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# Co-Semisimple Modules and Generalized Injectivity

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# CO-SEMISIMPLE MODULES AND GENERALIZED INJECTIVITY

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## 1. Introduction.

Let $R$ be a ring with identity and $M$ a left $R$-module. A left $R$-module $U$ is called *$M$-injective* if, for every submodule $N$ of $M$ and homomorphism $\phi : N \to U$, $\phi$ can be lifted to a homomorphism $\psi : M \to U$. A left $R$-module $M$ is called *co-semisimple* by Fuller [2] (and is called *$V$-module* by Ramamurthi [11] and Tominaga [12]) provided every submodule of $M$ is an intersection of maximal submodules. Fuller [2, Proposition 3.1] and Hirano [4, Proposition 3.1] or Dung, Huynh, Smith and Wisbauer [1] proved that $M$ is co-semisimple if and only if every simple left module is $M$-injective. Wisbauer [14] proved that $M$ is co-semisimple if and only if every finitely cogenerated left module in $\sigma[M]$ is $M$-injective. In this paper, we characterize co-semisimple left $R$-modules via generalized injectivity of some modules.

Let $\mathcal{F}$ be a left Gabriel topology on $R$ and $M$ a left $R$-module. We call $M$ *$\mathcal{F}$-co-semisimple* if every $\mathcal{F}$-cocritical left $R$-module $C$ in $\sigma[M]$ is dense in its $M$-injective hull $I(C)$. Let $\mathcal{X}$ be a specified class of left $R$-modules (for example, the class of all quasi-continuous left $R$-modules in $\sigma[M]$, or, the class of all quasi-injective left $R$-modules in $\sigma[M]$, etc.). We show that if the left Gabriel topology $\mathcal{F}$ is such that all left $R$-modules in$\mathcal{X}$ are $\mathcal{F}$-injective, then $M$ is $\mathcal{F}$-co-semisimple if and only if every $\mathcal{F}$-torsionfree $\mathcal{F}$-finitely cogenerated left $R$-module $N$ in $\sigma[M]$ is dense in its some essential extensions which are in $\mathcal{X}$.

As a corollary we show that a left $R$-module $M$ is a co-semisimple module if and only if every finitely cogenerated left $R$-module in $\sigma[M]$ is continuous if and only if every finitely cogenerated left $R$-module in $\sigma[M]$ is quasi-continuous if and only if every finitely cogenerated left $R$-module in $\sigma[M]$ is direct-injective.

Page and Yousif [10] proved that for a finitely generated left $R$-module $M$, $M$ is a noetherian co-semisimple module if and only if every semisimple left $R$-module is $M$-injective. In this paper we also show that a left $R$-module $M$ is a locally noetherian co-semisimple module if and only if every semisimple left $R$-module (in $\sigma[M]$ is $M$-injective if and only if every semisimple left $R$-module (in $\sigma[M]$) is the direct sum of a finitely cogenerated module and an $M$-injective module if and only if every essential extension in $\sigma[M]$ of every semisimple left $R$-module in $\sigma[M]$ is an $\mathcal{X}$-module, where $\mathcal{X}$ is a specified class of left $R$-modules.

## 2. Preliminaries.

Let $M$ be a left $R$-module. We say that a left $R$-module $N$ is *subgenerated* by $M$, or that $M$ is a subgenerator for $N$, if $N$ is isomorphic to a submodule of an $M$-generated module. Following [14], we denote by $\sigma[M]$ the full subcategory of $R$-Mod whose objects are all $R$-modules subgenerated by $M$. By [14, 17.9], every module $N$ in $\sigma[M]$ has an injective hull $I(N)$ in $\sigma[M]$, which is also called an *$M$-injective hull* of $N$. It is known that the $M$-injective hulls of a left $R$-module in $\sigma[M]$ are unique up to isomorphism. In the following, we always denote by $I(N)$ the $M$-injective hull of $N$ for any left $R$-module $N \in \sigma[M]$.

Let $\mathcal{X}$ be a class of left $R$-modules, that is a collection of left $R$-modules such that if $M \in \mathcal{X}$ then any left $R$-module isomorphic to $M$ belongs to $\mathcal{X}$. Any member of $\mathcal{X}$ is called an $\mathcal{X}$-module.

DEFINITION 2.1. Let $M$ be a left $R$-module and $\mathcal{X}$ a class of left $R$-modules in $\sigma[M]$. We call $\mathcal{X}$ an *$I$-class* in the category $\sigma[M]$ if it contains all $M$-injective left $R$-modules in $\sigma[M]$ and for any $N \in \sigma[M]$, if there exists an $M$-injective left $R$-module $L$ in $\sigma[M]$ such that $N \leq L$ and $N \oplus L$ is in $\mathcal{X}$, then $N$ is $L$-injective.

If $M = R$, then any $I$-class in the category $\sigma[M]$ is called an $I$-class of left $R$-modules.

Following [19], we call $\mathcal{X}$ an injectivity class in the category $\sigma[M]$ if it is closed under direct summands, contains all quasi-injective left $R$-modules in $\sigma[M]$ and $N \oplus I(N) \in \mathcal{X}$ implies that $N$ is $M$-injective. We claim that every injectivity class in the category $\sigma[M]$ is an $I$-class. In fact, if $\mathcal{X}$ is an injectivity class, then $\mathcal{X}$ contains all $M$-injective left $R$-modules in the category $\sigma[M]$. Suppose that $N$ is in $\sigma[M]$ and $L$ an $M$-injective left $R$-module in $\sigma[M]$ such that $N \leq L$ and $N \oplus L \in \mathcal{X}$. Then $L$ is an injective object of the category $\sigma[M]$. Thus there exists a homomorphism $g : I(N) \to L$ such that $g|_N = \tau$, the natural inclusion map $N \to L$. Now it follows that $g : I(N) \to L$ is a monomorphism since $N$ is essential in $I(N)$. Thus we have $L = I(N) \oplus P$ for a left $R$-module $P$. Therefore $N \oplus L = N \oplus I(N) \oplus P \in \mathcal{X}$. Since $\mathcal{X}$ is closed under direct summands, we obtain that $N \oplus I(N) \in \mathcal{X}$. Now it follows that $N$ is $M$-injective.

A left $R$-module $N$ is called a *CS-module* if every submodule of $M$ is essential in a summand of $M$. $N$ is called *continuous* if it is a $CS$-module and every submodule isomorphic to a summand of $M$ is itself a summand. $N$ is called *quasi-continuous* if it is a $CS$-module and if $N_1$ and $N_2$ are summands of $N$ with $N_1 \cap N_2 = 0$, then $N_1 \oplus N_2$ is a summand of $N$. A left $R$-module $N$ is an *SQC-module* [18] if and only if for any submodule $L$ of $N$ such that there exists a non-zero complement submodule $C$ of $N$ which is isomorphic to a factor module of $L$, any $R$-homomorphism from $L$ into $N$ may be extended to an endomorphism of $N$. $N$ is *E-injective* [16] if and only if for any non-zero complement submodule $C$ of $N$ and relative complement $K$ of $C$, any essential submodule $E$ of $N$ containing $K \oplus C$, any $R$-monomorphism $g : E \to N$ and $R$-homomorphism $f : E \to N$, there exists an endomorphism $h$ of ${}_RN$ such that $hg = f$. $N$ is a *NCI-module* [17] if and only if for any submodule $P$ containing a non-zero complement submodule of $N$ and any submodule $L$ of $N$ which is isomorphic to $P$, every $R$-homomorphism from $L$ into $P$ extends to an endomorphism of $N$. $N$ is called *direct-injective* if, for every direct summand $L$ of $N$, every monomorphism $L \to N$ splits.

The following proposition gives some examples of $I$-classes.

PROPOSITION 2.2. *The class of all quasi-injective (respectively, continuous, quasi-continuous, direct-injective, NCI, SQC, E-injective) left R-modules in* $\sigma[M]$ *is an I-class.*

PROOF. It follows from [5], [9], [15], ][17], 18] and [19].

Let $\mathcal{F}$ be a left Gabriel topology on $R$. The quotient category $(R, \mathcal{F})$-Mod, associated with $\mathcal{F}$, is the full subcategory of $R$-Mod whose objects are the $\mathcal{F}$-closed (i.e., $\mathcal{F}$-torsionfree and $\mathcal{F}$-injective) left $R$-modules, and it is a Grothendieck category. The inclusion functor $i : (R, \mathcal{F})\text{-Mod} \to R\text{-Mod}$ has a left adjoint $a : R\text{-Mod} \to (R, \mathcal{F})\text{-Mod}$ which is exact and assigns to each $M \in R$-Mod its module of quotients $M_{\mathcal{F}}$.

## 3. $\mathcal{F}$-Co-semisimple Modules.

DEFINITION 3.1. Let $\mathcal{F}$ be a left Gabriel topology on $R$, $M$ a left $R$-module and $N$ in $\sigma[M]$.

(1) We say that $N$ is *$\mathcal{F}$-cocyclic* in $\sigma[M]$ if there exists a $\mathcal{F}$-cocritical left $R$-module $C \in \sigma[M]$ such that

$$0 \to N_{\mathcal{F}} \to I(C)_{\mathcal{F}}$$

is exact. $N$ is called $\sigma - \mathcal{F}$-cocyclic in $\sigma[M]$ if it is a finite direct sum of $\mathcal{F}$-cocyclic left $R$-modules in $\sigma[M]$.

(2) $N$ is called *$\mathcal{F}$-finitely cogenerated* in $\sigma[M]$ if there exist $\mathcal{F}$-cocritical left $R$-

modules $C_1, \dots, C_n$ in $\sigma[M]$ such that the sequence

$$0 \to N_{\mathcal{F}} \to \bigoplus_{i=1}^{n} I(C_i)_{\mathcal{F}}$$

is exact.

If $M = R$ and $\mathcal{F} = \{R\}$, then $\mathcal{F}$-cocyclic modules in $\sigma[M]$ (resp. $\mathcal{F}$-finitely cogenerated modules in $\sigma[M]$) are precisely cocyclic (resp. finitely cogenerated) modules in the usual sense (see [14] and [7]).

DEFINITION 3.2. Let $M$ be a left $R$-module and $\mathcal{F}$ a left Gabriel topology on $R$. We call $M$ a *$\mathcal{F}$-co-semisimple* module if every $\mathcal{F}$-cocritical left $R$-module $C$ in $\sigma[M]$ is dense in its $M$-injective hull $I(C)$.

Note that if $\mathcal{F} = \{R\}$, then the $\mathcal{F}$-co-semisimple left $R$-modules are precisely the co-semisimple left $R$-modules. On the other hand, if $\mathcal{F}$ is a perfect Gabriel topology, then the inclusion functor $j : (R, \mathcal{F})\text{-Mod} \to R_{\mathcal{F}}\text{-Mod}$ is an equivalence; hence, for every left $R$-module $M$, $M$ is $\mathcal{F}$-co-semisimple if and only if $M_{\mathcal{F}}$ is co-semisimple. If $\mathcal{F}$ is a left Gabriel topology on $R$ such that for every left $R$-module $N$ in $\sigma[M]$, $N_{\mathcal{F}}$ is an injective object of $(R, \mathcal{F})$-Mod, then $M$ is a $\mathcal{F}$-co-semisimple module. In particular, if $\mathcal{F}$ is a left Gabriel topology on $R$ such that $(R, \mathcal{F})$-Mod is a spectral category (that is, every object is injective), then clearly every left $R$-module $M$ is $\mathcal{F}$-co-semisimple. Thus, if $\mathcal{G}$ denotes the left Goldie topology, then every left $R$-module $M$ is $\mathcal{G}$-co-semisimple.

THEOREM 3.3. *Let $M$ be a left $R$-module and $\mathcal{F}$ a left Gabriel topology on $R$. If $\mathcal{X}$ is an $I$-class in the category $\sigma[M]$ such that every $\mathcal{X}$-module is $\mathcal{F}$-injective, then the following conditions are equivalent.*

*(1) $M$ is $\mathcal{F}$-co-semisimple.*

*(2) Every $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated left $R$-module in $\sigma[M]$ is dense in its some essential extensions which are $\mathcal{X}$-modules.*

*(3) Every $\mathcal{F}$-torsionfree and $\sigma - \mathcal{F}$-cocyclic left $R$-module in $\sigma[M]$ is dense in its some essential extensions which are $\mathcal{X}$-modules.*

*(4) For every $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated left $R$-module $N$ in $\sigma[M]$, there exists an $\mathcal{X}$-module $L$ with essential submodule $N$ such that* $\mathrm{Rad}_{\mathcal{F}}(L) = 0$.

*(5) For every $\mathcal{F}$-torsionfree and $\sigma - F$-cocyclic left $R$-module $N$ in $\sigma[M]$, there exists an $\mathcal{X}$-module $L$ with essential submodule $N$ such that* $\mathrm{Rad}_{\mathcal{F}}(L) = 0$.

PROOF. (1) $\Rightarrow$ (2). Let left $R$-module $N \in \sigma[M]$ be $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated in $\sigma[M]$. Then there exist $\mathcal{F}$-cocritical left $R$-modules $C_1, \ldots, C_m \in \sigma[M]$ such that

$$0 \to N_{\mathcal{F}} \xrightarrow{g} \bigoplus_{i=1}^{m} I(C_i)_{\mathcal{F}}.$$

Consider the following diagram

$$\begin{array}{ccc} N_{\mathcal{F}} & \xrightarrow{\tau} & I(N)_{\mathcal{F}} \\ {\scriptstyle g}\downarrow & & \\ \bigoplus_{i=1}^{m} I(C_i)_{\mathcal{F}} & & \end{array}$$

It is clear that $\bigoplus_{i=1}^{m} I(C_i) \in \sigma[M]$ is $M$-injective. Thus $\bigoplus_{i=1}^{m} I(C_i)$ is an $\mathcal{X}$-module since $\mathcal{X}$ is an $I$-class in the category of $\sigma[M]$, and so is $\mathcal{F}$-injective by assumption. It is easy to see that $\bigoplus_{i=1}^{m} I(C_i)$ is $\mathcal{F}$-torsionfree. Thus $\bigoplus_{i=1}^{m} I(C_i)$ is $\mathcal{F}$-closed. This implies that $i\left(\left(\bigoplus_{i=1}^{m} I(C_i)\right)_{\mathcal{F}}\right) \simeq \bigoplus_{i=1}^{m} I(C_i)$. Thus $i\left(\left(\bigoplus_{i=1}^{m} I(C_i)\right)_{\mathcal{F}}\right)$ is $M$-injective. A similar argument gives that $i(I(N)_{\mathcal{F}}) \simeq I(N) \in \sigma[M]$. Thus $i(N_{\mathcal{F}}) \in \sigma[M]$. Now, by [14, 16.3], there exists a homomorphism $f : i(I(N)_{\mathcal{F}}) \to i\left(\left(\bigoplus_{i=1}^{m} I(C_i)\right)_{\mathcal{F}}\right)$ such that $i(g) = fi(\tau)$. Thus we have $g = a(f)\tau$. Since $N$ is essential in $I(N)$ and $N$ is $\mathcal{F}$-torsionfree, by [3, Lemma 0.1], it follows that $N_{\mathcal{F}}$ is an essential subobject of $I(N)_{\mathcal{F}}$. Thus $a(f) : I(N)_{\mathcal{F}} \to \bigoplus_{i=1}^{m} I(C_i)_{\mathcal{F}}$ is a monomorphism. Since $M$ is $\mathcal{F}$-co-semisimple, we have $(C_i)_{\mathcal{F}} = I(C_i)_{\mathcal{F}}$, $i = 1, \ldots, m$. Thus $I(N)_{\mathcal{F}}$ is isomorphic to a subobject of semisimple object

$\bigoplus_{i=1}^{m}(C_i)_{\mathcal{F}}$ of $(R,\mathcal{F})$-Mod. Hence $I(N)_{\mathcal{F}}$ is semisimple. This implies that $N_{\mathcal{F}} = I(N)_{\mathcal{F}}$. Clearly $I(N)$ is an $\mathcal{X}$-module since it is $M$-injective.

(2) $\Rightarrow$ (3). Let $N$ be a $\mathcal{F}$-torsionfree and $\sigma - \mathcal{F}$-cocyclic left $R$-module in $\sigma[M]$. Then $N = \bigoplus_{i=1}^{n} N_i$, where $N_1, \ldots, N_n$ are $\mathcal{F}$-cocyclic left $R$-modules in $\sigma[M]$. Thus there exist $\mathcal{F}$-cocritical left $R$-modules $C_1, \ldots, C_n \in \sigma[M]$ such that the sequence

$$0 \to N_{\mathcal{F}} \to \bigoplus_{i=1}^{n} I(C_i)_{\mathcal{F}}$$

is exact. Thus $N$ is $\mathcal{F}$-finitely cogenerated in $\sigma[M]$.

(3) $\Rightarrow$ (1). Let $N$ be a $\mathcal{F}$-cocritical left $R$-module in $\sigma[M]$. Then $N$ is $\mathcal{F}$-torsionfree and $N_{\mathcal{F}}$ is a simple object of $(R,\mathcal{F})$-Mod. Since $I(N)$ is $M$-injective, it is easy to see that $I(N)$ is $\mathcal{F}$-closed. Thus $i(I(N)_{\mathcal{F}}) \simeq I(N) \in \sigma[M]$; and hence $i(N_{\mathcal{F}})$, a submodule of $i(I(N)_{\mathcal{F}})$, is in $\sigma[M]$. Since the sequence

$$0 \to (i(N_{\mathcal{F}}))_{\mathcal{F}} \to (i(I(N)_{\mathcal{F}}))_{\mathcal{F}}$$

is exact, and $(i(I(N)_{\mathcal{F}}))_{\mathcal{F}} \simeq I(N)_{\mathcal{F}}$, we see that $i(N_{\mathcal{F}})$ is $\mathcal{F}$-cocyclic in $\sigma[M]$. It is easy to see that $i(I(N)_{\mathcal{F}}) \in \sigma[M]$ is $\mathcal{F}$-cocyclic, too. Thus $i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}})$ is $\sigma - \mathcal{F}$-cocyclic. Clearly $i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}})$ is $\mathcal{F}$-torsionfree. Thus, by condition (3), there exists an $\mathcal{X}$-module $L$ such that $i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}})$ is an essential submodule of $L$ and $(i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}}))_{\mathcal{F}} = L_{\mathcal{F}}$. Thus we get

$$L_{\mathcal{F}} \simeq N_{\mathcal{F}} \oplus I(N)_{\mathcal{F}}.$$

Clearly $L$ is $\mathcal{F}$-closed since it is $\mathcal{F}$-torsionfree and $\mathcal{F}$-injective. Thus

$$i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}}) \simeq i(L_{\mathcal{F}}) \simeq L.$$

This implies that $i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}})$ is an $\mathcal{X}$-module. By definition of $I$-classes it follows that $i(N_{\mathcal{F}})$ is $i(I(N)_{\mathcal{F}})$-injective. Thus there exists a homomorphism $f : i(I(N)_{\mathcal{F}}) \to i(N_{\mathcal{F}})$ such that $f|_{i(N_{\mathcal{F}})} = 1$. This implies that there exists a left $R$-module $H$ such that $i(I(N)_{\mathcal{F}}) = i(N_{\mathcal{F}}) \oplus H$. Thus $I(N)_{\mathcal{F}} = N_{\mathcal{F}} \oplus a(H)$. But $N_{\mathcal{F}}$ is essential in $I(N)_{\mathcal{F}}$ since $N$ is essential in $I(N)$. Thus $N_{\mathcal{F}} = I(N)_{\mathcal{F}}$ and we are done.

(2) $\Rightarrow$ (4). Let a left $R$-module $N \in \sigma[M]$ be $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated in $\sigma[M]$. By (2), there exists an essential extension $L$ of $N$ such that $L$ is an $\mathcal{X}$-module and $N$ is dense in $L$. By [3, Proposition 1.2], we have

$$(\mathrm{Rad}_{\mathcal{F}}(L))_{\mathcal{F}} = \mathrm{Rad}(L_{\mathcal{F}}) = \mathrm{Rad}(N_{\mathcal{F}}).$$

Since $N$ is $\mathcal{F}$-finitely cogenerated in $\sigma[M]$, there exist $\mathcal{F}$-cocritical left $R$-modules $C_1, \ldots, C_n$ in $\sigma[M]$ such that the following sequence is exact:

$$0 \to N_{\mathcal{F}} \to \bigoplus_{i=1}^{n} I(C_i)_{\mathcal{F}}.$$

By the results proved above, $M$ is $\mathcal{F}$-co-semisimple when condition (2) holds. Therefore every $\mathcal{F}$-cocritical left $R$-module in $\sigma[M]$ is dense in its $M$-injective hull. It follows that $I(C_i)_{\mathcal{F}} = (C_i)_{\mathcal{F}}$, $i = 1, \ldots, n$. Thus we have

$$\mathrm{Rad}\left(\bigoplus_{i=1}^{n} I(C_i)_{\mathcal{F}}\right) = \bigoplus_{i=1}^{n} \mathrm{Rad}(I(C_i)_{\mathcal{F}}) = 0,$$

which implies that $\mathrm{Rad}(N_{\mathcal{F}}) = 0$. Thus $\mathrm{Rad}_{\mathcal{F}}(L) = t(\mathrm{Rad}_{\mathcal{F}}(L))$, where $t$ is a left exact radical corresponding to the Gabriel topology $\mathcal{F}$. This means that $\mathrm{Rad}_{\mathcal{F}}(L)$ is a $\mathcal{F}$-torsion module. On the other hand, $\mathrm{Rad}_{\mathcal{F}}(L)$ is a $\mathcal{F}$-saturated submodule of $L$, and so is $\mathcal{F}$-torsionfree. Therefore we get $\mathrm{Rad}_{\mathcal{F}}(L) = 0$, as required.

(4) $\Rightarrow$ (5). It is similar to the implication (2) $\Rightarrow$ (3).

(5) $\Rightarrow$ (3). Let $N$ be a $\mathcal{F}$-torsionfree and $\sigma - \mathcal{F}$-cocyclic left $R$-module in $\sigma[M]$. Then there exists an essential extension $L$ of $N$ such that $L$ is an $\mathcal{X}$-module and $\mathrm{Rad}_{\mathcal{F}}(L) = 0$. It is enough to show that $N$ is dense in $L$. By [3, Proposition 1.3], it follows that $L$ is cogenerated by a class of $\mathcal{F}$-cocritical left $R$-modules. Since $N$ is $\sigma - \mathcal{F}$-cocyclic in $\sigma[M]$, as the implication (2) $\Rightarrow$ (3), we see $N$ is $\mathcal{F}$-finitely cogenerated in $\sigma[M]$. Thus there exist $\mathcal{F}$-cocritical left $R$-modules $C_1, \ldots, C_n$ such that the sequence

$$0 \to N_{\mathcal{F}} \to \bigoplus_{i=1}^{n} I(C_i)_{\mathcal{F}}$$

is exact. By analogy with the implication (1) $\Rightarrow$ (2), we obtain that $I(N)_{\mathcal{F}}$ is isomorphic to a subobject of $\bigoplus_{i=1}^{n} I(C_i)_{\mathcal{F}}$. By [14, 17.10], it follows that $I(N) \simeq I(L)$ since $N$ is essential in $L$. Thus $I(N)_{\mathcal{F}} \simeq I(L)_{\mathcal{F}}$, which implies that $L_{\mathcal{F}}$ can be embedded in $\bigoplus_{i=1}^{n} I(C_i)_{\mathcal{F}}$. This means that $L$ is $\mathcal{F}$-finitely cogenerated in $\sigma[M]$. Thus clearly $L$ is $\mathcal{F}$-finitely cogenerated in $R$-mod. By [3, Proposition 1.7], every family of $\mathcal{F}$-torsionfree modules which cogenerates $L$, does cogenerate it finitely. Thus there exist finite $\mathcal{F}$-cocritical left $R$-modules $D_1, \ldots, D_k$ such that the sequence $0 \to L \to \bigoplus_{i=1}^{k} D_i$ is exact, which implies that the sequence $0 \to L_{\mathcal{F}} \to \bigoplus_{i=1}^{k} (D_i)_{\mathcal{F}}$ is exact. This means that $L_{\mathcal{F}}$ is a semisimple object of $(R, \mathcal{F})$-Mod. On the other hand, $N$ is an essential submodule of $L$; and so $N_{\mathcal{F}}$ is essential in $L_{\mathcal{F}}$ by [3, Lemma 0.1]. It is then easy to see that $N_{\mathcal{F}} = L_{\mathcal{F}}$, in other words $N$ is dense in $L$; and we are done.

The following corollary generalizes a corresponding result of [7].

COROLLARY 3.4. *Let $\mathcal{F}$ be a left Gabriel topology on $R$. If $\mathcal{X}$ is an I-class of left $R$-modules such that every $\mathcal{X}$-module is $\mathcal{F}$-injective, then the following conditions are equivalent.*

*(1) $R$ is a $\mathcal{F}$-V-ring.*

*(2) Every $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated left $R$-module is dense in its some essential extensions which are $\mathcal{X}$-modules.*

*(3) Every $\mathcal{F}$-torsionfree and $\sigma-\mathcal{F}$-cocyclic left $R$-module is dense in its some essential extensions which are $\mathcal{X}$-modules.*

*(4) For every $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated left $R$-module $M$, there exists an $\mathcal{X}$-module $L$ with essential submodule $M$ such that* $\mathrm{Rad}_{\mathcal{F}}(L) = 0$.

*(5) For every $\mathcal{F}$-torsionfree and $\sigma - \mathcal{F}$-cocyclic left $R$-module $M$, there exists an $\mathcal{X}$-module $L$ with an essential submodule $M$ such that* $\mathrm{Rad}_{\mathcal{F}}(L) = 0$.

COROLLARY 3.5. *Let $M$ be a left $R$-module and $\mathcal{X}$ an $I$-class in the category $\sigma[M]$. Then the following assertions are equivalent.*

*(1) $M$ is co-semisimple.*

*(2) Every finitely cogenerated left $R$-module in $\sigma[M]$ is an $\mathcal{X}$-module.*

*(3) Every $\sigma$-cocyclic left $R$-module in $\sigma[M]$ is an $\mathcal{X}$-module.*

For $M = R$, Corollary 3.5 gives characterizations of left $V$-rings by generalized injectivity.

Page and Yousif [10] proved that for a finitely generated left $R$-module $M$, $M$ is a noetherian co-semisimple module if and only if every semisimple left $R$-module is $M$-injective. Recall that a left $R$-module $M$ is locally noetherian if every finitely generated submodule of $M$ is noetherian. We have

PROPOSITION 3.6. *Let $M$ be a left $R$-module and $\mathcal{X}$ an $I$-class in the category $\sigma[M]$. Then the following conditions are equivalent.*

*(1) $M$ is locally noetherian co-semisimple left $R$-module.*

*(2) Every semisimple left $R$-module (in $\sigma[M]$) is $M$-injective.*

*(3) Every semisimple left $R$-module (in $\sigma[M]$) is the direct sum of a finitely cogenerated left $R$-module and an $M$-injective module.*

*(4) For every semisimple left $R$-module $N$ in $\sigma[M]$, every essential extension in $\sigma[M]$ of $N$ is an $\mathcal{X}$-module.*

*(5) For every semisimple left $R$-module $N$ in $\sigma[M]$, every submodule of an essential extension in $\sigma[M]$ of $N$ is an $\mathcal{X}$-module.*

*If $\mathcal{X}$ is closed under direct summands, then the following are also equivalent.*

(6) *For every semisimple left $R$-module $N$ in $\sigma[M]$, every essential extension in $\sigma[M]$ of $N$ is the direct sum of a finitely cogenerated module and an $\mathcal{X}$-module.*

(7) *For every semisimple left $R$-module $N$ in $\sigma[M]$, every submodule of an essential extension in $\sigma[M]$ of $N$ is the direct sum of a finitely cogenerated module and an $\mathcal{X}$-module.*

PROOF. The equivalence of (1), (2) and (3) is proved in [8].

(4) $\Rightarrow$ (2). Let $N$ be a semisimple left $R$-module in $\sigma[M]$. Then $N \oplus I(N)$ is an essential extension of $N \oplus N$. Thus $N \oplus I(N)$ is an $\mathcal{X}$-module by condition (4), which implies that $N$ is $I(N)$-injective by the definition of $I$-class. Now it is easy to see that $N = I(N)$ is $M$-injective.

(2) $\Rightarrow$ (5). Let $N$ be a semisimple left $R$-module in $\sigma[M]$ and $L$ a submodule of an essential extension $D$ in $\sigma[M]$ of $N$. Then $N$ is $M$-injective by (2). Thus $N$ is $D$-injective by [14, 16.3]. Now it is easy to see that $N = D$; and thus $D$ is semisimple and $M$-injective. Therefore $L$, a direct summand of $D$, is $M$-injective. Since $\mathcal{X}$ contains all $M$-injective left $R$-modules in $\sigma[M]$, it follows that every submodule of an essential extension in $\sigma[M]$ of a semisimple left $R$-module in $\sigma[M]$ is an $\mathcal{X}$-module.

The implications (5) $\Rightarrow$ (4), (7) $\Rightarrow$ (6) and (5) $\Rightarrow$ (7) are clear.

(6) $\Rightarrow$ (4). Not that the class of semisimple left $R$-modules is closed under direct sums; it is easy to see that every direct sum of essential extensions in $\sigma[M]$ of semisimple left $R$-modules in $\sigma[M]$ is an essential extension of a semisimple module. Now, by analogy with the proof of the main result of [6], we see that every essential extension in $\sigma[M]$ of a semisimple left $R$-module in $\sigma[M]$ is an $\mathcal{X}$-module.

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