$ be the distinct eigenvalues in $\mathbb{C}^+_{-\nu_0}$. For a distinct eigenvalue λ_j with algebraic multiplicity m_j where $1 \leq m_j < \infty$, we define a Jordan cell $J_j(\lambda_j) \in \mathbb{C}^{m_j \times m_j}$ by $J_j = \bigoplus_{l=1}^{g_j} J_j^l(\lambda_j)$ where $J_j^l \in \mathbb{C}^{m_j^l \times m_j^l}$ is given by $J_j^l = \lambda_j I_{m_j^l} + E_{m_j^l}$. Here, $I_{m_j^l}$ is the identity matrix of order m_j^l and

$$E_{m_j^l} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

is also of order m_j^l. Thus

$$m_j = \sum_{l=1}^{g_j} m_j^l$$

where g_j represents the number of sub-cells in J_j. Extending this to all the N distinct eigenvalues in $\mathbb{C}^+_{-\nu_0}$ yields

$$J = \bigoplus_{j=1}^{N} J_j = \bigoplus_{j=1}^{N} \bigoplus_{l=1}^{g_j} J_j^l(\lambda_j) \tag{2.1}$$

so that $\sigma(J) = \sigma(\mathcal{A}) \cap \mathbb{C}^+_{-\nu_0}$ and

$$M = \sum_{j=1}^{N} m_j.$$

Thus, the question of constructing a Jordan matrix, $J \in \mathbb{C}^{M \times M}$, such that $\sigma(J) = \sigma(\mathcal{A}) \cap \mathbb{C}^+_{-\nu_0}$ reduces to the question of knowledge of the number of sub-cells, g_j, and the orders of the sub-cells for each eigenvalue. This is clearly the Segre characteristic ([7, p.241]) and may be displayed as

$$\left\{ \left(m_1^1, m_1^2, \ldots, m_1^{g_1}\right), \left(m_2^1, m_2^2, \ldots, m_2^{g_2}\right), \ldots, \left(m_N^1, m_N^2, \ldots, m_N^{g_N}\right) \right\}.$$

The resolution of this problem lies in the following theorem.

Theorem 2.1 *Let $\lambda_j \in \sigma(\mathcal{A})$ be an eigenvalue of algebraic multiplicity, m_j. Then the characteristic matrix, $\Delta(\lambda)$, admits the representation:*

$$\Delta(\lambda) = E(\lambda)\ \mathrm{diag}[(\lambda - \lambda_j)^{\kappa_j^1}, (\lambda - \lambda_j)^{\kappa_j^2}, \ldots, (\lambda - \lambda_j)^{\kappa_j^n}]\ F(\lambda) \qquad (2.2)$$

where $E(\lambda)$ and $F(\lambda)$ are $n \times n$ matrix functions which are analytic and invertible in a neighborhood of λ_j. Furthermore, $\kappa_j^1 \geq \kappa_j^2 \cdots \geq \kappa_j^n \geq 0$ are integers (called *partial multiplicities) which are uniquely determined by $\Delta(\lambda)$ and satisfy the relation:*

$$\kappa_j^1 + \kappa_j^2 + \cdots + \kappa_j^n = m_j \qquad (2.3)$$

This theorem is based on Theorem 1.1.2 and Corollary 1.1.7 of [8]. Furthermore, the partial multiplicities in (2.2) can be explicitly determined with the help of the algorithm described in the proof of Theorem 1.1.2 in [8]. The importance of this result in the construction of the Jordan matrix lies in the fact that one can set

$$m_j^l = \kappa_j^l, \quad l = 1, 2, \ldots, g_j$$

where g_j is the number of non-zero partial multiplicities in the representation, (2.2), of $\Delta(\lambda)$. The next result summarizes a further generalization of the concept of the generalized left characteristic matrix equation (glcme) originally introduced in [5]. See [6] for further details and a related development in [9].

2.2 Left Jordan Chains

Theorem 2.2 *Given S_d and an arbitrary $\nu_0 \geq 0$, there exists a non-trivial $Q \in \mathbb{C}^{M\times n}$ satisfying the glcme, namely,*

$$JQ = \int_{-r}^{0} e^{J\theta} Q d\alpha(\theta) \tag{2.4}$$

where $J \in \mathbb{C}^{M\times M}$ is a Jordan matrix such that $\sigma(J) = \sigma(\mathcal{A}) \cap \mathbb{C}^{+}_{-\nu_0}$. More specifically, $0 \neq Q \in \mathbb{C}^{M\times n}$ satisfies (2.4) $\Leftrightarrow$ $co(Q_j^l) \in \ker U'_\alpha(\lambda_j; m_j^l)$. That is,

$$\left[co(Q_j^l)\right]' U_\alpha(\lambda_j; m_j^l) = 0, \quad j = 1,2,\ldots,N; \quad l = 1,2,\ldots,g_j \tag{2.5}$$

where $'$ denotes transposition,

$$U_\alpha(\lambda_j; m_j^l) = \left.\begin{pmatrix} \Delta(\lambda) & \frac{1}{1!}\frac{d\Delta}{d\lambda} & \frac{1}{2!}\frac{d^2\Delta}{d\lambda^2} & \cdots & \frac{1}{(m_j^l-1)!}\frac{d^{m_j^l-1}\Delta}{d\lambda^{m_j^l-1}} \\ 0 & \Delta(\lambda) & \frac{1}{1!}\frac{d\Delta}{d\lambda} & \cdots & \frac{1}{(m_j^l-2)!}\frac{d^{m_j^l-2}\Delta}{d\lambda^{m_j^l-2}} \\ 0 & 0 & \Delta(\lambda) & \cdots & \frac{1}{(m_j^l-3)!}\frac{d^{m_j^l-3}\Delta}{d\lambda^{m_j^l-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Delta(\lambda) \end{pmatrix}\right|_{\lambda=\lambda_j}, \tag{2.6}$$

$$Q = \mathrm{col}(Q_j, \quad j = 1,2,\ldots,N; \quad Q_j \in \mathbb{C}^{m_j\times n}), \tag{2.7}$$

$$Q_j = \mathrm{col}(Q_j^l, \quad l = 1,2,\ldots,g_j; \quad Q_j^l \in \mathbb{C}^{m_j^l\times n}) \tag{2.8}$$

and $co : \mathbb{C}^{p\times q} \longrightarrow \mathbb{C}^{1\times pq}$ denotes the isomorphism which carries a $p \times q$ matrix into a pq column vector by stringing the rows of the matrix one after another and then transposing.

PROOF. See [6].

Basic properties of the isomorphism, *co*, can be found in [10]. Using the above result and with the help of [8], the notion of a left eigenvector in [5] is generalized as follows.

Definition 2.1 *Let $(co\ Q_j^l)' = \left(q_j^1, q_j^2, \ldots, q_j^{m_j^l}\right)$ where $q_j^l \in \mathbb{C}^{1\times n}$ is the $l-th$ row of the matrix $Q_j^l \in \mathbb{C}^{m_j^l\times n}$. Corresponding to the eigenvalue, λ_j, of algebraic multiplicity*

m_j , *the ordered set of* <u>*non-zero*</u> *vectors,* $q_j^1, q_j^2, \ldots, q_j^{m_j^l}$ *is a left Jordan chain of order* m_j^l *if*

$$\left(q_j^1, q_j^2, \ \cdots \ , q_j^{m_j^l}\right) U_\alpha(\lambda_j; m_j^l) = 0 \tag{2.9}$$

The leading vector, q_j^1 *is called a left eigenvector while the rest of the set, namely,* $q_j^2, q_j^3 \ \cdots \ , q_j^{m_j^l}$, *are called generalized (left) eigenvectors. In the particular case where* $m_j = 1$, *the generalized left eigenvector becomes a left eigenvector. The maximum order,* $\mu_j = \max_{1 \leq l \leq g_j} m_j^l$ *is called the rank of the eigenvector,* q_j^1.

Remark 2.1 *Since the rows of* $Q \in \mathbb{C}^{M \times n}$ *are either left eigenvectors or generalized left eigenvectors of* $\Delta(\lambda)$, *it follows that* $Q \neq 0$. *As* $\nu_0 \to \infty$, $M \to \infty$ *so that the maximum rank attainable by* Q *is* n. *However, regardless of whether rank*$Q = n$ *or not, the pair* $(J, Q) \in \mathbb{C}^{M \times M} \times \mathbb{C}^{M \times n}$ *satisfying the glcme permits the reduction of* $\mathcal{S}_d$ *to an ordinary differential equation as follows.*

Theorem 2.3 *Let*

$$z(t) = Qx(t) + \int_{-r}^{0} \int_{t+\theta}^{t} e^{J(t+\theta-\tau)} Q d\alpha(\theta) x(\tau) d\tau, \quad t \geq 0 \tag{2.10}$$

where $x(\cdot)$ *is any solution of* $\mathcal{S}_d$ *and the pair* $(J, Q) \in \mathbb{C}^{M \times M} \times \mathbb{C}^{M \times n}$ *satisfies the glcme, (2.4). Then* $z(\cdot)$ *satisfies the ordinary differential equation,*

$$\mathcal{S}_0 : \quad \dot{z}(t) = Jz(t), \quad t \geq 0. \tag{2.11}$$

PROOF. Follows by direct differentiation.

3 Main Result

Theorem 3.1 *Let* $J \in \mathbb{C}^{M \times M}$ *be a Jordan matrix such that* $\sigma(J) = \sigma(\mathcal{A}) \cap \mathbb{C}^+_{-\nu_0}$ *where* $\nu_0 \in [0, \infty)$ *and* $Q \in \mathbb{C}^{M \times n}$ *a non-trivial solution of the glcme, (2.4). Assume* *that* $rankQ = n$. *Then* $\mathcal{S}_d$ *has a small solution if and only if* $\forall \nu_0 \geq 0$, *there exists a non-zero* $\phi \in \mathrm{C}([-r,0];\mathbb{R}^n)$ *such that*

$$Q\phi(0) + \int_{-r}^{0}\int_{\theta}^{0} e^{J(\theta-\tau)}Q d\alpha(\theta)\phi(\tau)d\tau = 0 \tag{3.12}$$

Such a small solution vanishes no later that $t = r$.

PROOF. $\Rightarrow$ Let $x(\cdot)$ be a small solution of $\mathcal{S}_d$. Write out the transformation, (2.10), in terms of its components, multiply by e^{kt} and then take limits to get

$$\begin{aligned}
\lim_{t\to\infty} e^{kt}z_i(t) &= \sum_{j=1}^{n} Q_{i,j} \lim_{t\to\infty} e^{kt}x_j(t) \\
&\quad + \lim_{t\to\infty} \sum_{j,l=1}^{n} \int_{-r}^{0} \left\{ \int_{\theta}^{0} \left(e^{J\tau}Q\right)_{ij} e^{kt}x_l(t+\theta-\tau)d\tau \right\} d\alpha_{jl}(\theta) \\
&= \sum_{j,l=1}^{n} \int_{-r}^{0} \lim_{t\to\infty} \left\{ \int_{\theta}^{0} \left(e^{J\tau}Q\right)_{ij} e^{kt}x_l(t+\theta-\tau)d\tau \right\} d\alpha_{jl}(\theta) \\
&= \sum_{j,l=1}^{n} \int_{-r}^{0} \left\{ \int_{\theta}^{0} \left(e^{J\tau}Q\right)_{ij} \lim_{t\to\infty} e^{kt}x_l(t+\theta-\tau)d\tau \right\} d\alpha_{jl}(\theta) \\
&= 0
\end{aligned} \tag{3.13}$$

since $\lim_{t\to\infty} e^{kt}x_l(t+\theta-\tau) = e^{k(\tau-\theta)}\lim_{t\to\infty} e^{kt}x_l(t) = 0$. The interchanges above between limits and integrals are justified by the uniform boundedness of the integrand. Thus, $\forall\epsilon > 0$, $\exists T(\epsilon)$ such that $t > T(\epsilon) \Rightarrow -\epsilon < e^{kt}z_i(t) < \epsilon$ or $-\epsilon e^{-kt} < z_i(t) < \epsilon e^{-kt}$. By considering $k \to \infty$, we find that $z(t) = 0\ \forall t > T(\epsilon)$. This implies that $e^{Jt}z(0) = 0\ \forall t > T(\epsilon)$ or $z(0) = 0$. The result, (3.12), follows on putting $t = 0$ in (2.10).

$\Leftarrow$ Suppose that there exists a non-trivial $\phi \in \mathrm{C}([-r,0];\mathbb{R}^n)$ such that (3.12) holds; that is, $z(0) = 0$. From (2.11), $z(t) = e^{Jt}z(0) = 0 \quad \forall t \geq 0$. Then the transformation, (2.10), gives

$$Qx(t) = -\int_{-r}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)}Q d\alpha(\theta)x(\tau)d\tau \qquad \forall t \geq 0. \tag{3.14}$$

Note that $t - r \leq t + \theta \leq \tau \leq t$ so that for $t \geq r$, $\tau \geq 0$ and the RHS of (3.14) does not explicitly contain the initial function, ϕ. Let $y(t) = Qx(t)$. Since $rankQ = n$,

$\exists Q_L \in \mathbb{C}^{n\times M}$ such that $Q_L Q = I_n$. Then for $t \geq r$, (3.14) can be written as

$$y(t) = -\int_{-r}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)}Qd\alpha(\theta)Q_L\, y(\tau)d\tau. \tag{3.15}$$

Let $\bigvee_{-r}^{0}\alpha_{lp}$ denote the variation of α_{lp} on $[-r,0]$ and $\|y(t)\| = \max_i |y_i(t)|$. Recall that for $f \in \mathrm{C}([a,b])$ and $g \in \mathrm{BV}([a,b])$, $|\int_a^b f(t)dg(t)| \leq \sup_{[a,b]}|f(t)|\bigvee_a^b g$. Using these facts and (3.15), we obtain

$$\|y(t)\| \leq \int_{t-r}^{t} k_0 \left\|e^{J(t-\tau)}\right\| \, \|y(\tau)\| \, d\tau$$

where k_0 is some constant. Since $t-r \leq \tau \leq t$ or $0 \leq t-\tau \leq r$, it follows that $\left\|e^{J(t-\tau)}\right\| \leq e^{\|J\|r}$. Thus, for $t \geq r$, we have

$$\|y(t)\| \leq \int_{0}^{t} k_0 e^{\|J\|r}\, \|y(\tau)\| \, d\tau \tag{3.16}$$

and by Gronwall's inequality, $\|y(t)\| = \|Qx(t)\| = 0$. That is, for $t \geq r$ and any solution, $x(\cdot)$, we have $0 \leq \|x(t)\| = \|Q_L Q x(t)\| \leq \|Q_L\|\, \|Qx(t)\| = 0$ or $x(t) = 0$ $\forall t \geq r$ whence $\lim_{t\to\infty} e^{kt}x(t) = 0$ for any $k \in \mathbb{R}$. That is, $x(\cdot)$ is a small solution and vanishes no later that $t = r$. This concludes the proof of the theorem.

Discussion. Even though in a typical delay system $Q \in \mathbb{C}^{M\times n}$ has full rank, it is not generally true that $rank Q = n$. Therefore, the assumption of $rank Q = n$ in the principal result necessarily limits our class of delay systems. It would be interesting to obtain a direct condition on $\alpha(\cdot)$ and r to delineate those systems for which the above rank assumption is true.

It is evident from the foregoing that the question of small solutions reduces to the question of solvability of (3.12). To pursue this matter a little further, employ Fubini's Theorem to write (3.12) as

$$\begin{aligned} Q\phi(0) &= -\int_{-r}^{0}\int_{\theta}^{0} e^{J(\theta-\tau)}Qd\alpha(\theta)\phi(\tau)d\tau \\ &= -\int_{-r}^{0} e^{-J\tau}\left[\int_{-r}^{\tau} e^{J\theta}Qd\alpha(\theta)\right]\phi(\tau)d\tau \\ &= -\int_{-r}^{0^-} F(\tau)\phi(\tau)d\tau \end{aligned} \tag{3.17}$$

where $F(\tau) \in \mathrm{BV}([-r,0];\mathbb{C}^{M\times n})$ is given by

$$F(\tau) = e^{-J\tau}\int_{-r}^{\tau} e^{J\theta} Q d\alpha(\theta). \tag{3.18}$$

Thus, (3.12) is solvable for $\phi \in C([-r,0];\mathbb{R}^n)$ if and only if the system of linear equations, (3.17), is solvable for $\phi(0) \in \mathbb{R}^n$. By the Fredholm alternative Theorem, (3.17) is solvable if and only if

$$\int_{-r}^{0} vF(\tau)\phi(\tau)d\tau = 0 \tag{3.19}$$

$\forall v' \in \ker(Q')$. Partition $v \in \mathbb{C}^{1\times M}$ as $v = [v_1|v_2|\ldots|v_N]$ where $v_j \in \mathbb{C}^{1\times m_j}$ is given by $v_j = [v_j^1|v_j^2|\ldots|v_j^{g_j}]$ and $v_j^l \in \mathbb{C}^{1\times m_j^l}$, $l = 1,2,\ldots,g_j$. Using (2.1) to represent J, (2.7) and (2.8) to represent Q, we manipulate the LHS of (3.19) to the form

$$\int_{-r}^{0} vF(\tau)\phi(\tau)d\tau = \sum_{j=1}^{N}\sum_{l=1}^{g_j}\int_{-r}^{0} v_j^l e^{-\tau(\lambda_j I_{m_j^l}+E_{m_j^l})}\left[\int_{-r}^{\tau} e^{\theta(\lambda_j I_{m_j^l}+E_{m_j^l})} Q_j^l d\alpha(\theta)\right]\phi(\tau)d\tau \tag{3.20}$$

As $\nu_0 \to \infty$, $N \to \infty$ and (3.20) becomes an infinite series. Needless to say that this series is, in general, not zero as demanded by (3.19). Therefore, (3.17) is, in general, not solvable except under special conditions such as: i) $\sigma(\mathcal{A})$ is finite and $rank Q = n$. In this case, Q^{-1} exists and (3.12), or equivalently (3.17), is solvable for $\phi(0)$. ii) There exists a $\phi \in C([-r,0];\mathbb{R}^n)$ such that the RHS of (3.20) is identically zero. In that case the choice of $\phi(0) = 0$ is enough to satisfy (3.20). Both techniques are illustrated in the following example.

Example. Let $\mathcal{S}_d$: $\dot{x}(t) = A_0 x(t) + A_1 x(t-1)$ where

$$A_0 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Show that this system has small solutions and describe the set of initial functions which yields small solutions.

Solution. The characteristic matrix $\Delta(\lambda) = \begin{pmatrix} \lambda & -e^{-\lambda} & 1 \\ 0 & \lambda & 1-e^{-\lambda} \\ 0 & 0 & \lambda+1 \end{pmatrix}$, $\det \Delta(\lambda) = \lambda^2(\lambda+1)$ so that $\sigma(\mathcal{A}) = \{-1,0,0\} = \{\lambda_1,\lambda_2,\lambda_2\}$ is finite. Therefore, for any $\nu_0 > 1$, $\sigma(\mathcal{A}) \subset \mathbb{C}^+_{-\nu_0}$. Here, $N = 2$, $m_1 = 1$, $m_2 = 2$ and $M = 3$. To construct J requires knowledge of the Segre characteristic. For λ_1, $m_1 = 1$ so that, trivially, $m_1^1 = 1$. For λ_2, we employ the algorithm described in the proof of Theorem 1.1.2 of [8], to put $\Delta(\lambda)$ in the *local Smith form* as $\Delta(\lambda) = E(\lambda) diag\,(\lambda^2,1,1)\, F(\lambda)$ where $E(\lambda)$ and $F(\lambda)$ are $n \times n$ matrix functions invertible in some neighborhood of the origin of $\mathbb{C}$. This shows that $g_2 = 1$ and $m_2^1 (= \kappa_2^1) = 2$ so that the Segre characteristic is $\{(m_1^1),(m_2^1)\} = \{(1),(2)\}$.

The Jordan Matrix. $J_1^1 = (-1)$, $J_2^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow J = J_1 \oplus J_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Left Jordan Chains. For $m_1^1 = 1$, the Jordan chain reduces to a left eigenvector $Q_1^1 = (q_{11}, q_{12}, q_{13})$ given by (2.5) as $Q_1^1 \Delta(\lambda_1) = 0$ to give $Q_1 = Q_1^1 = (0,0,q_{13})$, $q_{13} \neq 0$. For $m_2^1 = 2$, we have a (left) Jordan chain of order 2 given by

$$\left(co\, Q_2^1\right)' \begin{pmatrix} \Delta(\lambda) & \frac{d\Delta}{d\lambda}(\lambda) \\ 0 & \Delta(\lambda) \end{pmatrix}_{\lambda=0} = 0.$$

Letting $(co\, Q_2^1)' = (q_{21}, q_{22}, q_{23}, q_{31}, q_{32}, q_{33})$ yields $q_{21} = 0$, $q_{23} = 0$, $q_{31} = q_{22}$, $2q_{22} + q_{33} = 0$ whence $Q_2 = Q_2^1 = \begin{pmatrix} q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} = \begin{pmatrix} 0 & q_{22} & 0 \\ q_{22} & q_{32} & -2q_{22} \end{pmatrix}$. Note that for a chain of order 2, $q_{22} \neq 0$. Therefore,

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & q_{13} \\ 0 & q_{22} & 0 \\ q_{22} & q_{32} & -2q_{22} \end{pmatrix}$$

and $rankQ = 3 = n$. With the above computations, it is verified that the glcme is satisfied. Since the spectrum is finite and $rankQ = n$, this system has small solutions. From (3.12), the set of initial functions which yield small solutions is given by $\phi \in C([-r,0];\mathbb{R}^n)$ such that

$$\phi(0) = -\int_{-r}^{0}\int_{\theta}^{0} Q^{-1}e^{J(\theta-\tau)}Q\,d\alpha(\theta)\phi(\tau)d\tau = -\int_{-r}^{0} Q^{-1}e^{-J(r+\tau)}QA_1\phi(\tau)d\tau. \quad (3.21)$$

This gives

$$\phi_3(0) = 0, \quad \phi_2(0) = -\int_{-1}^{0} \phi_3(\tau)d\tau, \quad \phi_1(0) = \int_{-1}^{0} [(1+\tau)\phi_3(\tau) - \phi_2(\tau)]\, d\tau.$$

Alternatively, choosing $\phi(\tau) = [\phi_1(\tau), 0, 0]$ yields $A_1\phi(\tau) = 0$ so that the RHS of (3.21) is identically zero. The satisfaction of (3.12) is then guaranteed by putting $\phi(0) = 0$.

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