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M. A. Chaudhary, Ashgar Qadir

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

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Proof of Riemann's Hypothesis

M. Aslam Chaudhry and Asghar Qadir [1]

Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia

Abstract

Results for two extensions of Riemann's zeta function are used to prove Riemann's hypothesis. The proof is also discussed in geometric terms.

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1. Introduction

In the process of studying the properties of his zeta function [10], Riemann conjectured that $\zeta(\sigma + i\tau) = 0$ implies that $\sigma = 1/2$ in the positive half-plane, i.e. in the complex plane of $\alpha = \sigma + i\tau$, the non-trivial zeros of the zeta function lie on the critical line $\sigma = 1/2$. Two important results were obtained [10] in this regard: (a) that the zeros *do* lie in the critical strip $0 < \sigma < 1$; and (b) that if α_1 is a zero of $\zeta(\alpha)$ so is $\alpha_2 = 1 - \alpha_1$. Though the hypothesis has been repeatedly verified [9] and the critical strip has been narrowed [7, 10], an analytical proof is still needed [8, 9], particularly in view of the large body of theory and applications [7, 10] based on this conjecture.

Following on the lines of a generalization of the complete and incomplete gamma functions by the introduction of a regularizing factor [2, 3, 5] which found many applications [3, 4] and a

[1] On leave of absence from the Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.

subsequent similar extension of the beta function [1], two extensions of the zeta function have been proposed and their properties investigated [6]. In this paper we use those extensions (and their property) to prove Riemann's hypothesis. Before proceeding to do so we represent the key points of earlier attempts (for our purposes) geometrically, so that we can discuss our proof in those terms.

We can regard $\zeta(\alpha)$ as a surface lying over the complex plane and restrict our attention to the critical strip. If $\zeta(\alpha)$ has a zero at $\alpha = \alpha_1$ the surface will touch the critical strip at α_1. It will then also touch at $\alpha = \alpha_2 = 1 - \alpha$, see Fig. 1. The problem in proving Riemann's hypothesis is that this constraint is inadequate to ensure that the contributions from the imaginary components of α_1 and α_2 do not arrange themselves to "cancel out" the shift of the real part from the critical line. Since $\zeta(\alpha)$ is a complex analytic function in the critical strip, $\alpha_3 = \overline{\alpha}_1$ and $\alpha_4 = \overline{\alpha}_2 = 1 - \overline{\alpha}_1$ are also zeros of $\zeta(\alpha)$. Thus we have zeros appearing in sets of 4 as shown in Fig. 1.

2. Proof of the Conjecture

To be able to state our proof we need to give the definitions (and a property) of two extensions of Riemann's zeta function. The reasoning behind the introduction of these extensions (and the proof of the property) is given elsewhere [6].

The (first) *extended zeta function*, $\zeta_b(\alpha)$, is defined by

$$\zeta_b(\alpha) = \frac{1}{\Gamma(\alpha)} \int_{t=0}^{\infty} t^{\alpha-1}(e^t - 1)^{-1} e^{-b/t} dt, \qquad (b > 0; b = 0, \sigma > 1), \tag{1}$$

which exactly matches $\zeta(\alpha)$ for $b = 0$ in the domain $\sigma > 1$. However, it does not extend the zeta function into the critical strip, as is obvious from the fact that $\zeta(\sigma) < 0$ and $\zeta_b(\sigma) > 0$ for $0 < \sigma < 1$. To extend into the critical strip we define the *second extension*,

$$\zeta_b^*(\alpha) = \frac{1}{\Gamma(\alpha)} \frac{1}{(1 - 2^{1-\alpha})} \int_{t=0}^{\infty} t^{\alpha-1}(e^t + 1)^{-1} e^{-b/t} dt, \qquad (b > 0; b = 0, \sigma > 0), \tag{2}$$

which exactly matches $\zeta(\alpha)$ for $b = 0$ in the domain $\sigma > 0$, and is related to the first extension by [6]

$$\zeta_b^*(\alpha) = \frac{\zeta_b(\alpha) - 2^{1-\alpha}\zeta_{2b}(\alpha)}{1 - 2^{1-\alpha}}, \qquad (b > 0; b = 0, \sigma > 0). \tag{3}$$

Theorem. *The non-trivial zeros of the zeta function lie on the critical line, $\sigma = 1/2$.*

Proof. The integral in (2) is uniformly convergent for any $\sigma \in (0, \infty)$ and $b \geq 0$ (the singularity of the ζ or ζ_b^* at $\alpha = 1$ coming from the factor outside the integral). Hence the limit of $\zeta_b^*(\alpha)$ as $b \to 0^+$ is $\zeta(\alpha)$. Remember that we are limiting our attention to the critical strip $0 < \sigma < 1$, where we have $\zeta_b^*(\alpha) \to \zeta(\alpha)$ uniformly as $b \to 0^+$. Moreover, if $\alpha_1 = \sigma_1 + i\tau_1$ is a zero of $\zeta(\alpha)$, so will be $\alpha_2 = \overline{\alpha}_1 = \sigma_1 - i\tau_1$ (as ζ is analytic) and $\alpha_3 = 1 - \alpha_1 = (1 - \sigma_1) - i\tau_1$, [10] and hence $\alpha_4 = \overline{\alpha}_3 = (1 - \sigma_1) + i\tau_1$. Thus, if $\alpha_1 = \sigma_1 + i\tau_1$ (and hence $1 - \overline{\alpha}_1$) is a zero of $\zeta(\alpha)$, we must have $\zeta_b^*(\alpha_1) \to 0$ and $\zeta_b^*(1 - \overline{\alpha}_1) \to 0$ as $b \to 0^+$. But then it follows from (3) that as $b \to 0^+$

$$\zeta_b(\alpha_1) \sim 2^{1-\alpha_1}\zeta_{2b}(\alpha_1), \tag{4}$$

$$\zeta_b(1 - \overline{\alpha}_1) \sim 2^{\overline{\alpha}_1}\zeta_{2b}(1 - \overline{\alpha}_1). \tag{5}$$

Dividing (4) by (5), as $\zeta_b(\alpha) \neq 0$ in the critical strip, we see that

$$\frac{\zeta_b(\alpha_1)}{\zeta_{2b}(\alpha_1)} \cdot \frac{\zeta_{2b}(1 - \overline{\alpha}_1)}{\zeta_b(1 - \overline{\alpha}_1)} \sim 2^{1-(\alpha_1 + \overline{\alpha}_1)} = 2^{1-2\sigma_1}. \tag{6}$$

However,

$$\lim_{b \to 0^+} \frac{\zeta_b(\alpha_1)\zeta_{2b}(1 - \overline{\alpha}_1)}{\zeta_{2b}(\alpha_1)\zeta_b(1 - \overline{\alpha}_1)} = 1. \tag{7}$$

Thus, in the limit $b \to 0^+$, (6) and (7) yield

$$2^{1-2\sigma_1} = 1 \quad \text{or} \quad \sigma_1 = 1/2,$$

for the zeros of $\lim_{b \to 0^+} \zeta_b^*(\alpha)$. But, since ζ_b^* converges uniformly to ζ as $b \to 0^+$, in the critical strip, the zeros of $\zeta(\alpha)$ *are* the zeros of $\lim_{b \to 0^+} \zeta_b^*(\alpha)$. Hence the zeros of $\zeta(\alpha)$ in the critical strip lie on the critical line, $\sigma = 1/2$.

3. Discussion of the Proof

Our proof may also be understood in geometrical terms. The second extension, $\zeta_b^*(\alpha)$, may be regarded as a deformation of $\zeta(\alpha)$ in the positive half-plane, and particularly in the critical strip. Fixing σ and varying τ moves along a vertical line, in Fig. 1, while varying σ moves horizontally. Now, varying τ in (2) only changes a phase for each value of t while varying σ changes a magnitude. The deformation of the zeta is not symmetrical about the critical line $\sigma = 1/2$. Our proof depends on the fact that though the change of phase, involved in going from α_1 to $\alpha_3 = \overline{\alpha}_1$, does not create a problem of consistency of the zeros of the deformed function coinciding with those of the original function, the change of magnitude, involved in going from α_1 to $\alpha_4 = 1 - \overline{\alpha}_1$ *does* create a problem due to the horizontal shift. Note that though the shift to $\alpha_2 = 1 - \alpha_1$ does have a horizontal shift, it is complicated by the associated vertical shift. The only way to avoid the inconsistency is to have α_4 and α_1 coincide, i.e. $\alpha_1 = 1 - \overline{\alpha}_1$ or $\alpha_1 + \overline{\alpha}_1 = 1$. Thus the zeros lie on the critical line $\sigma = 1/2$.

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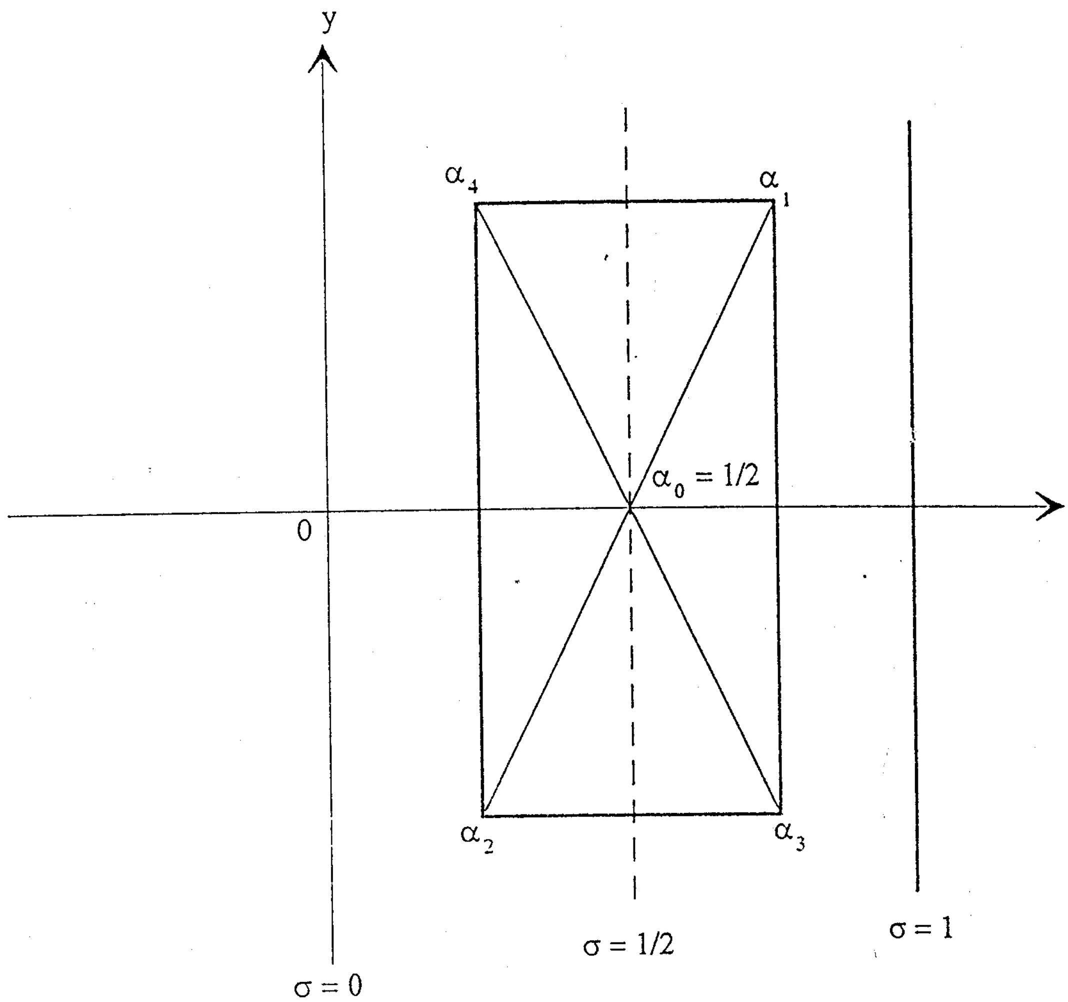

Figure 1: If α_1 is a zero of the zeta function so are $\alpha_2 = 1 - \alpha_1$, $\alpha_3 = \overline{\alpha}_1$ and $\alpha_4 = \overline{\alpha}_2$. These four points give the corners of a rectangle centered at $\alpha_0 = 1/2$ and symmetrical about $\sigma_0 = 1/2$. We can think of $\zeta(\alpha)$ as a surface "pinned down" at these four points.