Co-Semisimple Modules and Generalized Injectivity

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1. Introduction.

Let $R$ be a ring with identity and $M$ a left $R$-module. A left $R$-module $U$ is called $M$-injective if, for every submodule $N$ of $M$ and homomorphism $\phi : N \to U$, $\phi$ can be lifted to a homomorphism $\psi : M \to U$. A left $R$-module $M$ is called co-semisimple by Fuller [2] (and is called $V$-module by Ramamurthi [11] and Tominaga [12]) provided every submodule of $M$ is an intersection of maximal submodules. Fuller [2, Proposition 3.1] and Hirano [4, Proposition 3.1] or Dung, Huynh, Smith and Wisbauer [1] proved that $M$ is co-semisimple if and only if every simple left module is $M$-injective. Wisbauer [14] proved that $M$ is co-semisimple if and only if every finitely cogenerated left module in $\sigma[M]$ is $M$-injective. In this paper, we characterize co-semisimple left $R$-modules via generalized injectivity of some modules.

Let $\mathcal{F}$ be a left Gabriel topology on $R$ and $M$ a left $R$-module. We call $M$ $\mathcal{F}$-co-semisimple if every $\mathcal{F}$-cocritical left $R$-module $C$ in $\sigma[M]$ is dense in its $M$-injective hull $I(C)$. Let $\mathcal{X}$ be a specified class of left $R$-modules (for example, the class of all quasi-continuous left $R$-modules in $\sigma[M]$, or, the class of all quasi-injective left $R$-modules in $\sigma[M]$, etc.). We show that if the left Gabriel topology $\mathcal{F}$ is such that all left $R$-modules in $\mathcal{X}$ are $\mathcal{F}$-injective, then $M$ is $\mathcal{F}$-co-semisimple if and only if every $\mathcal{F}$-torsionfree $\mathcal{F}$-finitely cogenerated left $R$-module $N$ in $\sigma[M]$ is dense in its some essential extensions which are in $\mathcal{X}$.

As a corollary we show that a left $R$-module $M$ is a co-semisimple module if and
only if every finitely cogenerated left $R$-module in $\sigma[M]$ is continuous if and only if every finitely cogenerated left $R$-module in $\sigma[M]$ is quasi-continuous if and only if every finitely cogenerated left $R$-module in $\sigma[M]$ is direct-injective.

Page and Yousif [10] proved that for a finitely generated left $R$-module $M$, $M$ is a noetherian co-semisimple module if and only if every semisimple left $R$-module is $M$-injective. In this paper we also show that a left $R$-module $M$ is a locally noetherian co-semisimple module if and only if every semisimple left $R$-module (in $\sigma[M]$) is $M$-injective if and only if every semisimple left $R$-module (in $\sigma[M]$) is the direct sum of a finitely cogenerated module and an $M$-injective module if and only if every essential extension in $\sigma[M]$ of every semisimple left $R$-module in $\sigma[M]$ is an $\mathcal{X}$-module, where $\mathcal{X}$ is a specified class of left $R$-modules.

2. Preliminaries.

Let $M$ be a left $R$-module. We say that a left $R$-module $N$ is subgenerated by $M$, or that $M$ is a subgenerator for $N$, if $N$ is isomorphic to a submodule of an $M$-generated module. Following [14], we denote by $\sigma[M]$ the full subcategory of $R$-Mod whose objects are all $R$-modules subgenerated by $M$. By [14, 17.9], every module $N$ in $\sigma[M]$ has an injective hull $I(N)$ in $\sigma[M]$, which is also called an $M$-injective hull of $N$. It is known that the $M$-injective hulls of a left $R$-module in $\sigma[M]$ are unique up to isomorphism. In the following, we always denote by $I(N)$ the $M$-injective hull of $N$ for any left $R$-module $N \in \sigma[M]$.

Let $\mathcal{X}$ be a class of left $R$-modules, that is a collection of left $R$-modules such that if $M \in \mathcal{X}$ then any left $R$-module isomorphic to $M$ belongs to $\mathcal{X}$. Any member of $\mathcal{X}$ is called an $\mathcal{X}$-module.

**Definition 2.1.** Let $M$ be a left $R$-module and $\mathcal{X}$ a class of left $R$-modules in $\sigma[M]$. We call $\mathcal{X}$ an $I$-class in the category $\sigma[M]$ if it contains all $M$-injective left $R$-modules.
in $\sigma[M]$ and for any $N \in \sigma[M]$, if there exists an $M$-injective left $R$-module $L$ in $\sigma[M]$ such that $N \leq L$ and $N \oplus L$ is in $\mathcal{X}$, then $N$ is $L$-injective.

If $M = R$, then any $I$-class in the category $\sigma[M]$ is called an $I$-class of left $R$-modules.

Following [19], we call $\mathcal{X}$ an injectivity class in the category $\sigma[M]$ if it is closed under direct summands, contains all quasi-injective left $R$-modules in $\sigma[M]$ and $N \oplus I(N) \in \mathcal{X}$ implies that $N$ is $M$-injective. We claim that every injectivity class in the category $\sigma[M]$ is an $I$-class. In fact, if $\mathcal{X}$ is an injectivity class, then $\mathcal{X}$ contains all $M$-injective left $R$-modules in the category $\sigma[M]$. Suppose that $N$ is in $\sigma[M]$ and $L$ an $M$-injective left $R$-module in $\sigma[M]$ such that $N \leq L$ and $N \oplus L \in \mathcal{X}$. Then $L$ is an injective object of the category $\sigma[M]$. Thus there exists a homomorphism $g:I(N) \to L$ such that $g|_N = \tau$, the natural inclusion map $N \to L$. Now it follows that $g:I(N) \to L$ is a monomorphism since $N$ is essential in $I(N)$. Thus we have $L = I(N) \oplus P$ for a left $R$-module $P$. Therefore $N \oplus L = N \oplus I(N) \oplus P \in \mathcal{X}$. Since $\mathcal{X}$ is closed under direct summands, we obtain that $N \oplus I(N) \in \mathcal{X}$. Now it follows that $N$ is $M$-injective.

A left $R$-module $N$ is called a $CS$-module if every submodule of $M$ is essential in a summand of $M$. $N$ is called continuous if it is a $CS$-module and every submodule isomorphic to a summand of $M$ is itself a summand. $N$ is called quasi-continuous if it is a $CS$-module and if $N_1$ and $N_2$ are summands of $N$ with $N_1 \cap N_2 = 0$, then $N_1 \oplus N_2$ is a summand of $N$. A left $R$-module $N$ is an $SQC$-module [18] if and only if for any submodule $L$ of $N$ such that there exists a non-zero complement submodule $C$ of $N$ which is isomorphic to a factor module of $L$, any $R$-homomorphism from $L$ into $N$ may be extended to an endomorphism of $N$. $N$ is $E$-injective [16] if and only if for any non-zero complement submodule $C$ of $N$ and relative complement $K$ of $C$, any essential submodule $E$ of $N$ containing $K \oplus C$, any $R$-monomorphism $g:E \to N$ and $R$-homomorphism $f:E \to N$, there exists an endomorphism $h$ of $R_N$ such that $hg = f$. $N$ is a $NCI$-module [17] if and only if for any submodule $P$ containing a
non-zero complement submodule of $N$ and any submodule $L$ of $N$ which is isomorphic to $P$, every $R$-homomorphism from $L$ into $P$ extends to an endomorphism of $N$. $N$ is called direct-injective if, for every direct summand $L$ of $N$, every monomorphism $L \to N$ splits.

The following proposition gives some examples of $I$-classes.

**Proposition 2.2.** The class of all quasi-injective (respectively, continuous, quasi-continuous, direct-injective, $NCI$, $SQC$, $E$-injective) left $R$-modules in $\sigma[M]$ is an $I$-class.

**Proof.** It follows from [5], [9], [15], [17], [18] and [19].

Let $\mathcal{F}$ be a left Gabriel topology on $R$. The quotient category $(R, \mathcal{F})\text{-Mod}$, associated with $\mathcal{F}$, is the full subcategory of $R\text{-Mod}$ whose objects are the $\mathcal{F}$-closed (i.e., $\mathcal{F}$-torsionfree and $\mathcal{F}$-injective) left $R$-modules, and it is a Grothendieck category. The inclusion functor $i : (R, \mathcal{F})\text{-Mod} \to R\text{-Mod}$ has a left adjoint $a : R\text{-Mod} \to (R, \mathcal{F})\text{-Mod}$ which is exact and assigns to each $M \in R\text{-Mod}$ its module of quotients $M_\mathcal{F}$.

3. $\mathcal{F}$-Co-semisimple Modules.

**Definition 3.1.** Let $\mathcal{F}$ be a left Gabriel topology on $R$, $M$ a left $R$-module and $N$ in $\sigma[M]$. 

(1) We say that $N$ is $\mathcal{F}$-cocyclic in $\sigma[M]$ if there exists a $\mathcal{F}$-cocritical left $R$-module $C \in \sigma[M]$ such that 

$$0 \to N_\mathcal{F} \to I(C)_\mathcal{F}$$

is exact. $N$ is called $\sigma - \mathcal{F}$-cocyclic in $\sigma[M]$ if it is a finite direct sum of $\mathcal{F}$-cocyclic left $R$-modules in $\sigma[M]$.

(2) $N$ is called $\mathcal{F}$-finitely cogenerated in $\sigma[M]$ if there exist $\mathcal{F}$-cocritical left $R$-
modules $C_1, \ldots, C_n$ in $\sigma[M]$ such that the sequence

$$0 \to N_{\mathcal{F}} \to \bigoplus_{i=1}^n I(C_i)_{\mathcal{F}}$$

is exact.

If $M = R$ and $\mathcal{F} = \{R\}$, then $\mathcal{F}$-cocyclic modules in $\sigma[M]$ (resp. $\mathcal{F}$-finitely cogenerated modules in $\sigma[M]$) are precisely cocyclic (resp. finitely cogenerated) modules in the usual sense (see [14] and [7]).

**Definition 3.2.** Let $M$ be a left $R$-module and $\mathcal{F}$ a left Gabriel topology on $R$. We call $M$ a $\mathcal{F}$-co-semisimple module if every $\mathcal{F}$-cocritical left $R$-module $C$ in $\sigma[M]$ is dense in its $M$-injective hull $I(C)$.

Note that if $\mathcal{F} = \{R\}$, then the $\mathcal{F}$-co-semisimple left $R$-modules are precisely the co-semisimple left $R$-modules. On the other hand, if $\mathcal{F}$ is a perfect Gabriel topology, then the inclusion functor $j : (R, \mathcal{F})\text{-Mod} \to R_{\mathcal{F}}\text{-Mod}$ is an equivalence; hence, for every left $R$-module $M$, $M$ is $\mathcal{F}$-co-semisimple if and only if $M_{\mathcal{F}}$ is co-semisimple. If $\mathcal{F}$ is a left Gabriel topology on $R$ such that for every left $R$-module $N$ in $\sigma[M]$, $N_{\mathcal{F}}$ is an injective object of $(R, \mathcal{F})\text{-Mod}$, then $M$ is a $\mathcal{F}$-co-semisimple module. In particular, if $\mathcal{F}$ is a left Gabriel topology on $R$ such that $(R, \mathcal{F})\text{-Mod}$ is a spectral category (that is, every object is injective), then clearly every left $R$-module $M$ is $\mathcal{F}$-co-semisimple. Thus, if $\mathcal{G}$ denotes the left Goldie topology, then every left $R$-module $M$ is $\mathcal{G}$-co-semisimple.

**Theorem 3.3.** Let $M$ be a left $R$-module and $\mathcal{F}$ a left Gabriel topology on $R$. If $\mathcal{X}$ is an $I$-class in the category $\sigma[M]$ such that every $\mathcal{X}$-module is $\mathcal{F}$-injective, then the following conditions are equivalent.

1. $M$ is $\mathcal{F}$-co-semisimple.
2. Every $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated left $R$-module in $\sigma[M]$ is dense in its some essential extensions which are $\mathcal{X}$-modules.
(3) Every $\mathcal{F}$-torsionfree and $\sigma - \mathcal{F}$-cocyclic left $R$-module in $\sigma[M]$ is dense in its
some essential extensions which are $\mathcal{X}$-modules.

(4) For every $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated left $R$-module $N$ in $\sigma[M],
there exists an $\mathcal{X}$-module $L$ with essential submodule $N$ such that $\operatorname{Rad}_{\mathcal{F}}(L) = 0$.

(5) For every $\mathcal{F}$-torsionfree and $\sigma - F$-cocyclic left $R$-module $N$ in $\sigma[M]$, there exists
an $\mathcal{X}$-module $L$ with essential submodule $N$ such that $\operatorname{Rad}_{\mathcal{F}}(L) = 0$.

PROOF. (1) $\Rightarrow$ (2). Let left $R$-module $N \in \sigma[M]$ be $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely
cogenerated in $\sigma[M]$. Then there exist $\mathcal{F}$-cocritical left $R$-modules $C_1, \ldots, C_m \in \sigma[M]
such that

$$0 \to N_{\mathcal{F}} \xrightarrow{g} \bigoplus_{i=1}^{m} I(C_i)_{\mathcal{F}}.$$

Consider the following diagram

$$
\begin{array}{ccc}
N_{\mathcal{F}} & \xrightarrow{r} & I(N)_{\mathcal{F}} \\
\downarrow{g} & & \\
\bigoplus_{i=1}^{m} I(C_i)_{\mathcal{F}} & &
\end{array}
$$

It is clear that $\bigoplus_{i=1}^{m} I(C_i) \in \sigma[M]$ is $M$-injective. Thus $\bigoplus_{i=1}^{m} I(C_i)$ is an $\mathcal{X}$-module since
$\mathcal{X}$ is an $I$-class in the category of $\sigma[M]$, and so is $\mathcal{F}$-injective by assumption. It is
easy to see that $\bigoplus_{i=1}^{m} I(C_i)$ is $\mathcal{F}$-torsionfree. Thus $\bigoplus_{i=1}^{m} I(C_i)$ is $\mathcal{F}$-closed. This implies
that $i \left( \bigoplus_{i=1}^{m} I(C_i) \right)_{\mathcal{F}} \simeq \bigoplus_{i=1}^{m} I(C_i)$. Thus $i \left( \bigoplus_{i=1}^{m} I(C_i) \right)_{\mathcal{F}}$ is $M$-injective. A similar
argument gives that $i(I(N)_{\mathcal{F}}) \simeq I(N) \in \sigma[M]$. Thus $i(N_{\mathcal{F}}) \in \sigma[M]$. Now, by [14, 16.3],
there exists a homomorphism $f : i(I(N)_{\mathcal{F}}) \to i \left( \bigoplus_{i=1}^{m} I(C_i) \right)_{\mathcal{F}}$ such that $i(g) = f i(\tau)$.
Thus we have $g = a(f) \tau$. Since $N$ is essential in $I(N)$ and $N$ is $\mathcal{F}$-torsionfree, by
[3, Lemma 0.1], it follows that $N_{\mathcal{F}}$ is an essential subobject of $I(N)_{\mathcal{F}}$. Thus $a(f) : I(N)_{\mathcal{F}} \to \bigoplus_{i=1}^{m} I(C_i)_{\mathcal{F}}$ is a monomorphism. Since $M$ is $\mathcal{F}$-co-semisimple, we have $(C_i)_{\mathcal{F}} = I(C_i)_{\mathcal{F}}$, $i = 1, \ldots, m$. Thus $I(N)_{\mathcal{F}}$ is isomorphic to a subobject of semisimple object
⊕(C_i)_{\mathcal{F}} of (R, \mathcal{F})-\text{Mod}. Hence I(N)_{\mathcal{F}} is semisimple. This implies that \( N_{\mathcal{F}} = I(N)_{\mathcal{F}} \).

Clearly \( I(N) \) is an \( \mathcal{X} \)-module since it is \( M \)-injective.

(2) \( \Rightarrow \) (3). Let \( N \) be a \( \mathcal{F} \)-torsionfree and \( \sigma - \mathcal{F} \)-cyclic left \( R \)-module in \( \sigma[M] \).

Then \( N = \bigoplus_{i=1}^n N_i \), where \( N_1, \ldots, N_n \) are \( \mathcal{F} \)-cyclic left \( R \)-modules in \( \sigma[M] \). Thus there exist \( \mathcal{F} \)-cyclic left \( R \)-modules \( C_1, \ldots, C_n \in \sigma[M] \) such that the sequence

\[
0 \to N_{\mathcal{F}} \to \bigoplus_{i=1}^n I(C_i)_{\mathcal{F}}
\]

is exact. Thus \( N \) is \( \mathcal{F} \)-finitely cogenerated in \( \sigma[M] \).

(3) \( \Rightarrow \) (1). Let \( N \) be a \( \mathcal{F} \)-cyclic left \( R \)-module in \( \sigma[M] \). Then \( N \) is \( \mathcal{F} \)-torsionfree and \( N_{\mathcal{F}} \) is a simple object of \( (R, \mathcal{F})-\text{Mod} \). Since \( I(N) \) is \( M \)-injective, it is easy to see that \( I(N) \) is \( \mathcal{F} \)-closed. Thus \( i(I(N)_{\mathcal{F}}) \cong I(N) \in \sigma[M] \); and hence \( i(N_{\mathcal{F}}) \), a submodule of \( i(I(N)_{\mathcal{F}}) \), is in \( \sigma[M] \). Since the sequence

\[
0 \to i(N_{\mathcal{F}})_{\mathcal{F}} \to i(I(N)_{\mathcal{F}})_{\mathcal{F}}
\]

is exact, and \( (i(I(N)_{\mathcal{F}}))_{\mathcal{F}} \cong I(N)_{\mathcal{F}} \), we see that \( i(N_{\mathcal{F}}) \) is \( \mathcal{F} \)-cyclic in \( \sigma[M] \). It is easy to see that \( i(I(N)_{\mathcal{F}}) \in \sigma[M] \) is \( \mathcal{F} \)-cyclic, too. Thus \( i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}}) \) is \( \sigma - \mathcal{F} \)-cyclic. Clearly \( i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}}) \) is \( \mathcal{F} \)-torsionfree. Thus, by condition (3), there exists an \( \mathcal{X} \)-module \( L \) such that \( i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}}) \) is an essential submodule of \( L \) and \( (i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}}))_{\mathcal{F}} = L_{\mathcal{F}} \). Thus we get

\[
L_{\mathcal{F}} \cong N_{\mathcal{F}} \oplus I(N)_{\mathcal{F}}.
\]

Clearly \( L \) is \( \mathcal{F} \)-closed since it is \( \mathcal{F} \)-torsionfree and \( \mathcal{F} \)-injective. Thus

\[
i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}}) \cong i(L_{\mathcal{F}}) \cong L.
\]

This implies that \( i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}}) \) is an \( \mathcal{X} \)-module. By definition of \( I \)-classes it follows that \( i(N_{\mathcal{F}}) \) is \( i(I(N)_{\mathcal{F}}) \)-injective. Thus there exists a homomorphism \( f : i(I(N)_{\mathcal{F}}) \to i(N_{\mathcal{F}}) \) such that \( f\vert_{i(N_{\mathcal{F}})} = 1 \). This implies that there exists a left \( R \)-module \( H \) such
that \( i(I(N)) = i(N) \oplus H \). Thus \( I(N) = N \oplus a(H) \). But \( N \) is essential in \( I(N) \) since \( N \) is essential in \( I(N) \). Thus \( N = I(N) \) and we are done.

(2) \( \Rightarrow \) (4). Let a left \( R \)-module \( N \in \sigma[M] \) be \( \mathcal{F} \)-torsionfree and \( \mathcal{F} \)-finitely cogenerated in \( \sigma[M] \). By (2), there exists an essential extension \( L \) of \( N \) such that \( L \) is an \( \mathcal{X} \)-module and \( N \) is dense in \( L \). By [3, Proposition 1.2], we have

\[
(\text{Rad}_\mathcal{F}(L))_\mathcal{F} = \text{Rad}(L) = \text{Rad}(N).
\]

Since \( N \) is \( \mathcal{F} \)-finitely cogenerated in \( \sigma[M] \), there exist \( \mathcal{F} \)-cocritical left \( R \)-modules \( C_1, \ldots, C_n \) in \( \sigma[M] \) such that the following sequence is exact:

\[
0 \to N \to \bigoplus_{i=1}^n I(C_i) = 0.
\]

By the results proved above, \( M \) is \( \mathcal{F} \)-co-semisimple when condition (2) holds. Therefore every \( \mathcal{F} \)-cocritical left \( R \)-module in \( \sigma[M] \) is dense in its \( M \)-injective hull. It follows that \( I(C_i) = (C_i)_\mathcal{F}, \ i = 1, \ldots, n \). Thus we have

\[
\text{Rad} \left( \bigoplus_{i=1}^n I(C_i) \right) = \bigoplus_{i=1}^n \text{Rad}(I(C_i)) = 0,
\]

which implies that \( \text{Rad}(N) = 0 \). Thus \( \text{Rad}_\mathcal{F}(L) = t(\text{Rad}_\mathcal{F}(L)) \), where \( t \) is a left exact radical corresponding to the Gabriel topology \( \mathcal{F} \). This means that \( \text{Rad}_\mathcal{F}(L) \) is a \( \mathcal{F} \)-torsion module. On the other hand, \( \text{Rad}_\mathcal{F}(L) \) is a \( \mathcal{F} \)-saturated submodule of \( L \), and so is \( \mathcal{F} \)-torsionfree. Therefore we get \( \text{Rad}_\mathcal{F}(L) = 0 \), as required.

(4) \( \Rightarrow \) (5). It is similar to the implication (2) \( \Rightarrow \) (3).

(5) \( \Rightarrow \) (3). Let \( N \) be a \( \mathcal{F} \)-torsionfree and \( \sigma - \mathcal{F} \)-cocyclic left \( R \)-module in \( \sigma[M] \). Then there exists an essential extension \( L \) of \( N \) such that \( L \) is an \( \mathcal{X} \)-module and \( \text{Rad}_\mathcal{F}(L) = 0 \). It is enough to show that \( N \) is dense in \( L \). By [3, Proposition 1.3], it follows that \( L \) is cogenerated by a class of \( \mathcal{F} \)-cocritical left \( R \)-modules. Since \( N \) is \( \sigma - \mathcal{F} \)-cocyclic in \( \sigma[M] \), as the implication (2) \( \Rightarrow \) (3), we see \( N \) is \( \mathcal{F} \)-finitely cogenerated in \( \sigma[M] \). Thus there exist \( \mathcal{F} \)-cocritical left \( R \)-modules \( C_1, \ldots, C_n \) such that the sequence

\[
0 \to N \to \bigoplus_{i=1}^n I(C_i)
\]
is exact. By analogy with the implication (1) ⇒ (2), we obtain that \( I(N)_\mathcal{F} \) is isomorphic to a subobject of \( \bigoplus_{i=1}^{n} I(C_i)_\mathcal{F} \). By [14, 17.10], it follows that \( I(N) \cong I(L) \) since \( N \) is essential in \( L \). Thus \( I(N)_\mathcal{F} \cong I(L)_\mathcal{F} \), which implies that \( L_{\mathcal{F}} \) can be embedded in \( \bigoplus_{i=1}^{n} I(C_i)_\mathcal{F} \). This means that \( L \) is \( \mathcal{F} \)-finitely cogenerated in \( \sigma[M] \). Thus clearly \( L \) is \( \mathcal{F} \)-finitely cogenerated in \( R\text{-mod} \). By [3, Proposition 1.7], every family of \( \mathcal{F} \)-torsionfree modules which cogenerated \( L \), does cogenereate it finitely. Thus there exist finite \( \mathcal{F} \)-cocritical left \( R \)-modules \( D_1, \ldots, D_k \) such that the sequence \( 0 \to L \to \bigoplus_{i=1}^{k} D_i \) is exact, which implies that the sequence \( 0 \to L_{\mathcal{F}} \to \bigoplus_{i=1}^{k} (D_i)_{\mathcal{F}} \) is exact. This means that \( L_{\mathcal{F}} \) is a semisimple object of \( (R, \mathcal{F})\text{-Mod} \). On the other hand, \( N \) is an essential submodule of \( L \); and so \( N_{\mathcal{F}} \) is essential in \( L_{\mathcal{F}} \) by [3, Lemma 0.1]. It is then easy to see that \( N_{\mathcal{F}} = L_{\mathcal{F}} \), in other words \( N \) is dense in \( L \); and we are done.

The following corollary generalizes a corresponding result of [7].

**Corollary 3.4.** Let \( \mathcal{F} \) be a left Gabriel topology on \( R \). If \( \mathcal{X} \) is an \( I \)-class of left \( R \)-modules such that every \( \mathcal{X} \)-module is \( \mathcal{F} \)-injective, then the following conditions are equivalent.

1. \( R \) is a \( \mathcal{F} \)-\text{V-ring}.

2. Every \( \mathcal{F} \)-torsionfree and \( \mathcal{F} \)-finitely cogenerated left \( R \)-module is dense in its some essential extensions which are \( \mathcal{X} \)-modules.

3. Every \( \mathcal{F} \)-torsionfree and \( \sigma - \mathcal{F} \)-cocyclic left \( R \)-module is dense in its some essential extensions which are \( \mathcal{X} \)-modules.

4. For every \( \mathcal{F} \)-torsionfree and \( \mathcal{F} \)-finitely cogenerated left \( R \)-module \( M \), there exists an \( \mathcal{X} \)-module \( L \) with essential submodule \( M \) such that \( \text{Rad}_{\mathcal{F}}(L) = 0 \).

5. For every \( \mathcal{F} \)-torsionfree and \( \sigma - \mathcal{F} \)-cocyclic left \( R \)-module \( M \), there exists an \( \mathcal{X} \)-module \( L \) with an essential submodule \( M \) such that \( \text{Rad}_{\mathcal{F}}(L) = 0 \).
Corollary 3.5. Let $M$ be a left $R$-module and $X$ an $I$-class in the category $\sigma[M]$. Then the following assertions are equivalent.

(1) $M$ is co-semisimple.

(2) Every finitely cogenerated left $R$-module in $\sigma[M]$ is an $X$-module.

(3) Every $\sigma$-cyclic left $R$-module in $\sigma[M]$ is an $X$-module.

For $M = R$, Corollary 3.5 gives characterizations of left $V$-rings by generalized injectivity.

Page and Yousif [10] proved that for a finitely generated left $R$-module $M$, $M$ is a noetherian co-semisimple module if and only if every semisimple left $R$-module is $M$-injective. Recall that a left $R$-module $M$ is locally noetherian if every finitely generated submodule of $M$ is noetherian. We have

Proposition 3.6. Let $M$ be a left $R$-module and $X$ an $I$-class in the category $\sigma[M]$. Then the following conditions are equivalent.

(1) $M$ is locally noetherian co-semisimple left $R$-module.

(2) Every semisimple left $R$-module (in $\sigma[M]$) is $M$-injective.

(3) Every semisimple left $R$-module (in $\sigma[M]$) is the direct sum of a finitely cogenerated left $R$-module and an $M$-injective module.

(4) For every semisimple left $R$-module $N$ in $\sigma[M]$, every essential extension in $\sigma[M]$ of $N$ is an $X$-module.

(5) For every semisimple left $R$-module $N$ in $\sigma[M]$, every submodule of an essential extension in $\sigma[M]$ of $N$ is an $X$-module.

If $X$ is closed under direct summands, then the following are also equivalent.
(6) For every semisimple left $R$-module $N$ in $\sigma[M]$, every essential extension in $\sigma[M]$ of $N$ is the direct sum of a finitely cogenerated module and an $\mathcal{X}$-module.

(7) For every semisimple left $R$-module $N$ in $\sigma[M]$, every submodule of an essential extension in $\sigma[M]$ of $N$ is the direct sum of a finitely cogenerated module and an $\mathcal{X}$-module.

**Proof.** The equivalence of (1), (2) and (3) is proved in [8].

(4) $\Rightarrow$ (2). Let $N$ be a semisimple left $R$-module in $\sigma[M]$. Then $N \oplus I(N)$ is an essential extension of $N \oplus N$. Thus $N \oplus I(N)$ is an $\mathcal{X}$-module by condition (4), which implies that $N$ is $I(N)$-injective by the definition of $I$-class. Now it is easy to see that $N = I(N)$ is $M$-injective.

(2) $\Rightarrow$ (5). Let $N$ be a semisimple left $R$-module in $\sigma[M]$ and $L$ a submodule of an essential extension $D$ in $\sigma[M]$ of $N$. Then $N$ is $M$-injective by (2). Thus $N$ is $D$-injective by [14, 16.3]. Now it is easy to see that $N = D$; and thus $D$ is semisimple and $M$-injective. Therefore $L$, a direct summand of $D$, is $M$-injective. Since $\mathcal{X}$ contains all $M$-injective left $R$-modules in $\sigma[M]$, it follows that every submodule of an essential extension in $\sigma[M]$ of a semisimple left $R$-module in $\sigma[M]$ is an $\mathcal{X}$-module.

The implications (5) $\Rightarrow$ (4), (7) $\Rightarrow$ (6) and (5) $\Rightarrow$ (7) are clear.

(6) $\Rightarrow$ (4). Not that the class of semisimple left $R$-modules is closed under direct sums; it is easy to see that every direct sum of essential extensions in $\sigma[M]$ of semisimple left $R$-modules in $\sigma[M]$ is an essential extension of a semisimple module. Now, by analogy with the proof of the main result of [6], we see that every essential extension in $\sigma[M]$ of a semisimple left $R$-module in $\sigma[M]$ is an $\mathcal{X}$-module.
REFERENCES


(ja1961)