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Bicyclic Extensions - the non-group congruences

R.J. Warne

Abstract

Let T be a monoid with the group of units U. Let θ be a homomorphism of T into U (θ^0 denoting the indentity automorphism of T). Let P and K be disjoint sets and let γ be a homomorphism of T into G_K, the full tranformation group on K. Let $S = ((I^0 \times \{0\}) \times (T \times P)) \cup ((I^0 \times N) \times (T \times K))$ where $N(I^0)$ are the natural numbers (non-negative integers) under the multiplication $((n, k), (g, p))\ ((r, s), (h, q))) = ((n + r - t, k + s - t), (g\theta^{r-t}\ h\theta^{k-t}, x))$ where $t = \min(k, r)$ and $x = q$ or $p(h\theta^{k-r-1}\gamma)$ according to whether $k \leq r$ or $k > r$. We term S a bicyclic extension of T. A congruence $\mathcal{P}$ on S such that $((m, n), (a, r))\ \mathcal{P}\ ((p, q), (b, s))$ implies $m = p$ and $n = q$ is termed a separating congruence. A congruence $\mathcal{P}$ on S which is not a group congruence ($S/\mathcal{P}$ is a group) or a separating congruence is termed a crossover congruence. We characterized the group congruences in a previous paper. In this paper, we characterize the crossover congruences and the separating congruences. These results generalize the corresponding results of Picochi [7] for Bruck-Reilly extensions ($|P| = |K| = 1$) to bicyclic extensions.

The bicyclic semigroup C has played a central role in the theory of simple semigroups. Any simple semigroup S whose structure is not given by the Rees Theorem [3] is locally bicyclic (in the sense that every element of $E(S)$, the set of idempotents of S, is the identity of some bicyclic subsemigroup of S). In fact, many papers have described various classes of simple semigroups as essentially split extensions of groups or unions of groups by C or generalizations of C. For examples see the references [5, 6, 9, {10, 11}, and 12 - 21]. Of the above constructions, the E-bisimple construction [16, 22] is the most preferable in regard to the combined criteria of elegance, simplicity, generality, and applicability. So, we choose it as the prototype for developing a more general structure theory of simple semigroups, and we begin by replacing "group" with "monoid" in this construction. We termed the resulting construction a bicyclic extension in [23, 24]. In [23, 24], we characterized bicyclic extensions of finite chains of groups as simple regular semigroups S such that $E(S)$ is a strong ω-chain of bands ($P_n \times Y : n \in I^0$, the non-negative integers) where P_n is a right zero semigroup ($xy = y$) and Y is a finite chain. This theorem generalizes the structure theorems in [5, 9, {10, 11}, 16]. In [23, 24], we also determined the group congruences of a bicyclic extension. In this paper, we determine the non-group congruences on a bicyclic extension S. A congruence $\mathcal{P}$ on S such that $((m, n), (a, r))\ \mathcal{P}\ ((p, q), (b, s))$ implies $m = p$ and $n = q$ is termed a separating congruence. A congruence $\mathcal{P}$ on S which is not a group congruence or a separating congruence is termed a crossover congruence (following

Proof. First, we show $\mathcal{P}_T$ is a θ-admissible congruence on T. Clearly $\mathcal{P}_T$ is an equivalence relation on T. If $a\ \mathcal{P}_T\ b$ and $c \in T$, $((0,0),(a\,c,p_0)) = ((0,0),(a,p_0))\ ((0,0),(c,p_0))\ \mathcal{P}\ ((0,0),(b,p_0))\ ((0,0),(c,p_0)) = ((0,0),(b\,c,p_0))$. So, $a\,c\ \mathcal{P}_T\ b\,c$. Similarly, $ca\ \mathcal{P}_T\ cb$. So, $\mathcal{P}_T$ is a congruence on T. Next, we show $\mathcal{P}_T$ is θ-admissible. Suppose $g\ \mathcal{P}_T\ h$. Then, $((0,0),(g\theta,p_0)) = ((0,1),(1,q))\,((0,0),(g,p_0))\,((1,0),(1,p_0))\ \mathcal{P}\ ((0,1),(1,q))\ ((0,0),(h,p_0))\,((1,0),(1,p_0)) = ((0,0),(h\theta,p_0))$. So, $g\theta\ \mathcal{P}_T\ h\theta$. It is easily checked that $\mathcal{P}_{T\times P}$ is a left $\mathcal{P}_T$-invariant equivalence relation on $T \times P$. Next, we show $\mathcal{P}_{T\times K}$ is a left $\mathcal{P}_T$-invariant equivalence relation on $T \times K$ such that $(u,v) \in \mathcal{P}_T$ and $(a,r)\ \mathcal{P}_T\ (b,s)$ implies $(a(u\theta), r(u\gamma))\ \mathcal{P}_T\ (b(v\theta), s(v\gamma))$. Clearly, $\mathcal{P}_{T\times K}$ is an equivalence relation on $T \times K$. Suppose $a\ \mathcal{P}_T\ b$ and $(c,r)\ \mathcal{P}_{T\times K}\ (d,s)$. Then, $((0,1),(a\,c,r)) = ((0,0),(a,p_o))\,((0,1),(c,r))\ \mathcal{P}\ ((0,0),(b,p_0))\,((0,1),(d,s)) = ((0,1),(b\,d,s))$. So, $\mathcal{P}_{T\times K}$ is left $\mathcal{P}_T$-invariant. Suppose, $(u,v) \in \mathcal{P}_T$ and $(a,r)\ \mathcal{P}_{T\times K}\ (b,s)$. Thus, $((0,1),(a(u\theta),r(u\gamma))) = ((0,1),(a,r))\,((0,0),(u,p_0))\ \mathcal{P}\ ((0,1),(b,s))\,((0,0),(v,p_0)) = ((0,1),(b(v\theta),s(v\gamma)))$. So, $(a(u\theta),r(u\gamma))\ \mathcal{P}_{T\times K}\ (b(v\theta),s(v\gamma))$.

Theorem 2. *Let $S = (T,P,K,\theta,\gamma)$ be a bicyclic extension. Let Γ be a θ-admissible congruence relation on T, let δ be a left Γ-invariant equivalence relation on $T \times P$, and let λ be a left Γ-invariant equivalence relation on $T \times K$ such that $(u,v) \in \Gamma$ and $(a,k)\ \lambda\ (b,t)$ imply $(a(u\theta),k(u\gamma))\ \lambda\ (b(v\theta),t(v\gamma))$. The following relation $\mathcal{P} = (\Gamma,\delta,\lambda)$ is a separating congruence on S. $((m,n),(a,r)\ \mathcal{P}\ ((p,q),(b,s))$ if and only if $m = p$, $n = q$, $a\ \Gamma\ b$, $(a,r)\ \delta\ (b,s)$ if $n = 0$, and $(a,r)\ \lambda\ (b,s)$ if $n \neq 0$. Conversely, every separating congruence $\mathcal{P}$ on S can be so constructed by means of an appropriate triple (Γ,δ,λ) (in fact, we may take $\Gamma = \mathcal{P}_T$, $\delta = \mathcal{P}_{T\times P}$, and $\lambda = \mathcal{P}_{T\times K}$).*

Proof. We first show $\tau = (\Gamma,\delta,\lambda)$ is a congruence relation on S. It is easily checked that τ is an equivalence relation. Next, we show that τ is compatible. Suppose $((m,n),(a,r))\ \tau\ ((p,q),(b,s))$ and $((h,k),(c,w)) \in S$. Thus, $m = p$, $n = q$, $a\ \Gamma\ b$, and $(a,r)\ \delta\ (b,s)$ if $n = 0$, and $(a,r)\ \lambda\ (b,s)$ if $n \neq 0$. We first show $A = ((m,n),(a,r))\ ((h,k),(c,w))\ \tau\ ((m,n),(b,s))\ ((h,k),(c,w)) = B$. If $h \geq n$, $A = ((m+h-n,k),(a\theta^{h-n}c,w))$ and $B = ((m+h-n,k),(b\theta^{h-n}c,w))$. Since Γ is a θ-admissible congruence, $a\theta^{h-n}c\ \Gamma\ b\theta^{h-n}c$. If $k \neq 0$, then $a\theta^{h-n}c\ \Gamma\ b\theta^{h-n}c$ and $(1,w)\ \lambda\ (1,w)$ imply $(a\theta^{h-n}c,w)\ \lambda\ (b\theta^{h-n}c,w)$ since λ is a left Γ-invariant equivalence relation. If $k = 0$, $(a\theta^{h-n}c,w)\ \delta\ (b\theta^{h-n}c,w)$ since δ is a left Γ-invariant equivalence relation. Next, assume $n > h$. Then, $A = ((m,n+k-h),(a(c\theta^{n-h}),r(c\theta^{n-h-1}\gamma))$ and $B = ((m,n+k-h),(b(c\theta^{n-h}),s(c\theta^{n-h-1}\gamma))$.

Note, $a(c\theta^{n-h})\ \Gamma\ b(c\theta^{n-h})$ and $n+k-h = k+(n-h) \neq 0$. Since $(a,r)\ \lambda\ (b,s)$ (as $n \neq 0$) and $c\theta^{n-h-1}\ \Gamma\ c\theta^{n-h-1}$, $(a(c\theta^{n-h}), r(c\theta^{n-h-1}\gamma))\ \lambda\ (b(c\theta^{n-h}), s(c\theta^{n-h-1}\gamma))$. So $A\ \tau\ B$. Similarly, τ is a left congruence, so τ is a congruence.

Let $\mathcal{P}$ be any congruence on S such that $((m,n),(a,r))\ \mathcal{P}\ ((p,q),(b,s))$ implies $m=p$ and $n=q$. So, $((0,m),(1,x))\ ((m,n),(a,r))\ ((n,0),(1,p_0))\ \mathcal{P}\ ((0,m),(1,x))\ ((m,n),(b,s))\ ((n,0),(1,p_0))$. Thus, $((0,0),(a,p_0))\ \mathcal{P}\ ((0,0),(b,p_0))$. Hence, $a\ \mathcal{P}_T\ b$. Suppose, $n \neq 0$. Note $((0,m),(1,x))\ ((m,n),(a,r))\ \mathcal{P}\ ((0,m),(1,x))\ ((m,n),(b,s))$. Thus, $((0,n),(a,r))\ \mathcal{P}\ ((0,n),(b,s))$. So, $((0,n),(a,r))\ ((n-1,0),(1,p_0))\ \mathcal{P}\ ((0,n),(b,s))\ ((n-1,0),(1,p_0))$. Thus, $((0,1),(a,r))\ \mathcal{P}\ ((0,1),(b,s))$. Thus, $(a,r)\ \mathcal{P}_{T\times K}\ (b,s)$. If $n=0$, $((m,0),(a,r))\ \mathcal{P}\ ((m,0),(b,s))$. Hence, $((0,m),(1,x))\ ((m,0),(a,r))\ \mathcal{P}\ ((0,m),(1,x))\ ((m,0),(b,s))$. So, $((0,0),(a,r))\ \mathcal{P}\ ((0,0),(b,s))$. Hence, $(a,r)\ \mathcal{P}_{T\times P}\ (b,s)$. So $\mathcal{P} \subseteq (\mathcal{P}_T, \mathcal{P}_{T\times P}, \mathcal{P}_{T\times K})$. Conversely, suppose $((m,n),(a,r))\ (\mathcal{P}_T, \mathcal{P}_{T\times P}, \mathcal{P}_{T\times K})\ ((p,q),(b,s))$. Thus, $m=p$, $n=q$, $a\ \mathcal{P}_T\ b$, $(a,r)\ \mathcal{P}_{T\times P}\ (b,s)$ if $n=0$, and $(a,r)\ \mathcal{P}_{T\times K}\ (b,s)$ if $n \neq 0$. Suppose $n=0$. Then, $((0,0),(a,r))\ \mathcal{P}\ ((0,0),(b,s))$. So, $((m,0),(1,p_0))\ ((0,0),(a,r))\ \mathcal{P}\ ((m,0),(1,p_0))\ ((0,0),(b,s))$. Hence, $((m,0),(a,r))\ \mathcal{P}\ ((m,0),(b,s))$. So, $((m,n),(a,r))\ \mathcal{P}\ ((m,n),(b,s))$. Suppose $n \neq 0$. Then, $((0,1),(a,r))\ \mathcal{P}\ ((0,1),(b,s))$. Suppose $k>0$ and $((0,k),(a,r))\ \mathcal{P}\ ((0,k),(b,s))$. Then, $((0,k),(a,r))\ ((0,1)(1,x))\ \mathcal{P}\ ((0,k),(b,s))\ ((0,1)(1,x))$. Hence, $((0,k+1),(a,r))\ \mathcal{P}\ ((0,k+1),(b,s))$. So, $((0,n),(a,r))\ \mathcal{P}\ ((0,n),(b,s))$. Hence, $((m,0),(1,p_0)) \cdot ((0,n),(a,r))\ \mathcal{P}\ ((m,0),(1,p_0))\ ((0,n),(b,s))$. So, $((m,n),(a,r))\ \mathcal{P}\ ((m,n),(b,s))$. So, $((m,n),(a,r))\ \mathcal{P}\ ((p,q),(b,s))$. Thus, $(\mathcal{P}_T, \mathcal{P}_{T\times P}, \mathcal{P}_{T\times K}) \subseteq \mathcal{P}$. So, $\mathcal{P} = (\mathcal{P}_T, \mathcal{P}_{T\times P}, \mathcal{P}_{T\times K})$. Apply Lemma 1.

Let S be a bicyclic extension of a monoid T. We write $S = (T,P,K,\theta,\gamma)$.

If T is a group, S is termed an E-bisimple semigroup [16,22]. In this case, a congruence $\mathcal{P}$ on S is termed block separating if $((n,n),(1,s))\ \mathcal{P}\ ((m,m),(1,t))$ implies $n=m$.

Lemma 3 [22, Theorem 3.1]). *Let $\mathcal{P}$ be a congruence relation on an E-bisimple semigroup. Then, either $\mathcal{P}$ is a group congruence ($S/\mathcal{P}$ is a group) or $\mathcal{P}$ is a block separating congruence.*

Let $\overline{C} = (I^0 \times \{0\}) \times (\{1\} \times P) \cup (I^0 \times N) \times (\{1\} \times K)$ under the multiplication $((m,n),(1,r))\ ((p,q),(1,s)) = (m+p-t, n+q-t),(1,x))$ where $x=s$ if $n \leq p$ and $x=r$ if $p>n$. Then, $\overline{C}$ is an E-bisimple semigroup ([22, Theorem 1.1]). Thus, any congruence $\mathcal{P}$ on $\overline{C}$ is either a group congruence or a block separating congruence by Lemma 3.

If $\mathcal{P}$ is a congruence on S, let $\mathcal{P}_{\overline{C}} = \mathcal{P}/\overline{C}$. If X is a semigroup, $E(X)$ will denote its set of idempotents.

Theorem 4. *For a congruence $\mathcal{P}$ on S, exactly one of the following conditions hold:*

1) *$\mathcal{P}$ is a group congruence. This is valid if and only if $\mathcal{P}_{\overline{C}}$ is a group congruence on $\overline{C}$.*

2) *$((m,n),(a,r))\ \mathcal{P}\ ((p,q),(b,s))$ if and only if $m = p$, $n = q$, $a\ \mathcal{P}_T\ b$, $(a,r)\ \mathcal{P}_{T\times P}\ (b,s)$ if $n = 0$, and $(a,r)\ \mathcal{P}_{T\times K}\ (b,s)$ if $n \neq 0$ (i.e. $\mathcal{P} = (\mathcal{P}_T, \mathcal{P}_{T\times P}, \mathcal{P}_{T\times K})$ is a separating congruence).*

3) *$\mathcal{P}$ is a not group congruence and there exists at least one pair of elements of S such that $((m,n),(a,r))\ \mathcal{P}\ ((p,q),(b,s))$ with $m \neq p$ or $n \neq q$ (i.e. $\mathcal{P}$ is a crossover congruence).*

For any such pair, $m = p+1$, $n = q+1$ and, $a\ \mathcal{P}_T\ b\theta$; or $p = m+1$, $q = n+1$ and $b\ \mathcal{P}_T\ a\theta$.

For all such pairs $((m',n'),(a',r'))\ \mathcal{P}\ ((p',q'),(b',s'))$, $m' = p'$ if and only if $n' = q'$ and then $a'\ \mathcal{P}_T\ b'$, $(a',r')\ \mathcal{P}_{T\times P}\ (b',s')$ if $n' = 0$, $(a',r')\ \mathcal{P}_{T\times K}\ (b',s')$ if $n' \neq 0$.

$(1,k_1)\ \mathcal{P}_{T\times K}\ (1,k_2)$ for all $k_1, k_2 \in K$.

Proof. It is easy to see that (1), (2), and (3) are mutually exclusive. We first show that $\mathcal{P}$ is a group congruence if and only if $\mathcal{P}_{\overline{C}}$ is a group congruence. Suppose $\mathcal{P}$ is a group congruence on S. Then $e\ \mathcal{P}_{\overline{C}}\ f$ for all $e, f \in E(\overline{C})$. So, $\overline{C}/\mathcal{P}_{\overline{C}}$ is a regular semigroup with precisely one idempotent and hence a group. Conversely, suppose $\mathcal{P}_{\overline{C}}$ is a group congruence on $\overline{C}$. Thus noting [22, Lemma 2.1(c)], $((n,n),(1,p))\ \mathcal{P}\ ((k,k),(1,q))$ for all $n, k \in I^0$, $p \in P$ $(n = 0)$ $(p \in K$ $(n \neq 0))$, $q \in P$ $(k = 0)$ $(q \in K$ $(k \neq 0))$. Let $a = ((m,n),(g,p)) \in S$. Thus, for $s \in K$ $((n,m),(g,p))\ ((m+1,n+1),((g\theta)^{-1},s)) = ((n+1,n+1),(g\theta(g\theta)^{-1},s)) = ((n+1,n+1),(1,s))$. Let $x = ((m+1,n+1),((g\theta)^{-1},s))$ and let $e_0 = ((0,0),(1,p_0))$. So, $a\mathcal{P}x\mathcal{P} = e_0\mathcal{P}$. Furthermore, $((m+1,n+1),((g\theta)^{-1},u))\ ((n,m),(g,p)) = ((m+1,m+1),((g\theta)^{-1}g\theta, u(g\gamma))) = ((m+1,m+1),(1,u(g\gamma)))$. Let $y = ((m+1,n+1),((g\theta)^{-1},u))$. So, $y\mathcal{P}a\mathcal{P} = e_0\mathcal{P}$. Let $b = ((r,s),(h,t)) \in S$. Then, $((r,s),(h,t))\ ((s,s),(1,t)) = ((r,s),(h,t)) = ((r,r),(1,v)) \cdot ((r,s),(h,t))$. So, $b\mathcal{P}e_0\mathcal{P} = e_0\mathcal{P}b\mathcal{P} = b\mathcal{P}$. Hence, $S/\mathcal{P}$ is a group.

Suppose $\mathcal{P}$ is not a group congruence on S. Then, $\mathcal{P}_{\overline{C}}$ is not a group congruence on $\overline{C}$. Suppose $\mathcal{P}|S_0$ where $S_0 = (U, P, K, \theta|U, \gamma|U)$ is a group congruence. Thus, $((k,k),(1,s))\ \mathcal{P}\ ((q,q),(1,t))$ for $k, q \in I^0$, $s \in K(k \neq 0)$, $s \in P(k = 0)$, $t \in K(q \neq 0)$, $t \in P(q = 0)$. So, $\mathcal{P}_{\overline{C}}$ is a group congruence on $\overline{C}$, a contradiction. Thus, $\mathcal{P}|S_0$ is a block separating congruence on S_0 by Lemma 3. Suppose $((m,n),(a,r))\ \mathcal{P}\ ((p,q),(b,s))$ (in S). Let $k > \max(n,q)$. Thus, $((m+k-n,k),(a\theta^{k-n},s)) = ((m,n),(a,r))\,((k,k),(1,s))\ \mathcal{P}\ ((p,q),(b,s))\,((k,k),(1,s)) = ((p+k-q,k),(b\theta^{k-q},s))$. Hence, by [22, Lemma 4.4], $m+k-n = p+k-q$ or $m-n = p-q$. So, $m-p = n-q$. Thus, $m = p$ if and only if $n = q$. Suppose $((m,n),(a,r))\ \mathcal{P}\ ((p,q),(b,s))$ implies $m = p$ and $n = q$. Hence, $\mathcal{P}$ is a separating congruence and condition (2) follows from Theorem 2. Suppose condition (2) is not valid. Then, again by Theorem 2, there exist $((m,n),(a,s)) \in S$, $((p,q),(b,t)) \in S$ such that $((m,n),(a,s))\ \mathcal{P}\ ((p,q),(b,t))$ with $m \neq p$ or $n \neq q$. Hence $m \neq p$ and $n \neq q$ with $m-p = n-q$. Suppose $q < n$ (hence $p < m$).

We first show $a\ \mathcal{P}_T\ \theta^{m-p}$. Note, for $v \in P$, $((0,0),(a,v)) = ((0,m),(1,z))\,((m,n),(a,s))\,((n,0),(1,v))\ \mathcal{P}\ ((0,m),(1,z))\,((p,q),(b,t))\,((n,0),(1,v)) = ((0,m+q-p),(b\theta^{m-p},z(b\theta^{m-p-1}\gamma)))\,((n,0),(1,v)) = ((0,0),(b\theta^{m-p},v))$. So

(a) $\quad a\ \mathcal{P}_T\ \theta^{m-p}$.

Furthermore, for $w \in P$, $((0,0),(b,w)) = ((0,p),(1,r))\,((p,q),(b,t))\,((q,0),(1,w))\ \mathcal{P}\ ((0,p),(1,r))\,((m,n),(a,s))\,((q,0),(1,w)) = ((m-p,n),(a,s))\,((q,0),(1,w)) = ((m-p,n-q),(a,s))$. So,

(b) $\quad ((m-p,m-p),(a,s))\ \mathcal{P}\ ((0,0),(b,w))$ for all $w \in P$.

Using (a), for $\overline{p}_0 \in K$, $((m-p,m-p),(a,\overline{p}_0)) = ((m-p,0),(1,p_0))\,((0,0),(a,p_0))((0,m-p),(1,\overline{p}_0))\ \mathcal{P}\ ((m-p,0),(1,p_0))\,((0,0),(b\theta^{m-p},p_0))\,((0,m-p),(1,\overline{p}_0)) = ((m-p,m-p),(b\theta^{m-p},\overline{p}_0))$. Hence, $((m-p,m-p),(1,\overline{p}_0)) = ((m-p,m-p),(b\theta^{m-p},\overline{p}_0))\,((m-p,m-p),((b\theta^{m-p})^{-1},\overline{p}_0))\ \mathcal{P}\ ((m-p,m-p),(a,\overline{p}_0))((m-p,m-p),((b\theta^{m-p})^{-1},\overline{p}_0)) = ((m-p,m-p),(a(b\theta^{m-p})^{-1},\overline{p}_0))$. So,

(c) $\quad ((m-p,m-p),(1,\overline{p}_0))\ \mathcal{P}\ ((m-p,m-p),(a(b\theta^{m-p})^{-1},\overline{p}_0))$ for all $\overline{p}_0 \in K$.

Note that for $\overline{p}_0 \in K$,

$$((m-p,m-p),(a,s))\cdot((1,1),((b\theta)^{-1},\overline{p}_0)) = \begin{cases} ((m-p,m-p),(a(b\theta^{m-p})^{-1},s(b\theta^{m-p-1})^{-1}\gamma)) & \text{if } m-p > 1 \\ ((m-p,m-p),(a(b\theta^{m-p})^{-1},\overline{p}_0)) & \text{if } m-p = 1 \end{cases}$$

Thus, using (c), $((m-p,m-p),(a,s))\,((1,1),((b\theta)^{-1},\overline{p}_0))\ \mathcal{P}\ ((m-p,m-p),(1,\overline{\overline{p}}_0))$ for some $\overline{\overline{p}}_0 \in K$. Hence using (b), $((m-p,m-p),(1,\overline{\overline{p}}_0))\ \mathcal{P}\ ((0,0),(b,w))\,((1,1),((b\theta)^{-1},\overline{p}_0)) = ((1,1),(1,\overline{p}_0))$. Since $\mathcal{P}_{\overline{C}}$ is a block separating congruence, $m-p = 1$. So, $n-q = 1$. Thus, $m = p+1$, $n = q+1$, and by (a) $a\ \mathcal{P}_T\ b\theta$. We have shown that if $((m',n'),(a',r'))\ \mathcal{P}\ ((p',q'),(b',s'))$, then $m' = p'$ iff $n' = q'$. Using the proof of Theorem 2, then $a\ \mathcal{P}_T\ b$, $(a',r')\ \mathcal{P}_{T\times P}\ (b',s')$ if $n' = 0$, and $(a',r')\ \mathcal{P}_{T\times K}\ (b',s')$ if $n' \neq 0$.

Finally, we show $((1,k_1)\mathcal{P}_{T\times K}(1,k_2)$ for all $k_1,k_2 \in K$. We have shown there exists at least one pair of elements of S such that $((p+1,q+1),(a,r))\ \mathcal{P}\ ((p,q),(b,s))$. Thus for $v \in P$, $((1,1),(a,r)) = ((0,p),(1,u)) \cdot ((p+1,q+1),(a,r))\,((q,0),(1,v))\ \mathcal{P}\ ((0,p),(1,u))\,((p,q),(b,s))\,((q,0),(1,v)) = ((0,0),(b,v))$. So, for $t \in K$, $((0,1),(a,r)) = ((0,1),(1,t))\,((1,1),(a,r))\ \mathcal{P}\ ((0,1),(1,t))\,((0,0),(b,v)) = ((0,1),(b\theta,t(b\gamma)))$. So, let $k \in K$. Thus, $k = t(b\gamma)$ for some $t \in K$ since $b\gamma \in G_K$. Hence $((0,1),(a,r))\ \mathcal{P}\ ((0,1),(b\theta,k))$ for all $k \in K$. Thus if $k_1,k_2 \in K$, $((0,1),(1,k_1)) = ((0,0),((b\theta)^{-1},p_0))\,((0,1),(b\theta,k_1))\ \mathcal{P}\ ((0,0),((b\theta)^{-1},p_0))\,((0,1),(b\theta,k_2)) = ((0,1),(1,k_2))$. The congruence given in condition (3) is termed a crossover congruence by Piochi [7].

Lemma 5. *Let $\mathcal{P}$ be a crossover congruence on $S = (T,P,K,\theta,\gamma)$ and let $B = \{b \in T \mid ((1,1),((b\theta),t)\ \mathcal{P}\ ((0,0),(b,s))$ for all $t \in K$ and all $s \in P\}$. Then, B is a non-empty ideal of T different from T, B is $\mathcal{P}_T$ saturated (i.e. $u\ \mathcal{P}_T\ v$ and $u \in B$ implies $v \in B$), and for $x,y \in B$, $x\ \mathcal{P}_T\ y$ iff $x\theta\ \mathcal{P}_T\ y\theta$.*

Proof. We first show that $B \neq \emptyset$. By the proof of Theorem 4, there exists $a,b \in T$ and $r \in K$ such that $((1,1),(a,r))\ \mathcal{P}\ ((0,0),(b,v))$ for all $v \in P$. Thus, for $t \in K$, $((1,1),(a,r)) = ((1,1),(1,t))\,((1,1),(a,r))\ \mathcal{P}\ ((1,1),(1,t))\,((0,0),(b,v)) = ((1,1),(b\theta,t(b\gamma)))$. So, $((1,1),(b\theta,k))\ \mathcal{P}\ ((0,0),(b,v))$ for all $k \in K$ and all $v \in P$. Hence, $b \in B$ and $B \neq \emptyset$.

We next show B is an ideal of T. Let $b \in B$ and $c \in T$. For all $s \in P$ and $t \in K$, $((0,0),(b\,c,s)) = ((0,0),(b,s))\,((0,0),(c,s))\ \mathcal{P}\ ((1,1),(b\theta,t))\,((0,0),(c,s)) = ((1,1),((b\,c)\theta,t(c\gamma)))$. So, $((0,0),(b\,c,s))\ \mathcal{P}\ ((1,1),((b\,c)\theta,k))$ for all $s \in P$ and $k \in K$. Thus, $b\,c \in B$. Similarly, $cb \in B$. We next show $B \neq T$. Suppose $1 \in B$. Then, $((1,1),(1,t))\ \mathcal{P}\ ((0,0),(1,s))$ for all $t \in K$ and $s \in P$. **This is impossible** since $\mathcal{P}_{\overline{C}}$ is not a group congruence in $\overline{C}$ by Theorem 4 and hence must be a block separating congruence on $\overline{C}$ by Lemma 3. So, $B \neq T$. Next, we show B is is $\mathcal{P}_T$ saturated. Let $x,y \in B$ and suppose $x\ \mathcal{P}_T\ y$ and $x \in B$. Then, for all $s \in P$ and $k \in K$, $((0,0),(y,s))\ \mathcal{P}\ ((0,0),(x,s))\ \mathcal{P}\ ((1,1),(x\theta,k)) = ((0,0),(x,s))\,((1,1),(1,k))\ \mathcal{P}\ ((0,0),(y,s))\,((1,1),(1,k)) = ((1,1),(y\theta,k))$. Hence, $y \in B$. Finally, let $x,y \in B$ and assume $x\theta\ \mathcal{P}_T\ y\theta$. So, for all $k \in K$, $((1,1),(x\theta,k)) = ((1,0),(1,p_0))\,((0,0),(x\theta,p_0))\,((0,1),(1,k))\ \mathcal{P}\ ((1,0),(1,p_0))\,((0,0),(y\theta,p_0))\,((0,1),(1,k)) = ((1,1),(y\theta,k))$. Hence, $((0,0),(x,p_0))\ \mathcal{P}\ ((1,1),(x\theta,k))\ \mathcal{P}\ ((1,1),(y\theta,k))\ \mathcal{P}\ ((0,0),(y,p_0))$. So, $x\ \mathcal{P}_T\ y$. By Theorem 1, $x\ \mathcal{P}_T\ y$ implies $x\theta\ \mathcal{P}_T\ y\theta$.

Lemma 6. *Let $\mathcal{P}$ be a crossover congruence on S. If $a \in B$ and $a\ \mathcal{P}_T\ b$, then $((0,0),(a,r))\ \mathcal{P}\ ((0,0),(b,s))$ for all $r, s \in P$.*

Proof. First note $b \in B$ and $a\theta\ \mathcal{P}_T\ b\theta$ by Lemma 5 and Lemma 1. Thus, for $k \in K$, $((1,1),(a\theta,k)) = ((1,0),(1,p_0))\,((0,0),(a\theta,p_0))\,((0,1),(1,k))\ \mathcal{P}\ ((1,0),(1,p_0))\,((0,0),(b\theta,p_0))\,((0,1),(1,k)) = ((1,1),(b\theta,k))$. Hence, for all $r, s \in P$, $((0,0),(a,r))\ \mathcal{P}\ ((1,1),(a\theta,k))\ \mathcal{P}\ ((1,1),(b\theta,k))\ \mathcal{P}\ ((0,0),(b,s))$.

Lemma 7. *Let $\mathcal{P}$ be a crossover congruence relation on S and suppose $((p+1,q+1),(a,r))\ \mathcal{P}\ ((p,q),(b,s))$. Then, $b \in B$.*

Proof. Let $v \in P$. Then, $((1,1),(a,r)) = ((0,p),(1,u))\,((p+1,q+1),(a,r))\,((q,0),(1,v))\ \mathcal{P}\ ((0,p),(1,u)) \cdot ((p,q),(b,s))\,((q,0),(1,v)) = ((0,0),(b,v))$. Let $t_0 \in K$. Then, $((1,1),(a,r)) = ((1,1),(1,t_0)) \cdot ((1,1),(a,r))\ \mathcal{P}\ ((1,1),(1,t_0))\,((0,0),(b,v)) = ((1,1),(b\theta,t_0(b\gamma)))$. So, $((0,0),(b,v))\ \mathcal{P}\ ((1,1),(a,r))\ \mathcal{P}\ ((1,1),(b\theta,t))$ for all $v \in P$ and $t \in K$. Hence, $b \in B$.

Theorem 8. *Let $S = (T,P,K,\theta,\gamma)$. Let Γ be a congruence relation on T, δ be an equivalence relation on $T \times P$, and B be a non-empty ideal of T such that*

1. *Γ is θ-admissible*
2. *δ is left Γ-invariant*
3. *$B \neq T$ and B is Γ-saturated*
4. *For $x, y \in B$, $x\ \Gamma\ y$ if and only if $x\theta\ \Gamma\ y\theta$*

Then, the following relation $\mathcal{P} = (\Gamma,\delta,B)$ is a crossover congruence relation on S:

$$((m,n),(a,r))\ \mathcal{P}\ ((p,q),(b,s)) \Leftrightarrow \begin{cases} m=p,\ n=q,\ a\ \Gamma\ b, \text{ and if } n=0,\ a \in B \text{ or } (a,r)\ \delta\ (b,s) & (I) \\ m=p+1,\ n=q+1,\ a\ \Gamma\ b\theta,\ b \in B & (II) \\ p=m+1,\ q=n+1,\ b\ \Gamma\ a\theta,\ a \in B & (III) \end{cases}$$

Conversely, every crossover congruence $\mathcal{P}$ on S can be so constructed by means of an appropriate triple (Γ,δ,B).

Proof. Let $\mathcal{P}$ be a crossover congruence on S. Let $B = \{b \in T : ((0,0),(b,u))\ \mathcal{P}\ ((1,1),(b\theta,k))$ for all $u \in P$ and $k \in K\}$. Let $\Gamma = \mathcal{P}_T$ and let $\delta = \mathcal{P}_{T\times P}$. By Lemma 1 and Lemma 5, $\Gamma = \mathcal{P}_T$ is a θ-admissible congruence on T and $\delta = \mathcal{P}_{T\times P}$ is a left $\mathcal{P}_T$-invariant equivalence relation on $T \times P$, and

B is a non-empty ideal of T such that 3-4 is valid. Let $\lambda = (\mathcal{P}_T, \mathcal{P}_{T\times P}, B)$ and suppose $((m,n),(a,r))\ \lambda\ ((p,q),(b,s))$. First assume $m = p$, $n = q$, $a\ \mathcal{P}_T\ b$, and if $n = 0$, $a \in B$ or $(a,r)\ \mathcal{P}_{T\times P}\ (b,s)$. First, suppose $n \neq 0$. Then, by Theorem 4, $((0,1),(1,r))\ \mathcal{P}\ ((0,1),(1,s))$. So, if $((0,k),(1,r))\ \mathcal{P}\ ((0,k),(1,s))$, $((0,k+1),(1,r)) = ((0,1),(1,r))\,((0,k),(1,r))\ \mathcal{P}\ ((0,1),(1,s))\,((0,k),(1,s)) = ((0,k+1),(1,s))$ for $k \geq 1$. Thus, $((0,n),(1,r))\ \mathcal{P}\ ((0,n),(1,s))$. Hence, $((m,n),(a,r)) = ((m,0),(1,p_0))\,((0,0),(a,p_0))\,((0,n),(1,r))\ \mathcal{P}\ ((m,0),(1,p_0))\,((0,0),(b,p_0))\,((0,n),(1,s)) = ((m,n),(b,s))$. Next, assume $n = 0$. If $a \in B$, $((0,0),(a,r))\ \mathcal{P}\ ((0,0),(b,s))$ by Lemma 6. Thus, $((m,0),(a,r)) = ((m,0),(1,p_0))\,((0,0),(a,r))\ \mathcal{P}\ ((m,0),(1,p_0)) \cdot ((0,0),(b,s)) = ((m,0),(b,s))$. Similarly, if $a \notin B$, $(a,r)\ \mathcal{P}_{T\times P}\ (b,s)$ implies $((m,0),(a,r))\ \mathcal{P}\ ((m,0),(b,s))$. Next, assume $m = p+1$, $n = q+1$, $a\ \mathcal{P}_T\ b\theta$, and $b \in B$. Thus, ((m,n),(a,r)) $= ((p,0),(1,p_0))\,((1,1),(a,r)) \cdot ((0,q),(1,s))\ \mathcal{P}\ ((p,0),(1,p_0))\,((1,1),(b\theta,r))\,((0,q),(1,s))\ \mathcal{P}\ ((p,0),(1,p_0))\,((0,0),(b,p_0))\,((0,q),(1,s)) = ((p,q),(b,s))$. Dually, $p = m+1$, $q = n+1$, $b\ \mathcal{P}_T\ a\theta$ and $a \in B$ imply $((p,q),(b,s))\ \mathcal{P}\ ((m,n),(a,r))$. Thus, $\lambda \subseteq \mathcal{P}$. Conversely, let $((m,n),(a,r))\ \mathcal{P}\ ((p,q),(b,s))$. First assume $m = p$ and $n = q$. Then, pre-multiplying by $((0,m),(1,x))$ and post-multiplying by $((n,0),(1,p_0))$ gives $((0,0),(a,p_0))\ \mathcal{P}\ ((0,0),(b,p_0))$ or $a\ \mathcal{P}_T\ b$. If $n = 0$ and $a \notin B$, pre-multiplying by $((0,m),(1,x))$ gives $(a,r)\ \mathcal{P}_{T\times P}\ (b,s)$. Suppose $m \neq p$ or $n \neq q$. Then, by Theorem 4 and Lemma 7, $m = p+1$, $n = q+1$, $a\ \mathcal{P}_T\ b\theta$, and $b \in B$ or dually $p = m+1$, $q = n+1$, $b\ \mathcal{P}_T\ a\theta$, and $a \in B$. Thus, $\mathcal{P} \subseteq \lambda$ and $\mathcal{P} = (\mathcal{P}_T, \mathcal{P}_{T\times P}, B)$.

We now show $\mathcal{P} = (\Gamma, \delta, B)$ is a congruence relation on S. Trivially, $\mathcal{P}$ is reflexive. Since $a \in B$ and $a\ \Gamma\ b$ imply $b \in B$ by (3), $\mathcal{P}$ is trivially symmetric. We next show that $\mathcal{P}$ is transitive. Let $x = ((m,n),(a,r))$, $y = ((p,q),(b,s))$, $z = ((u,v),(c,w))$, and suppose $x\ \mathcal{P}\ y$ and $y\ \mathcal{P}\ z$. We consider several cases. Assume $x\ \mathcal{P}\ y$ by (I) and $y\ \mathcal{P}\ z$ by (I). Thus, $m = u$, $n = v$, and $a\ \Gamma\ c$. If $n \neq 0$, $x\ \mathcal{P}\ z$ by (I). Suppose $n = 0$. By (3), $a \in B$ iff $b \in B$ iff $c \in B$. So, if $a \notin B$, $(a,r)\ \delta\ (b,s)\ \delta\ (c,w)$. Hence, $x\ \mathcal{P}\ z$ by (I). If $a \in B$, $x\ \mathcal{P}\ z$ by (I). Suppose $x\ \mathcal{P}\ y$ by (II) and $y\ \mathcal{P}\ z$ by (I). Then, $m = u+1$ and $n = v+1$. Since $b\ \Gamma\ c$, $b\theta\ \Gamma\ c\theta$ by (1). So, $a\ \Gamma\ c\theta$. Since $b \in B$ and $b\ \Gamma\ c$, $c \in B$ by (3). Thus, $x\ \mathcal{P}\ z$ by (II). Suppose $x\ \mathcal{P}\ y$ by (III) and $y\ \mathcal{P}\ z$ by (I). Then, $u = m+1$, $v = n+1$, $c\ \Gamma\ b\ \Gamma\ a\theta$ and $a \in B$. Suppose $x\ \mathcal{P}\ y$ by (III) and $y\ \mathcal{P}\ z$ by (II). Thus, $m = p-1 = u$ and $n = q-1 = v$. Since $a\theta\ \Gamma\ b\ \Gamma\ c\theta$ and $a, c \in B$, $a\ \Gamma\ c$ by (4). So, if $n \neq 0$, $x\ \mathcal{P}\ z$ by (I). Since $a \in B$, this is also valid for $n = 0$. The dual cases are similar. Assume $x\ \mathcal{P}\ y$ by (II) and $y\ \mathcal{P}\ z$ by (II). Since $b\ \Gamma\ c\theta$ and $b \in B$, $c\theta \in B$ by (3). So, $1 = (c\theta)(c\theta)^{-1} \in B$

and $B = T$ which contradicts (3). Hence this case is impossible. Similarly, the case $x \mathcal{P} y$ by (III) and $y \mathcal{P} z$ by (III) is impossible. Hence, $\mathcal{P}$ is an equivalence relation. We next show $\mathcal{P}$ is a congruence relation. First assume $((m,n),(a,r)) \mathcal{P} ((p,q),(b,s))$ by (I). Thus, $m = p$, $n = q$ and $a \Gamma b$, and if $n = 0$, $a \in B$ or $(a,r) \delta (b,s)$. We show $C = ((h,k),(c,w))((m,n),(a,r)) \mathcal{P} ((h,k),(c,w))((m,n),(b,s)) = D$. If $k \leq m$, $C = ((h+m-k,n),(c\theta^{m-k}a,r))$ and $D = ((h+m-k,n),(c\theta^{m-k}b,s))$. If $n \neq 0$, $C \mathcal{P} D$ by (I). If $n = 0$ and $a \in B$, $C \mathcal{P} D$ by (I). If $n = 0$ and $(a,r) \delta (b,s)$, then $(c\theta^{m-k}a,r) \delta (c\theta^{m-k}b,s)$ by (2). So, $C \mathcal{P} D$ by (I). If $k > m$, $C = ((h,k+n-m),(c(a\theta^{k-m}),w(a\theta^{k-m-1}\gamma)))$ and $D = ((h,k+n-m),(c(b\theta^{k-m}),w(b\theta^{k-m-1}\gamma)))$. Since $k > m$, $k+n-m > 0$ and, hence, $C \mathcal{P} D$ by (I). So, in case (I) $\mathcal{P}$ is a left congruence. Similarly, $\mathcal{P}$ is a right congruence in this case. Next, assume $((m,n),(a,r)) \mathcal{P} ((p,q),(b,s))$ by (II). Hence, $m = p+1$, $n = q+1$, $a \Gamma b\theta$ and $b \in B$. We will show $E = ((m,n),(a,r))((h,k),(c,w)) \mathcal{P} ((p,q),(b,s))((h,k),(c,w)) = F$. Assume $h \geq n > q$. Then, $E = ((m+h-n,k),(a\theta^{h-n}c,w))$ and $F = ((p+h-q,k),(b\theta^{h-q}c,w))$. So, $m+h-n = p+1+h-q-1 = p+h-q$, $k = k$, and $a\theta^{h-n}c = a\theta^{h-q-1}c \,\Gamma\, (b\theta)\theta^{h-q-1}c = b\theta^{h-q}c$. Thus, if $k \neq 0$, $E \mathcal{P} F$ by (I). Let $k = 0$. Since $a\theta^{h-n}c \,\Gamma\, b\theta^{h-q}c$ and $(1,w) \delta (1,w)$, $(a\theta^{h-n}c,w) \delta (b\theta^{h-q}c,w)$ by (2). So, $E \mathcal{P} F$ by (I). Next, assume $h < q < n$. Then, $E = ((m,n+k-h),(a(c\theta^{n-h}),r(c\theta^{n-h-1}\gamma)))$ and $F = ((p,q+k-h),(b(c\theta^{q-h}),s(c\theta^{q-h-1}\gamma)))$. Since $m = p+1$, $k-h+n = (k-h+q)+1$, $a(c\theta^{n-h}) \,\Gamma\, (b\theta)(c\theta^{n-h}) = (b(c\theta^{n-h-1}))\theta = (b(c\theta^{q-h}))\theta$, and $b(c\theta^{q-h}) \in B$, $E \mathcal{P} F$ by (II). Finally, assume $h = q < n$. Then, $E = ((m,n+k-h),(a(c\theta^{n-h}),r(c\theta^{n-h-1}\gamma)))$ and $F = ((p,k),(bc,w))$. Since $m = p+1$, $n+k-h = q+1+k-h = k+1$, $a(c\theta^{n-h}) = a(c\theta) \,\Gamma\, (bc)\theta$, and $(bc) \in B$, $E \mathcal{P} F$ by (II). So, in case (II), $\mathcal{P}$ is a right congruence. Similarly, $\mathcal{P}$ is a left congruence in this case. The proof that $\mathcal{P}$ is a congruence in case (III) is similar to the proof of case (II). Since $1 \notin B$ (by (3)), $((0,0),(1,p_0)) \mathcal{P} ((1,1),(1,r))$ for any $p_0 \in P$ or $r \in K$ is impossible. So $\mathcal{P}$ is not a group congruence. Thus, since $B \neq \emptyset$, $\mathcal{P}$ is a crossover congruence. Let $a \in B$, $b = a\theta$, $p_0 \in P$, and $r \in K$. Then, $((0,0),(a,p_0)) \mathcal{P} ((1,1),(b,r))$.

References

1. A.H. Clifford, *Extensions of semigroups*, Trans. Amer. Math. Soc. 68(1950), 165-173.
2. A.H. Clifford, *A class of d-simple semigroups*, Amer. J. Math. 75(1953), 547-556.
3. A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, Math. Surveys No. 7 (Amer. Math. Soc., Providence, R.I., 1961).
4. J.M. Howie, *An introduction to semigroup theory*, (Academic Press, London, 1976)
5. B.P. Koçin, *Structure of inverse ideal simple ω-semigroups*, Vestnik Leningrad Univ. 23, No. 7 (1968), 41-50 (in Russian).
6. W.D. Munn, *Regular ω-semigroups*, Glasgow Math. J. 9(1968), 46-66.
7. Brunetto Piochi, *Congruences on Bruck-Reilly extensions of monoids*, Semigroup Forum 50(1995), 179-191.
8. D. Rees, *On the ideal structure of a semigroup satisfying the cancellation law*, Quarterly J. Math. Oxford Ser. 19(1948), 101-108.
9. N.R. Reilly, *Bisimple ω-semigroups*, Proc. Glasgow Math. Assoc. 7(1966), 160-169.
10. R.J. Warne, *Homomorphisms of d-simple inverse semigroups with identity*, Pacific J. Math. 14(1964), 1111-1122.
11. R.J. Warne, *A class of bisimple inverse semigroups*, Pacific J. Math. 18(1966), 563-577.
12. R.J. Warne, *Bisimple inverse semigroups mod groups*, Duke Math. J. 34(1967),787-812.
13. R.J. Warne, *I-bisimple semigroups*, Trans. Amer. Math. Soc. 150(1968), 367-368.
14. R.J. Warne, *I-regular semigroups*, Math. Japon. 15(1970), 91-100.
15. R.J. Warne, ω^n *I-bisimple semigroups*, Acta Math. Acad. Sci. Hung. 21(1970), 121-150.
16. R.J. Warne, *E-bisimple semigroups*, Journal of Natural Sciences and Math. 11(1971), 51-81.
17. R.J. Warne, $\omega Y-\mathcal{L}-$*unipotent semigroups*, Jñānabha 3(1973), 99-118.
18. R.J. Warne, *Generalized* $\omega-\mathcal{L}-$*unipotent bisimple semigroups*, Pacific J. Math. 50(1974), 102-118.
19. R.J. Warne, *Standard regular semigroups*, Pacific J. Math. 65(1976), 539-562.
20. R.J. Warne, *Natural regular semigroups*, Colloq. Math. Soc. Janos Boylai, Szeged 20(1976), 685-720.
21. R.J. Warne, *Embedding of regular semigroups in wreath products*, Journal of Pure and Applied Algebra 29(1983), 177-207.
22. R.J. Warne, *Some consequences of structure theorem for E-bisimple semigroups*, in Words, Languages, and Combinatorics, II, M. Ito and H. Jurgensen (eds), London, 1994, World Scientific, 425-442.
23. R.J. Warne, *Bicyclic Extensions*, Technical Report No. 201, Dept. of Math. Sci., King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia, March 1996, 41 pages.
24. R.J. Warne, *Bicyclic Extensions*, to appear.
25. R.J. Warne and Syed Omar, *Ideal extensions of bicyclic extensions*, to appear.

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