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Extended Riemann's Zeta Functions

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Abstract

Analogous to recent useful generalizations of the family of gamma functions and beta functions, two extensions of Riemann's zeta function are presented, for which the usual properties and representations are naturally and simply extended. In addition, incomplete zeta and generalized zeta functions are defined and extended. Hurwitz–type formulae for these extended zeta functions are also proved.

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1. Introduction

The zeta function, though originally introduced by Euler, was independently used by Riemann to attack a problem in the theory of prime numbers [3, 9, 10, 11]. It was known that prime numbers become progressively sparser for large values but no explicit expression explaining how they become so was available until the time of Legendre and Gauss. Writing the number of primes less than or equal to n as

$$\pi(n) = \sum_{p \leq n} 1, \tag{1}$$

with summation running over primes only. They conjectured that as $n \to \infty$,

$$\pi(n) \sim n/\log n. \tag{2}$$

In an attempt to prove the conjecture, Riemann used the zeta function extended to complex variables. (It may be mentioned in passing that though Riemann's proof was incomplete, there is now available an elementary proof of this result and it is commonly known as the prime number theorem [4, 11].

Studying the properties of the zeta function, Riemann conjectured that the non-trivial zeros of $\zeta(\alpha), \alpha = \sigma + i\tau$, lie on the critical line $\sigma = 1/2$ in the complex plane. Though it is proved that they are restricted to the strip, $0 < \sigma < 1$ (Hardy managed to show that there are infinitely many zeros on the critical line), there is still no proof of Riemann's conjecture [4, 9, 11]. There is also no counter-example contradicting it.

In this paper, we present two extensions of Riemann's zeta functions which are closely related to each other. There being infinitely many new functions and infinitely many extensions or generalizations possible for well known functions, one needs some clear-cut criteria to determine whether a given extension is worthwhile or not. If it arises in diverse problems or interesting new relations turn up between it and other functions, or new insights are provided for the original functions, or particularly elegant results can be found for the new function, it would be worthwhile. In our case, we do find natural extensions of the previous results, obtain new results for the extensions and expect that there will be a wider applicability of these extended zeta functions.

Our extensions were motivated by the wide applications of the generalization of the family of gamma functions [1, 2] and the beta function [3]. We use integral representations of the zeta function to extend them analogously to the generalized gamma function. We also apply our extension procedure to the generalized zeta function, $\zeta(\alpha, q)$, and obtain the corresponding extensions for it. The incomplete zeta functions and their corresponding extensions are also introduced.

For clarity of presentation, we state and prove our results as theorems and take $\alpha = \sigma + i\tau$.

2. The Extended Zeta Function

The original definition of the zeta function by Euler [11] was

$$\zeta(\sigma) = \sum_{n=1}^{\infty} n^{-\sigma}, \quad (\sigma > 1). \tag{3}$$

It was known that this function has the integral representation for $\sigma > 1$ [11]

$$\zeta(\alpha) = \frac{1}{\Gamma(\alpha)} \int_{t=0}^{\infty} t^{\alpha-1}(1-e^{-t})^{-1}e^{-t}dt, \quad (\sigma > 1). \tag{4}$$

The integral in (4) $\zeta(\alpha)$ becomes singular for $\sigma \leq 1$ because of the singularity of the integrand at $t = 0$. This integrand is similar to that appearing in Euler's gamma function [2] which was generalized to

$$\Gamma_b(\alpha) = \int_{t=0}^{\infty} t^{\alpha-1} e^{-t-b/t} dt, \quad (b > 0;\ b = 0,\ \sigma > 0). \tag{5}$$

This function reduces to the usual gamma function for $b = 0$. The corresponding extended zeta function is defined as

$$\zeta_b(\alpha) = \frac{1}{\Gamma(\alpha)} \int_{t=0}^{\infty} t^{\alpha-1}(1-e^{-t})^{-1} e^{-t-b/t} dt, \quad (b > 0;\ b = 0,\ \sigma > 1). \tag{6}$$

It avoids the singularity of the integrand at $t = 0$ for $b > 0$ as it is exponentially suppressed, see Fig 1. As such, it allows us to continue this function into the domain $\sigma < 1$. Clearly, in the limit $b \to 0$ we recover the original zeta function for $\sigma > 1$.

Theorem 1.

$$\zeta_b(\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{n=1}^{\infty} \Gamma_{nb}(\alpha) n^{-\alpha}, \quad (b > 0;\ b = 0,\ \sigma > 1). \tag{7}$$

Proof. Expanding $(1-e^{-t})^{-1}$ in (6) as a power series in e^{-t}, we obtain

$$\zeta_b(\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{n=1}^{\infty} \int_{t=0}^{\infty} t^{\alpha-1} e^{-nt-b/t} dt. \tag{8}$$

Rescaling the integration variable by n^{-1}, we prove the theorem. Clearly, as $\Gamma_0(\alpha) = \Gamma(\alpha)$, setting $b = 0$ in (7) yields (3).

Corollary. *Using the reflection property of the generalized gamma function [2]*

$$\Gamma_b(-\alpha) = b^{-\alpha}\Gamma_b(\alpha), \quad (b > 0), \tag{9}$$

we immediately obtain the relationship

$$\sum_{n=1}^{\infty} \Gamma_{nb}(\alpha) = b^{\alpha}\Gamma(-\alpha)\zeta_b(-\alpha), \quad (b > 0). \tag{10}$$

Further, since [2]

$$\Gamma_b(\alpha) = 2b^{\alpha/2} K_\alpha(2\sqrt{b}), \quad (b > 0), \tag{11}$$

we can relate the extended zeta function to the Macdonald function by

$$\zeta_b(\alpha) = \frac{2b^{\alpha/2}}{\Gamma(\alpha)} \sum_{n=1}^{\infty} n^{-\alpha/2} K_\alpha(2\sqrt{nb}), \quad (b > 0). \tag{12}$$

Theorem 2.

$$\zeta_b(\alpha)\Gamma(\alpha) = 2^{\alpha-1} \int_{t=0}^{\infty} t^{\alpha-1} e^{-t-b/2t} \text{csch }(t) dt, \quad (b > 0;\ b = 0, \sigma > 1). \tag{13}$$

Proof. This result follows by changing t to $2t$ in (6) and using the fact that

$$(e^{2t} - 1)^{-1} = e^{-t} \text{csch }(t)/2.$$

Putting $b = 0$ gives

$$\zeta(\alpha)\Gamma(\alpha) = 2^{\alpha-1} \int_{t=0}^{\infty} t^{\alpha-1} e^{-t} \text{csch }(t) dt, \tag{14}$$

which is a standard result for Riemann's zeta function [5, p. 32(4)].

Theorem 3.

$$\zeta_b(\alpha) - 2^{-\alpha}\zeta_{2b}(\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{n=1}^{\infty} \frac{\Gamma_{(2n-1)b}(\alpha)}{(2n-1)^\alpha}, \quad (b > 0; b = 0, \sigma > 1). \tag{15}$$

Proof. Expanding out the left-hand side as a summation of the generalized gamma function, we have

$$\text{LHS} = \frac{1}{\Gamma(\alpha)} \left[\sum_{n=1}^{\infty} \frac{\Gamma_{nb}(\alpha)}{n^\alpha} - \frac{1}{2^\alpha} \sum_{n=1}^{\infty} \frac{\Gamma_{n(2b)}(\alpha)}{n^\alpha} \right]. \tag{16}$$

We can combine 2^α with n^α to obtain $(2n)^\alpha$ and express $\Gamma_{2(nb)}$ as $\Gamma_{(2n)b}$. Thus the right-hand side of (16) becomes the difference between the sum over all natural numbers and even natural numbers, which is simply the sum over all odd natural numbers, namely, the right-hand side of (15). This proves the result.

Corollary.

$$\sum_{n=1}^{\infty}(2n-1)^{-\alpha} = (1-2^{-\alpha})\zeta(\alpha), \quad (\sigma > 1). \tag{17}$$

Proof. This standard result for the zeta function [5, p. 32(3)] is directly obtained by setting $b = 0$ in (15).

Theorem 4.

$$\zeta_b(\alpha) - 2^{1-\alpha}\zeta_{2b}(\alpha) = \frac{1}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\Gamma_{nb}(\alpha)}{n^{\alpha}}, \quad (b > 0; b = 0, \sigma > 0). \tag{18}$$

Proof. Expanding the right-hand side of (18), we get a summation over odd natural numbers minus that over even natural numbers. Thus,

$$\text{RHS} = \frac{1}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\frac{\Gamma_{(2n-1)b}(\alpha)}{(2n-1)^{\alpha}} - \frac{2^{-\alpha}}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\frac{\Gamma_{n(2b)}(\alpha)}{n^{\alpha}}. \tag{19}$$

Now using (7), with b replaced by $2b$, and (15), we see that

$$\text{RHS} = \zeta_b(\alpha) - 2^{-\alpha}\zeta_{2b}(\alpha) - 2^{-\alpha}\zeta_{2b}(\alpha). \tag{20}$$

Simplifying the right-hand side in (20), we get the left-hand side of (18), thus proving the result.

Corollary.

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\alpha} = (1-2^{1-\alpha})\zeta(\alpha), \quad (\sigma > 0). \tag{21}$$

Proof. This standard result, [11, p. 21], is directly obtained by setting $b = 0$ in (18).

Theorem 5.

$$\zeta_b(\alpha) - 2^{1-\alpha}\zeta_{2b}(\alpha) = \frac{1}{\Gamma(\alpha)}\int_{t=0}^{\infty}t^{\alpha-1}(1+e^{-t})^{-1}e^{-t-b/t}dt, \quad (b > 0; b = 0, \sigma > 0). \tag{22}$$

Proof. Using the definition of generalized gamma function, given by (5), and rescaling the variable of integration and b by n, we get

$$n^{-\alpha}\Gamma_{nb}(\alpha) = \int_{t=0}^{\infty}t^{\alpha-1}e^{-nt-b/t}dt. \tag{23}$$

Whence, multiplying by $(-1)^{n-1}$ and summing both sides over positive natural numbers n, we obtain

$$\sum_{n=1}^{\infty}(-1)^{n-1}\Gamma_{nb}(\alpha)n^{-\alpha} = \int_{t=0}^{\infty} t^{\alpha-1}(1+e^{-t})^{-1}e^{-t-b/t}dt. \tag{24}$$

Using (24) and the previous theorem, we arrive at (22). The above argument directly generalizes to give

$$\zeta_b(\alpha) - m^{1-\alpha}\zeta_{mb}(\alpha) = \frac{1}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\Gamma_{bk}(\alpha)}{[n(m-1)-k]^{\alpha}}. \tag{25}$$

Corollary.

$$\zeta_b(\alpha) - 2^{1-\alpha}\zeta_{2b}(\alpha) = \frac{2^{\alpha-1}}{\Gamma(\alpha)}\int_{t=0}^{\infty} t^{\alpha-1}e^{-t-b/2t}\operatorname{sech}(t)dt, \quad (b>0; b=0, \sigma>0). \tag{26}$$

Proof. Scaling the variable of integration in (22) by 2 directly gives the above result as

$$(1+e^{-2t})^{-1} = \frac{1}{2}e^{t}\operatorname{sech}(t).$$

Corollary.

$$(1-2^{1-\alpha})\zeta(\alpha) = \frac{2^{\alpha-1}}{\Gamma(\alpha)}\int_{t=0}^{\infty} t^{\alpha-1}e^{-t}\operatorname{sech}(t)dt, \quad (\sigma>0). \tag{27}$$

Proof. Obtained directly from (26) with $b=0$, is a standard result for the zeta function (see [5, p. 32(6)]).

Corollary

$$\zeta(\alpha) = \frac{1}{\Gamma(\alpha)(1-2^{1-\alpha})}\int_{t=0}^{\infty}\frac{t^{\alpha-1}}{1+e^{t}}dt, \quad (\sigma>0), \tag{28}$$

is another form of (27) (see, [5, p. 32(5)]).

3. Transforms and the Extended Zeta Function

It is of particular interest to find relationships between our extended zeta function and transforms of various functions. This may, alternatively, be thought of as the representation

Proof. In (6), replacing b by $(a+b)$ and expanding $e^{-a/t}$ as an infinite series, we get

$$\zeta_{a+b}(\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \int_{t=0}^{\infty} \frac{t^{(\alpha-n)-1} e^{-t-b/t}}{1-e^{-t}} dt. \tag{35}$$

This equation directly reduces to (34). Notice that we could have interchanged a and b.

There are various special cases of the above result which are of interest. Taking $b=0$, we get

$$\zeta_a(\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \zeta(\alpha-n)\Gamma(\alpha-n), \tag{36}$$

which gives the extended zeta function as an infinite series in terms of the original Riemann zeta function. Taking $a=-b$ gives

$$\zeta(\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{b^n}{n!} \zeta_b(\alpha-n)\Gamma(\alpha-n). \tag{37}$$

Notice that here it does not matter what (non-zero) choice of b is made. Further, taking $a=b$, we get

$$\zeta_{2b}(\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} \zeta_b(\alpha-n)\Gamma(\alpha-n). \tag{38}$$

Theorem 7.

$$\int_{t=0}^{\infty} \zeta_b(\alpha) b^{s-1} db = \frac{\Gamma(s)\Gamma(\alpha+s)}{\Gamma(\alpha)} \zeta(\alpha+s), \quad (s>0,\ \sigma>0). \tag{39}$$

Proof. For two functions $f(t)$ and $g(t)$ [6, p. 308(14)], we have

$$\mathcal{M}\left\{\int_{t=0}^{\infty} t^{\alpha-1} f(t) g(\tau/t) dt; s\right\} = \mathcal{M}\{f(t); s+\alpha\}\mathcal{M}\{g(t); s\}. \tag{40}$$

If $f(t) = (e^t-1)^{-1}$ and $g(t) = e^{-t}$, then, $\mathcal{M}\{g(t); s\} = \Gamma(s)$ and, in view of (32), $\mathcal{M}\{f(t); s+\alpha\} = \Gamma(\alpha+s)\zeta(\alpha+s)$. On using (6), (40) gives

$$\Gamma(\alpha) \int_{\tau=0}^{\infty} \zeta_\tau(\alpha) \tau^{s-1} d\tau = \Gamma(s)\Gamma(\alpha+s)\zeta(\alpha+s). \tag{41}$$

Rearranging and replacing τ by b, we get (39). □

This theorem is the continuous analogue of (37) in the sense that it regards $\zeta_b(\alpha)$ as a component in the decomposition of $\zeta(\alpha)$. However, while there, b could be any fixed arbitrary value, here b is the variable of integration and so summand over itself. Choosing the parameter for the "transform" $s = 1$, we get

$$\begin{aligned}\zeta(\alpha+1) &= \frac{1}{\alpha}\int_{b=0}^{\infty}\zeta_b(\alpha)db \\ &= \frac{1}{\Gamma(\alpha+1)}\int_{t=0}^{\infty}\int_{b=0}^{\infty} t^{\alpha-1}(1-e^{-t})^{-1}e^{-t-b/t}db\,dt. \end{aligned} \tag{42}$$

This equation provides a consistency check in that it is simply the integral of (6).

The above theorem is useful in providing a Fourier transform representation of a complex zeta function in terms of our extended zeta function for a real variable. Denoting the real and imaginary components of the complex variable α by σ and τ and the Fourier transform by $\mathcal{F}$, we get the following:

Theorem 8.

$$\zeta(\sigma+i\tau) = \frac{\Gamma(\sigma)}{\Gamma(\sigma+i\tau)\Gamma(i\tau)}\mathcal{F}\{\zeta_{e^x}(\sigma);\tau\}. \tag{43}$$

Proof. Putting $\alpha = \sigma$ and $s = i\tau$ in (39) and $\ln b = x$ gives the above result directly. □

Corollary

$$\zeta_{e^x}(\sigma) = \frac{1}{2\pi\Gamma(\sigma)}\int_{\tau=-\infty}^{\infty} e^{-i\tau x}\zeta(\sigma+i\tau)\Gamma(\sigma+i\tau)\Gamma(i\tau)d\tau. \tag{44}$$

Proof. This is simply obtained by inverting (43).

4. The Second Extended Zeta Function

It is well known (see [11, p. 30]) that the non-trivial zeros of the Riemann zeta function lie in the strip $0 < \sigma < 1$. Therefore, it is desirable to have an extension of Riemann's zeta function over $0 < \sigma < \infty$. Unfortunately the function $\zeta_b(\alpha)$, though it extends naturally and simply the usual properties of Riemann's zeta function, for $\sigma > 1$ ceases to extend these properties in the critical strip [8]. However, the situation is not totally hopeless. In view of (22) and (28), it seems natural to introduce the second extended zeta function

$$\zeta_b^*(\alpha) = \frac{1}{\Gamma(\alpha)(1-2^{1-\alpha})} \int_{t=0}^{\infty} t^{\alpha-1}(1+e^{-t})^{-1} e^{-t-b/t} dt, \quad (b > 0; b = 0, \sigma > 0), \tag{45}$$

whose graphs are shown in Fig. 2. It can be seen from (28) and (45) that

$$\zeta_0^*(\alpha) = \zeta(\alpha), \quad (\sigma > 0). \tag{46}$$

Moreover, in view of (22) and (45), the extended zeta functions are related via

$$\zeta_b^*(\alpha) = \frac{\zeta_b(\alpha) - 2^{1-\alpha}\zeta_{2b}(\alpha)}{1-2^{1-\alpha}}. \tag{47}$$

We notice from (47) that $\alpha = 1$ is the only simple pole of $\zeta_b^*(\alpha)$ with residue given by

$$\text{Res}\{\zeta_b^*(\alpha); 1\} = (\zeta_b(1) - \zeta_{2b}(1)]/\ln 2. \tag{48}$$

Further, putting $\alpha = 1$ and letting $b \to 0$ in (18), we get

$$\lim_{b\to 0} [\zeta_b(1) - \zeta_{2b}(1)] = \ln 2. \tag{49}$$

Thus, $\text{Res}\{\zeta_b^*(\alpha); 1\} \to 1$ as $b \to 0^+$.

Theorem 9.

$$\int_{b=0}^{\infty} \zeta_b^*(\alpha) b^{s-1} db = \left(\frac{1-2^{1-\alpha-s}}{1-2^{1-\alpha}}\right) \frac{\Gamma(s)\Gamma(\alpha+s)}{\Gamma(\alpha)} \zeta(\alpha+s),$$

$$(\text{Re}(s) > 0, \sigma > 0;\ \text{Re}(s) = 0, \text{Im}(s) \neq 0,\ \sigma > 0). \tag{50}$$

Proof. It is similar to the proof of theorem (7). In particular, the substitution $s = 1$ in (50) yields the interesting identity

$$\int_{b=0}^{\infty} \zeta_b^*(\alpha)db = \left(\frac{2^\alpha - 1}{2^\alpha - 2}\right) \alpha\zeta(1+\alpha). \tag{51}$$

Theorem 10.

$$\zeta_b^*(\alpha) = \frac{1}{\Gamma(\alpha)(1-2^{1-\alpha})} \sum_{n=1}^{\infty} (-1)^{n-1}\Gamma_{nb}(\alpha)n^{-\alpha}, \quad (b > 0; b = 0, \sigma > 0). \tag{52}$$

Proof. Straightforward. In particular, putting $b = 0$ in (52) yields the standard result [11, p. 21] for the zeta function.

5. The Incomplete Zeta Functions and Their Extensions

The generalization of the gamma function originally arose from the incomplete gamma functions defined by

$$\gamma(\alpha, x) = \int_{t=0}^{x} t^{\alpha-1}e^{-t}dt, \quad (\sigma > 0), \tag{53}$$

$$\Gamma(\alpha, x) = \int_{t=x}^{\infty} t^{\alpha-1}e^{-t}dt, \tag{54}$$

so that

$$\gamma(\alpha, x) + \Gamma(\alpha, x) = \Gamma(\alpha). \tag{55}$$

Since we are dealing with integral representations of the zeta function, it is natural to define incomplete zeta functions:

$$z(\alpha, x) = \frac{1}{\Gamma(\alpha)} \int_{t=0}^{x} t^{\alpha-1}(1-e^{-t})^{-1}e^{-t}dt, \quad (\sigma > 1), \tag{56}$$

$$Z(\alpha, x) = \frac{1}{\Gamma(\alpha)} \int_{t=x}^{\infty} t^{\alpha-1}(1-e^{-t})^{-1}e^{-t}dt, \tag{57}$$

so that

$$z(\alpha, x) + Z(\alpha, x) = \zeta(\alpha); \tag{58}$$

and define the extension of these functions

$$z_b(\alpha,x) = \frac{1}{\Gamma(\alpha)}\int_{t=0}^{x} t^{\alpha-1}(1-e^{-t})^{-1}e^{-t-b/t}dt, \quad (b>0; b=0, \sigma>1), \tag{59}$$

$$Z_b(\alpha,x) = \frac{1}{\Gamma(\alpha)}\int_{t=x}^{\infty} t^{\alpha-1}(1-e^{-t})^{-1}e^{-t-b/t}dt. \tag{60}$$

Clearly, $z_b(\alpha,x)+Z_b(\alpha,x)=\zeta_b(\alpha)$ and $z_b(\alpha,\infty)=Z_b(\alpha,0)=\zeta_b(\alpha)$. Similarly, we can define $z_b^*(\alpha,x)$ and $Z_b^*(\alpha,x)$. Results analogous to those for the extended zeta function hold also for the incomplete versions.

Theorem 11.

$$Z_b(\alpha,x) = \frac{1}{\Gamma(\alpha)}\sum_{n=1}^{\infty} n^{-\alpha}\Gamma(\alpha,nx;nb), \tag{61}$$

$$Z_b^*(\alpha,x) = \frac{1}{\Gamma(\alpha)(1-2^{1-\alpha})}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\alpha}\Gamma(\alpha,nx;nb), \tag{62}$$

where $\Gamma(\alpha,x;b)$ is the generalized incomplete gamma function [1].

Proof. Straightforward.

Theorem 12.

$$Z(\alpha+1,x)\Gamma(\alpha+1)=\alpha\sum_{n=1}^{\infty}n^{-\alpha-1}\Gamma(\alpha,nx)+x^{\alpha}\ln\left(\frac{e^x}{e^x-1}\right). \tag{63}$$

Proof. From [7, p. (8.356)(2)], we have

$$\Gamma(\alpha+1,nx)=n^{\alpha}x^{\alpha}e^{-nx}+\alpha\Gamma(\alpha,nx). \tag{64}$$

Therefore,

$$Z(\alpha+1,x)\Gamma(\alpha+1)=\alpha\sum_{n=1}^{\infty}n^{-\alpha-1}\Gamma(\alpha,nx)+x^{\alpha}\sum_{n=1}^{\infty}\frac{e^{-nx}}{n}. \tag{65}$$

Using the series representation for $\ln(1-e^{-x})$, we obtain (63).

6. Extensions of the Generalized Zeta Function

Of more interest than the incomplete zeta functions is the standard generalized zeta function, originally defined by [11, p. 36]

$$\zeta(\alpha, q) = \sum_{n=0}^{\infty}(n+q)^{-\alpha}, \quad (\sigma > 1, 0 < q \leq 1), \tag{66}$$

so that $\zeta(\alpha, 1) = \zeta(\alpha)$. It has the integral representation [11, p. 37]

$$\zeta(\alpha, q) = \frac{1}{\Gamma(\alpha)} \int_{t=0}^{\infty} t^{\alpha-1}(1-e^{-t})^{-1}e^{-qt}dt, \quad (\sigma > 1, 0 < q \leq 1). \tag{67}$$

Our extended generalized functions are, therefore,

$$\zeta_b(\alpha, q) = \frac{1}{\Gamma(\alpha)} \int_{t=0}^{\infty} t^{\alpha-1}(1-e^{-t})^{-1}e^{-qt-b/t}dt, \quad (b > 0; b = 0, \sigma > 1, 0 < q \leq 1), \tag{68}$$

and

$$\zeta_b^*(\alpha, q) = \frac{1}{\Gamma(\alpha)(1-2^{1-\alpha})} \int_{t=0}^{\infty} t^{\alpha-1}(1+e^{-t})^{-1}e^{-qt-b/t}dt, \quad (b > 0; b = 0, \sigma > 0, 0 < q \leq 1). \tag{69}$$

For completeness we mention the extended generalized incomplete zeta functions, $z_b(\alpha, q; x)$, $Z_b(\alpha, q; x)$ and $z_b^*(\alpha, q; x)$ and $Z_b^*(\alpha, q; x)$, defined by taking the appropriate limits of integration in (67) and (68).

Theorem 13.

$$\zeta_b^*(\alpha, q) = \frac{2^{1-\alpha}\zeta_{2b}(\alpha, q/2) - \zeta_b(\alpha, q)}{1-2^{1-\alpha}}. \tag{70}$$

Proof. Replacing b by $2b$ and q by $q/2$ in (68) and changing the variable of integration from t to $2t$, we get an expression for $\zeta_{2b}(\alpha, q/2)$ which has an extra factor of $2^{\alpha-1}$ in it. Multiplying this expression by the inverse of this factor and subtracting (68) from it, we find, in the integrand, the difference

$$(1-e^{-2t})^{-1} - (1-e^{-t})^{-1} = (1+e^{-t})^{-1}, \tag{71}$$

which gives the integrand in (69). However, the extra factor $(1-2^{1-\alpha})$ in (69) is not present in the expression. Dividing by this factor we obtain (70).

7. Hurwitz Type Theorems for Extended Generalized Zeta Functions

For the usual zeta function, in the domain $\sigma < 0$, the Hurwitz formula

$$\zeta(\alpha,q) = \frac{2\Gamma(1-\alpha)}{(2\pi)^{1-\alpha}}\left\{\sin\left(\frac{1}{2}\alpha\pi\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi qn)}{n^{1-\alpha}} + \cos\left(\frac{1}{2}\alpha\pi\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi qn)}{n^{1-\alpha}}\right\}, \tag{72}$$

applies. The Riemann functional equation

$$2^{1-\alpha}\Gamma(\alpha)\zeta(\alpha)\cos\left(\frac{1}{2}\alpha\pi\right) = \pi^{\alpha}\zeta(1-\alpha), \tag{73}$$

which may be obtained by putting $q=1$ in (72) is in fact more generally valid.

In this section we prove Hurwitz type formulas for the extended generalized zeta functions and deduce analogues of (72) and (73) as special cases.

Theorem 14. *For* $0 < \sigma, q < 1, b > 0$,

$$\zeta_b(\sigma,q) = \frac{2\Gamma(1-\sigma)}{(2\pi)^{1-\sigma}}\sum_{n=1}^{\infty} n^{\sigma-1}\sin\left(2\pi qn + \frac{\sigma\pi}{2} - \frac{b}{2n\pi}\right). \tag{74}$$

Proof. To evaluate the zeta function from (68) we need to convert to the complex plane and determine the corresponding contour integral of the function

$$f(z) = \frac{z^{\sigma-1}e^{-qz-b/z}}{1-e^{-z}}. \tag{75}$$

The contour needs to exclude the origin where there is an essential singularity and essentially stay on the positive half plane. For this purpose we take a contour consisting of two parts, one on the upper quarter plane and one on the lower, $C_n = C_n^+ \cup C_n^-$, see Fig. 3. Note that $f(z)$ has simple poles at $z = \pm 2\pi ki$, for non-zero natural numbers k. We take C_n^+ to start at some $(\sqrt{R^2-\epsilon^2}, \epsilon)$ and go *inwards* to $(\sqrt{\delta^2-\epsilon^2}, \epsilon)$ for some appropriately chosen δ, ϵ such that $R > 2\pi + \delta$, $\pi > \delta > \epsilon$. Now it goes in a nearly quarter circle, in a counter-cockwise direction, to $(0, \delta)$ and then straight up till it reaches $(0, 2\pi - \delta)$. It then goes in a semi-circle in a clockwise direction to $(0, 2\pi + \delta)$ and continues to surround the poles till it reaches $(0, R)$. The number of poles enclosed on the positive side is $n = [R/2\pi]$, (and $\delta < R - 2n\pi$). The contour C_n^- is similarly given with a reverse orientation. The sum of the residues for the k–th poles is

$$\begin{aligned}
&\text{Res}\{f, 2k\pi i\} + \text{ Res } \{f; -2k\pi i\} \\
&= (2k\pi)^{\sigma-1}\left[e^{i\{(\sigma-1)\frac{\pi}{2}-2kq\pi+\frac{b}{2k\pi}\}} + e^{i\{(\sigma-1)\frac{3\pi}{2}+2kq\pi-\frac{b}{2k\pi}\}}\right] \\
&= -2(2k\pi)^{\sigma-1}e^{i\sigma\pi}\sin\left(2k\pi q + \frac{\sigma\pi}{2} - \frac{b}{2k\pi}\right). \qquad (76)
\end{aligned}$$

In view of Lemma 1 in the appendix, and by the residue theorem, we get

$$\int_{C_n} f(z)dz = 2(2\pi i)e^{i\sigma\pi}\sum_{k=1}^{n}\sin\left(2k\pi q + \frac{\sigma\pi}{2} - \frac{b}{2k\pi}\right). \qquad (77)$$

We define our contour $C = \lim_{n\to\infty} C_n$. Clearly, as $n \to \infty$, $R \to \infty$. In this limit we obtain

$$\left(1 - e^{2\pi q\sigma i}\right)\int_0^\infty \frac{t^{\sigma-1}e^{-qt-b/t}}{1-e^{-t}}dt = -2(2\pi i)e^{i\sigma\pi}\sum_{k=1}^{\infty}(2k\pi)^{\sigma-1}\sin\left(2k\pi q + \frac{\sigma\pi}{2} - \frac{b}{2k\pi}\right). \qquad (78)$$

However,

$$e^{2\pi\sigma i} - 1 = 2\pi i\frac{e^{\pi\sigma i}}{\Gamma(\sigma)\Gamma(1-\sigma)}. \qquad (79)$$

From (78) and (79), we arrive at (74). This proves the result.

Remark. The series on the right hand side of (74) converges uniformly by the Dirichlet test for $0 < q_0 \le q \le q_1 < 1$, $b \ge 0$. Therefore the sum of this series is continuous for $b \ge 0$ and the function $\zeta_b(\sigma, q)$ can be extended by continuity at $b = 0$ for $0 < \sigma, \; q < 1$.

Theorem 15. *For* $0 < \sigma, q < 1, b > 0; -1 < \sigma < 0, 0 < q \leq 1, b \geq 0$

$$\zeta_b^*(\sigma, q) = \frac{2\pi^{\sigma-1}\Gamma(1-\sigma)}{1-2^{1-\sigma}} \sum_{k=0}^{\infty} (2k+1)^{\sigma-1} \sin\left((2k+1)\pi q + \frac{\sigma\pi}{2} - \frac{b}{(2k+1)\pi}\right). \tag{80}$$

Proof. The function $f(z) = z^{\sigma-1}\frac{e^{-qz-b/z}}{1+e^{-z}}$ has simple poles at $z = \pm(2k+1)\pi i$ where k is a natural number. The sum of the residues at these poles is

$$\begin{aligned}
&\text{Res}\{f; (2k+1)\pi i\} + \text{Res}\ \{f; -(2k+1)\pi i\} \\
&= [(2k+1)\pi]^{\sigma-1}\left[e^{i\left\{(\sigma-1)\frac{\pi}{2}-(2k+1)\pi q+\frac{b}{(2k+1)\pi}\right\}} + e^{i\left\{(\sigma-1)\frac{3\pi}{2}+(2k+1)\pi q-\frac{b}{(2k+1)\pi}\right\}}\right] \\
&= -2\pi^{\sigma-1}e^{i\sigma\pi}(2k+1))^{\sigma-1}\sin\left((2k+1)\pi q + \frac{\sigma\pi}{2} - \frac{b}{(2k+1)\pi}\right)
\end{aligned} \tag{81}$$

Using Lemma 2 in the appendix and by the residue theorem, we find that

$$(1-e^{2\pi\sigma i})\int_{t=0}^{\infty} \frac{t^{\sigma-1}e^{-qt-b/t}}{1+e^{-t}}dt = -2\pi i(2\pi^{\sigma-1})e^{i\sigma\pi}\sum_{k=1}^{\infty}(2k+1)^{\sigma-1}\sin\left((2k+1)q\pi + \frac{\sigma\pi}{2} - \frac{b}{(2k+1)\pi}\right). \tag{82}$$

From (79) and (82), we get

$$\int_{t=0}^{\infty} \frac{t^{\sigma-1}e^{-qt-b/t}}{1+e^{-t}}dt = 2\pi^{\sigma-1}\Gamma(\sigma)\Gamma(1-\sigma)\sum_{k=1}^{\infty}(2k+1)^{\sigma-1}\sin\left((2k+1)q\pi + \frac{\sigma\pi}{2} - \frac{b}{(2k+1)\pi}\right). \tag{83}$$

From (45) and (83), we arrive at (80). Now consider the series on the right hand side of (83), obtained for $b > 0$. Again by the Dirichlet test, we have uniform convergence for $0 < q_0 \leq q \leq q_1 < 1, \quad b \geq 0$. Therefore the sum of this series is continuous up to $b = 0$ for $0 < \sigma, \quad q < 1$. Moreover by considering the integral representation of the left hand side of (83), we are allowed to take the limit $b \to 0$. Therefore (83) is valid for $0 < \sigma, \quad q < 1$, $b \geq 0$. The same formula can be extended further when $-1 < \sigma < 0$. We have absolute and uniform convergence of the series for $0 < q \leq 1, \quad b \geq 0$. Also by considering the integral representation of the function $\zeta_b^*(\sigma, q)$ we obtain a formula valid for $0 < q \leq 1, \quad b \geq 0$. In particular we recover the Hurwitz formula (72) for $b = 0$.

Remark. Note that the above arguments for the Hurwitz type formulae remain valid for complex values of α by the principle of analytic continuation.

8. Discussion and Conclusion

We have seen that the generalization of the incomplete and complete gamma functions [1, 2] and the beta function [3] is useful for extending the zeta function. However, the extension does not continue to match the zeta function for all ranges of the real part of the argument. The first extension carries on beyond the singularity (at $\sigma = 1$) on the positive side, while the zeta function is negative (for $0 < \sigma < 1$). Since the extension was designed to be non-singular at $\sigma = 1$ it could not match the zeta function beyond that value.

To extend into the range $0 < \sigma < 1$, which is of special interest on account of Riemann's hypothesis (that the non-trivial zeros of his zeta function lie on the line $\sigma = 1/2$), we started with an integral representation of the zeta function valid for $0 < \sigma < \infty$ and used the "regularizer" to extend it. The singularity at $\sigma = 0$ (for a real argument) was retained by a factor outside the integral. This second extension carries many of the previous properties into the strip $0 < \sigma < 1$ and has elegant relations to the zeta function itself and its first extension.

There are infinitely many possible extensions of the zeta function. Of course *some* results analogous to those for the zeta and generalized zeta functions would hold for any extension. What we require is that the results be *naturally* and *simply* extended. This criterion is met by both extensions. It is expected that such natural extensions would turn out to be useful. We also defined some incomplete zeta functions, analogous to the incomplete gamma functions, whose properties would bear investigation.

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APPENDIX

Lemmas associated with the Hurwitz type Theorems:

Lemma 1. Let

$$f(z) = z^{\sigma-1}\frac{e^{-qz-b/z}}{1-e^{-z}}, \qquad \text{where } 0 < \sigma < 1, \quad 0 < q \leq 1, \quad b > 0$$

and

$$\begin{aligned}
&\gamma_0^+ : z = \epsilon e^{i\theta} && 0 \leq \theta \leq \tfrac{\pi}{2}, \\
&\gamma_0^- : z = \epsilon e^{i\theta} && -\tfrac{\pi}{2} \leq \theta \leq 0, \\
&C_n^+ : z = (2n+1)\pi e^{i\theta} && 0 \leq \theta \leq \tfrac{\pi}{2}, \\
&C_n^- : z = (2n+1)\pi e^{i\theta} && -\tfrac{\pi}{2} \leq \theta \leq 0, \\
&\ell_k^+ : z = i\ell && 2\pi k + \delta \leq \ell \leq 2\pi(k+1) - \delta \text{ for some small } \delta > 0, \\
&\ell_k^- : z = -i\ell,
\end{aligned}$$

then

$$\begin{aligned}
&\int_{\gamma_0^+} f(z)dz, \int_{\gamma_0^-} f(z)dz \to 0 \text{ as } \epsilon \to 0, \\
&\int_{C_n^+} f(z)dz, \int_{C_n^-} f(z)dz \to 0 \text{ as } n \to \infty, \\
&\int_{\ell_k^-} f(z)dz + \int_{\ell_k^+} f(z)dz = 0, \quad k = 1, 2, \ldots .
\end{aligned}$$

Proof. Since

$$\left|\frac{z}{1-e^{-z}}\right| \leq K.$$

Hence

$$\left|\int_{\gamma_0^+} f(z)dz\right| \leq K\epsilon^{\sigma-1}\int_{\theta=0}^{\pi/2} e^{-\frac{b}{\epsilon}\cos\theta}d\theta.$$

But

$$\cos\theta \geq \frac{2}{\pi}\left(\frac{\pi}{2}-\theta\right) \text{ for } 0 \leq \frac{\pi}{2}-\theta \leq \frac{\pi}{2}.$$

Thus,

$$
\begin{aligned}
K\epsilon^{\sigma-1}\int_0^{\pi/2} e^{\frac{-b}{\epsilon}\cos\theta}d\theta &\le K\epsilon^{\sigma-1}\int_{\theta=0}^{\pi/2} e^{-\frac{2b}{\pi\epsilon}\left(\frac{\pi}{2}-\theta\right)}d\theta \\
&= K\epsilon^{\sigma-1}\int_{\theta=0}^{\pi/2} e^{-\frac{2b}{\pi\epsilon}\theta}d\theta \\
&= K\epsilon^{\sigma}\frac{\pi}{2b}(1-e^{-b/\epsilon}) \\
&< \frac{K\pi}{2b}\epsilon^{\sigma}.
\end{aligned}
$$

Therefore

$$
\left|\int_{\gamma_0^+} f(z)dz\right| \to 0 \text{ as } \epsilon \to 0.
$$

Similarly

$$
\left|\int_{\gamma_0^-} f(z)dz\right| < \frac{K\pi}{2b}\epsilon^{\sigma} \to 0 \text{ as } \epsilon \to 0.
$$

On the upper quarter circle C_n^+

$$
\begin{aligned}
|1-e^{-z}|^{-1} &\le M, \\
\left|z^{\sigma-1}\frac{e^{-qz-b/z}}{1-e^{-z}}\right| &\le MR^{\sigma-1}e^{-\left(qR+\frac{b}{R}\right)\cos\theta}, \\
&\le MR^{\sigma-1}e^{-\frac{2}{\pi}\left(qR+\frac{b}{R}\right)\left(\frac{\pi}{2}-\theta\right)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left|\int_{C_m^+} f(z)dz\right| &\le MR^{\sigma}\int_{\theta=0}^{\pi/2} e^{-\frac{2}{\pi}\left(qR+\frac{b}{R}\right)\theta}d\theta \\
&< \frac{M\pi}{2q}\left(1+\frac{b}{qR^2}\right)^{-1}R^{\sigma-1} \to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Similarly

$$
\left|\int_{C_n^-} f(z)dz\right| \to 0 \text{ as } n \to \infty.
$$

Lemma 2. Let

$$
f(z) = z^{\sigma-1}\frac{e^{-qz-b/z}}{1+e^{-z}} \text{ where } |\sigma|<1,\ 0<q\le 1,\ \ b>0,
$$

and

γ_0^+, γ_0^- : as in Lemma 1,

C_n^+ : $z = (2n\pi)e^{i\theta}, 0 \le \theta \le \frac{\pi}{2}$,

C_n^- : $z = (2n\pi)e^{i\theta}, -\frac{\pi}{2} \le \theta \le 0$,

ℓ_k^+ : $z = i\ell, (2k-1)\pi + \delta \le \ell \le (2k+1)\pi - \delta$ for small $\delta > 0$,

ℓ_k^- : $z = -i\ell$,

then

$$\int_{\gamma_0^+} f(z)dz, \int_{\gamma_0^-} f(z)dz \to 0 \text{ as } \epsilon \to 0,$$
$$\int_{C_n^+} f(z)dz, \int_{C_n^-} f(z)dz \to 0 \text{ as } n \to \infty,$$
$$\int_{\ell_k^-} f(z)dz + \int_{\ell_k^+} f(z)dz = 0 \quad k = 0, 1, \ldots .$$

Proof. There is a significant change in this case in our estimate along γ_0^+, γ_0^-. But the rest of the proof is quite similar.

$$\begin{aligned} |1 + e^{-z}|^{-1} &\le K, \\ \left| \int_{\gamma_0^+} z^{\sigma-1} \frac{e^{-qz-b/z}}{1+e^{-z}} dz \right| &\le K\epsilon^\sigma \int_{\theta=0}^{\pi/2} e^{-\frac{b}{\epsilon}\cos\theta} d\theta, \\ &< \frac{K\pi}{2b}\epsilon^{\sigma+1} \to \text{ as } \epsilon \to 0. \end{aligned}$$

Similarly,

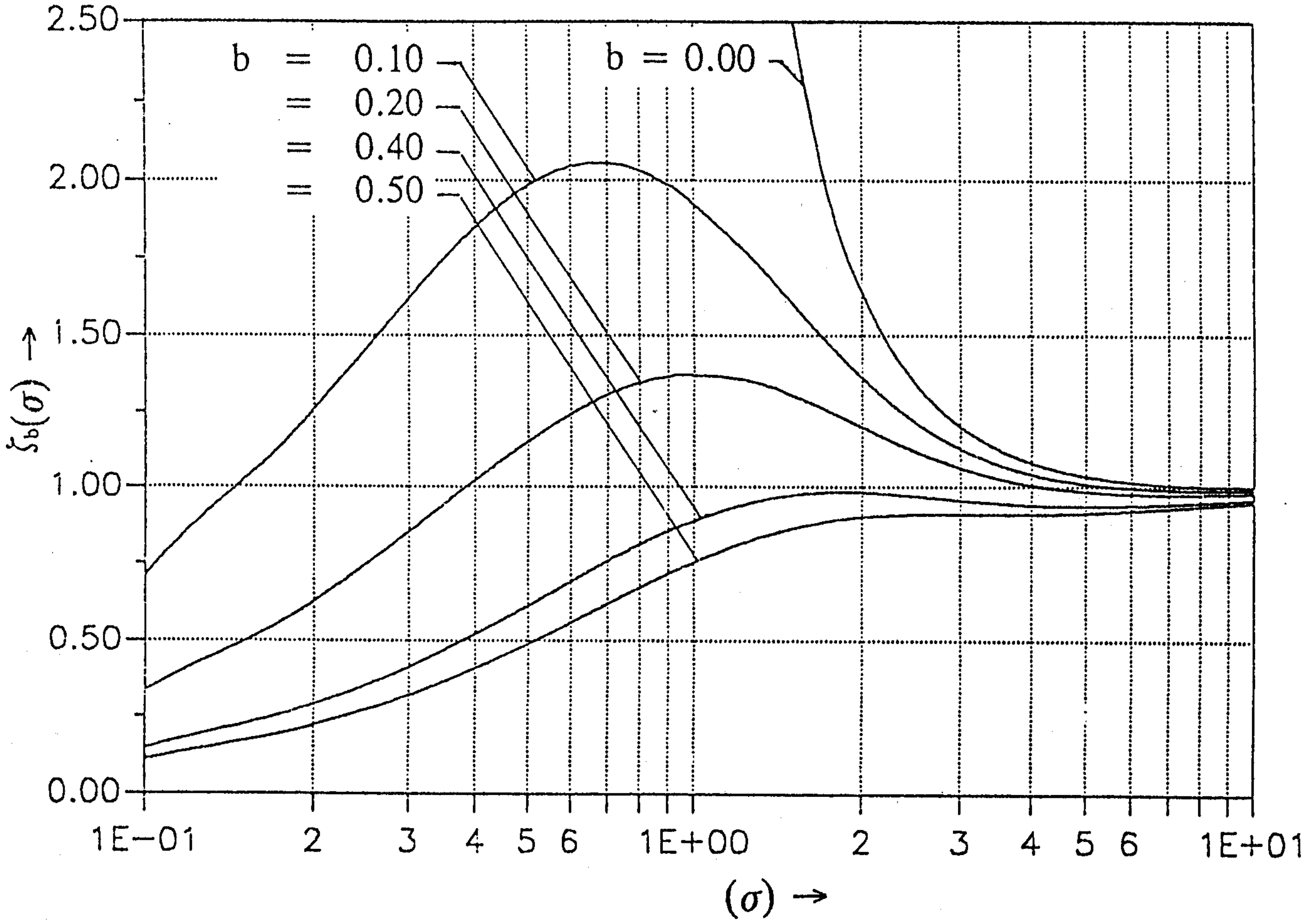
$$\left| \int_{\gamma_0^-} f(z)dz \right| \to 0 \text{ as } \epsilon \to 0.$$

Figure Captions

Fig. 1: The graph of $\zeta_b(\sigma)$ for $b = 0$ (the Riemann zeta function for $\sigma > 1$), $b = 0.1, 0.2, 0.4, 0.5$. Notice how increasing b "pulls down" the function. The graph was obtained by the numerical integration subroutine QDAGI [13].

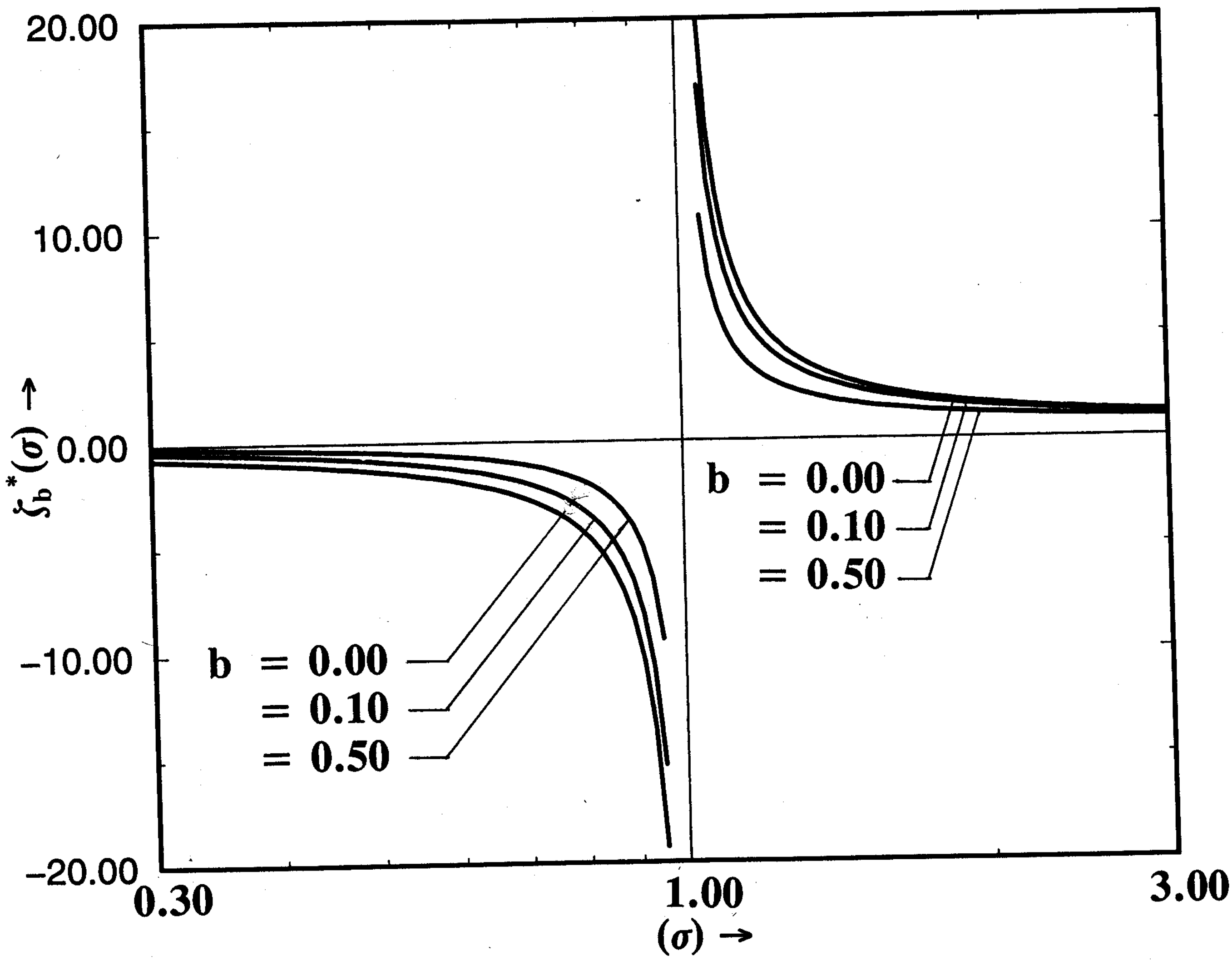
Fig. 2: The graph of $\zeta_b^*(\sigma)$ for $b = 0$ (the Riemann zeta function for $\sigma > 0$), $b = 0.1$ and 0.5. Note how increasing b "depresses" the function closer to the axes. The graph was obtained by the numerical integration subroutine QDAGI [13].

Fig. 3: The contour for integration, C, is the limit of $C_n = C_n^+ \cup C_n^-$ which avoids the essential singularity at $z = 0$ and surrounds all the simple poles on the upper half of the imaginary line (C_n^+) and the lower half (C_n^-).



2.50
2.00
1.50
1.00
0.50
0.00
b = 0.10
= 0.20
= 0.40
= 0.50
b = 0.00
ζb(σ) →
1E−01
2
3
4
5
6
1E+00
2
3
4
5
6
1E+01
(σ) →

540-2



20.00
10.00
0.00
−10.00
−20.00
$\zeta_b^*(\sigma)$ →
b = 0.00
= 0.10
= 0.50
b = 0.00
= 0.10
= 0.50
0.30
1.00
3.00
(σ) →

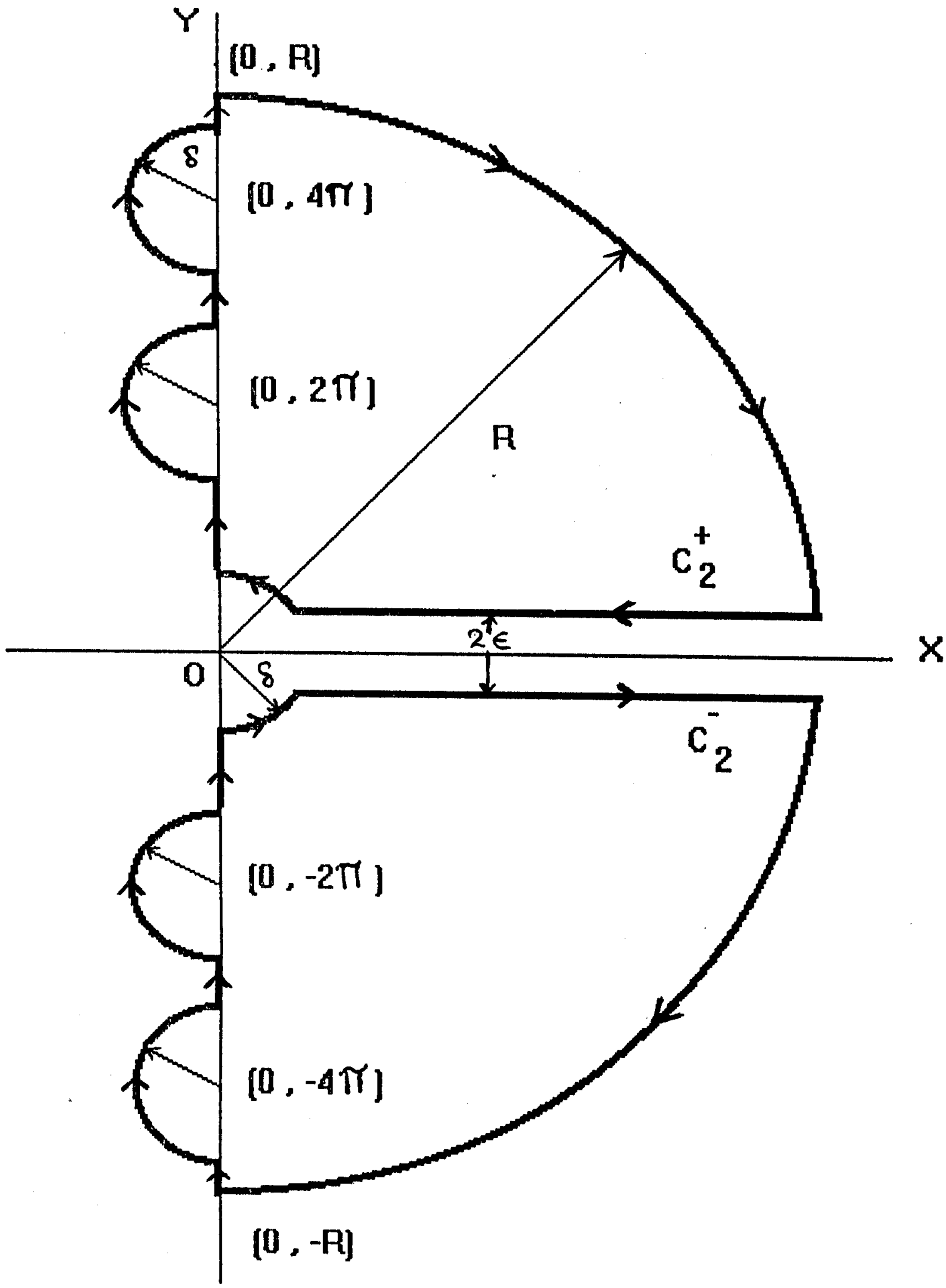


Fig. 3