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Abstract

A method is described for inverting the Mellin transform which uses an expansion in Laguerre polynomials and converts Mellin transform to Laplace transform, then the maximum likelihood regularization method is used to recover the original function of the Mellin transform.

The performance of he method is illustrated by the inversion of the test functions available in the literature [J. Inst. Math. & Appl. 20(1977) 73; Mathematics of Computation. 53(1989) 589]. Effectiveness of the method is shown by results obtained demonstrating by means of tables and diagrams.

AMS(MOS) Subject classification: 65R20

Key Words Inversion of Mellin transform, Laguerre polynomials, Laplace transform, regularization, convolution equation, maximum likelihood, filter function.

1. Introduction.

The Mellin transform defined by

$$M(\phi) = p(s) = \int_0^\infty t^{s-1}\phi(t)dt \tag{1.1}$$

is one of the most important integral transforms. This arises in a natural way in the solution of boundary value problems concerning an infinite wedge.

If $p(s)$ and $\phi(t)$ satisfy appropriate conditions (see Snedon [17]), one can prove that the relationship $p(s) = M\phi(t)$ defines a one–one correspondence between $p(s)$ and $\phi(t)$.

This problem (1.1), formally solved by the inversion formula

$$\phi(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} t^{-s}p(s)ds, \quad a < x < b, \tag{1.2}$$

cannot be solved analytically in most cases. Therefore, in practice, either the asymptotic expansions are employed or numerical methods are used. In the latter case, one of the methods is numerical quadrature of (1.2) having the obvious disadvantage that one numerical quadrature is required per value of t, thus turning out to be time-consuming when the value of ϕ is required at many points t.

Since the Mellin transform is essentially the two–sided Laplace transform, one can reduce the inversion of the Mellin transform to inversion of Laplace transforms.

In the present paper we study that equation (1.1) has the form of a Fredholm integral equation of the first kind and it is well known that the problem of solving such equations is basically ill–conditioned. Many physicists have discovered after much wasted effort that it is essential to understand this feature before attempting to compute solutions.

The ill–posedness of Laplace transform inversion, in the case where $\phi \in L^2(R_+)$

can be investigated by means of Mellin transform [12]. In practice, however, the case of an infinite set of equidistant points was investigated by Papoulis [13]. Several other methods [1–12] have been proposed and a detailed review and comparison is given in Davis [8] and Talbot [14].

The previous methods do not include regularization techniques. Regularization methods have been discussed by Brianzi [4], Essah and Delves [9], Ang [1] Gelfgat [10] and Wahba [19].

In particular, the theory is used to tackle the Laplace transform inversion in a well–conditioned (regularized) manner. This difficult numerical problem, which is frequently encountered by physicists and engineers, is still the subject of much attention in the literature. Finally, we include some specific numerical examples to illustrate and demonstrate clearly the need to consider information content in order to avoid obtaining meaningless results.

2. Conversion to the Laplace Transform.

We will consider an approach to reduce the inversion of the Mellin transform to inversion of Laplace transform. This approach is based on the series expansion of the original function $\phi(t)$. It has been proposed in [15].

For the numerical inversion of the Mellin transform, we consider the following expansion of Laguerre polynomials [18]

$$\phi(t) = \frac{1}{2} r e^{-rt/2} \sum_{j=0}^{\infty} A_j(r) \ell_j \left(\frac{1}{2} rt\right) \tag{2.1}$$

where

$$A_j(r) = \sum_{m=0}^{j} \binom{j}{m} \left(-\frac{1}{2} r\right)^m b_m \tag{2.2}$$

with

$$b_m = \frac{p(m+1)}{m!}, \qquad r \text{ being a free parameter.} \tag{2.3}$$

We consider the function $g(s)$ which is represented by

$$g(s) = \sum_{m=0}^{\infty} (-1)^m b_m s^m, \qquad s \in \mathbb{C} \text{ with } \mathrm{Re}(s) > 0 \tag{2.4}$$

where b_m is given by (2.3).

By using equation (1.1) in (2.4), one can easily see that $g(s)$ is a Laplace transform. In fact if we set

$$\mathcal{L}\{\phi; s\} = \int_0^{\infty} e^{-st} \phi(t) dt,$$

it follows that

$$g(s) = \mathcal{L}\{\phi; -s\}. \tag{2.5}$$

The integral in (2.5) may not exist; however, we may always provide values of α such that the Laplace transform related to $p(s + \alpha)$ is,

$$g(s) = \mathcal{L}\{\phi(t) t^{\alpha}; -s\} \text{ which does exist.}$$

Now, we have to consider

$$\int_0^{\infty} \phi(t) e^{-st} dt = g(s). \tag{2.6}$$

3. Solving Laplace Transform (2.6) by Regularization Method.

We are interested in solving equation (2.6). We make the following substitution.

Let

$$s = \beta^x \text{ and } t = \beta^{-y}, \quad \beta > 1. \tag{3.1}$$

Then (2.6) becomes

$$g(\beta^x) = \int_{-\infty}^{\infty} \log \beta e^{-\beta^{x-y}} \cdot \phi(\beta^{-y})\beta^{-y} dy. \tag{3.2}$$

Multiplying both sides by β^x, we obtain the convolution equation

$$\int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x) \tag{3.3}$$

where

$$\left.\begin{aligned} G(x) &= \beta^x g(\beta^x) = sg(s) \\ K(x) &= \log \beta e^{-\beta^x} \beta^x = \log \beta s e^{-s} \\ F(y) &= \phi(\beta^{-y}) = \phi(t). \end{aligned}\right\} . \tag{3.4}$$

In order that we can apply our deconvolution method to equation (3.3), $G(x)$ has usually compact support i.e. $G(x) \to 0$ as $x \to \pm\infty$.

In Tikhonov [16] regularization, the approximate solution F_λ in (3.3) is defined as

$$C(F;\lambda) = \left\{ \|KF - G\|_2^2 + \lambda\Omega(F) \right\} \tag{3.5}$$

which is minimized over the subspace $H^p \in L_2$ and $\lambda > 0$ is a regularization parameter. Here Ω is some nonnegative "stabilizing" functional which controls the sensitivity of the regularized solution F_λ to perturbation in G.

We shall restrict our attention to p–th order ($p = 2$) regularization of the form

$$C(F;\lambda) = \|KF - G\|_2^2 + \lambda\|F^{(p)}\|_2^2 \tag{3.6}$$

which is minimized over the subspace $H^p \in L^2$. Both norms in (3.6) are L_2. $F^{(p)}$ denotes the p–th derivative of F and λ the regularization parameter.

4. p–th Order Regularization.

Consider the smoothing functional $C(F;\lambda)$ of equation (3.6) with $\Omega(F) = \|F^{(p)}\|_2^2$. In the case of convolution equation, (3.3) can be written as

$$C(F;\lambda) = \|K(x) * F(x) - G(x)\|_2^2 + \lambda\|F^{(p)}\|_2^2 \tag{4.1}$$

where * denotes the convolution and both norms in (4.1) are L_2. Using Plancherel's identity, the convolution theorem for Fourier transforms and the property $(F^{(p)})\hat{} = (i\omega)^p\hat{F}$, we can write (4.1) as

$$C(F;\lambda) = \frac{1}{2\pi}\|\hat{K}(\omega)\hat{F}(\omega) - \hat{G}(\omega)\|_2^2 + \lambda\|(i\omega)^p\hat{F}(\omega)\|_2^2 \tag{4.2}$$

$C(F;\lambda)$ is minimized with respect to F when

$$\hat{F}(\omega) = \frac{\overline{\hat{K}}\hat{G}}{|\hat{K}|^2 + \lambda\omega^{2p}} = z(\omega;\lambda)\frac{\hat{G}(\omega)}{\hat{K}(\omega)} \tag{4.3}$$

where

$$z(\omega;\lambda) = \frac{|\hat{K}|^2}{|\hat{K}|^2 + \lambda\omega^{2p}} \tag{4.4}$$

$z(\omega;\lambda)$ is called the p–th order filter or stabilizer; therefore, (4.3) can be written as

$$F_\lambda(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} z(\omega;\lambda)\frac{\hat{G}(\omega)}{\hat{K}(\omega)}\exp(i\omega y)d\omega \tag{4.5}$$

$F_\lambda(y)$ in (4.5) is approximated by

$$F_{n,\lambda}(x) = \sum_{q=0}^{N-1} \frac{\hat{G}_{N,q}}{\hat{K}_{N,q}} Z_{q;\lambda}$$

where

$$z_{q;\lambda} = \frac{|\hat{K}_{N,q}|^2}{|\hat{K}_{N,q}|^2 + \lambda\omega_q^{2p}} \tag{4.6}$$

is a filter function dependent on a parameter λ and $\tilde{\omega}_q = 2\pi q$ (cut–off frequency)

$$\omega_q = \begin{cases} \tilde{\omega}_q, & 0 \le q < \frac{1}{2}N \\ \tilde{\omega}_{N-q}, & \frac{1}{2}N \le q \le N-1. \end{cases}$$

The optimal λ in (4.4) is still to be determined and N is the number of data points.

5. The Filter in a Stochastic Setting.

Several authors [4, 9, 10] have treated filters in various frameworks. In this section we relate the p–th order convolution filter (4.6) to certain spectral densities which play a role in the ML optimization of λ in the next section. Data function G_n and the underlying function $U_n \in T_N$ such that

$$G_n = U_n(x_n) + \epsilon_n \equiv U_n + \epsilon_n. \tag{5.1}$$

We identify both $\{U_n\}$ and $\{\epsilon_n\}$ with independent stationary stochastic processes. Since in general the expectation $E(g(x_j))$ is not zero. In the limit $N \to \infty, h \to 0$, for any discrete process x_n we may write (see, for example, [5])

$$x_n = \int_0^1 \exp(2\pi i\omega n) dS_x(\omega) \tag{5.2}$$

where $S_x(\omega)$ is a stochastic process defined on $[0,1)$. The essential property of S_x we require is

Lemma 5.1

The variance of any integral $\int \theta(\omega) dS_x(\omega)$ is given by $\int |\theta(\omega)|^2 dG_x(\omega)$ where $dG_x(\omega) = E(dS_x(\omega)|)$.

$G_x(\omega)$ may be interpreted as a spectral distribution function, and accordingly we shall write $dG_x(\omega) = P_x(\omega)d\omega$ where $P_x(\omega)$ is a spectral density.

Now consider $F_N \in T_N$ with $\underline{F} = (F_n) \equiv (F_N(x_n))$ defined by $(K\underline{F})_n = U_n, \quad n = 0, 1, \ldots N-1$, with K given by

$$K = \psi \text{ diag } (\hat{K}_{N,q})\psi^H \tag{5.3}$$

where ψ is the unitary matrix with elements

$$\psi_{rs} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi}{N} irs\right), \quad r, s = 0, 1, \ldots, N-1 \tag{5.4}$$

From (5.2) we have

$$\begin{aligned} F_n &= \sum_{m=0}^{N-1} \left\{(K^{-1})_{mn}\right\} \int_0^1 \exp(2\pi i\omega m) ds_u(\omega) \\ &= \int_0^1 [\hat{K}_N(\omega)]^{-1} \exp(2\pi i\omega m) dS_u(\omega) \end{aligned} \tag{5.5}$$

where

$$\hat{K}_N(\omega) + \frac{1}{N} \sum_{n=0}^{N-1} K_n \exp(2\pi i\omega n) dS_u(\omega). \tag{5.6}$$

Assume that F_n is estimated by $\sum_{m=0}^{N-1} \ell_m G_{n-m}$, where $\{\ell_m\}$ is a filter which we shall relate to $z_{q;\lambda}$ and $\{G_n\}$ is periodically continued for $n \notin [0, N)$. Then the error

$$F_n - \sum_{m=0}^{N-1} \ell_m G_{n-m} \tag{5.7}$$

is given by

$$\int_0^1 \exp(2\pi i\omega n)\left(\left[\hat{K}_N(\omega)\right]^{-1} - \hat{\ell}_N(\omega)\right) dS_U(\omega) - \int_0^1 \exp(2\pi i\omega n)\hat{\ell}_N(\omega) dS_\epsilon(\omega) \tag{5.8}$$

where $\hat{\ell}_N(\omega)$ is defined as in (5.6). From Lemma 5.1, the variance of this error is clearly

$$\int_0^1 \left|[\hat{K}_N(\omega)]^{-1} - \hat{\ell}_N(\omega)\right|^2 P_u(\omega) d\omega + \int_0^1 \left|\hat{\ell}_N(\omega)\right|^2 P_\epsilon(\omega) d\omega \tag{5.9}$$

which is minimized when

$$\hat{\ell}_N(\omega)\hat{K}_N(\omega) = \frac{P_u(\omega)}{P_u(\omega) + P_\epsilon(\omega)}. \tag{5.10}$$

Since the Fourier coefficients of the filtered solution must satisfy

$$\hat{F}_{N,q;\lambda} = \hat{\ell}_{N,q}\hat{G}_{N,q} = z_{q;\lambda}\hat{G}_{N,q}[\hat{K}_{N,q}]^{-1},$$

we find

$$z_{q;\lambda} = \hat{\ell}_{N,q}\, \hat{K}_{N,q}.$$

Using (5.10), we have

$$z_{q;\lambda} = \frac{P_u(qh)}{P_u(qh) + P_\epsilon(qh)} \tag{5.11}$$

6. Optimization by Maximum Likelihood (ML).

We now simply relate the filter (5.11) to the p–th order filter (4.6). Assuming that the errors are uncorrelated, $P_\epsilon(\omega)$ has the form

$$P_\epsilon(\omega) = \sigma^2 = \text{ constant} \tag{6.1}$$

where σ^2 is the unknown variance of the noise in the data. Choosing

$$P_u(\omega) = \frac{\sigma^2 |\hat{K}_N(\omega)|^2}{\lambda \tilde{\omega}^{2p}} \tag{6.2}$$

where

$$\tilde{\omega} = \begin{cases} 2N\pi\omega, & 0 \le \omega < \frac{1}{2} \\ 2N\pi(1-\omega), & \frac{1}{2} \le \omega < 1 \end{cases}$$

then yields (4.6) from (5.11). Moreover, the spectral density for $\{G_n\}$ is then

$$P_G(\omega) = P_u(\omega) + P_\epsilon(\omega) = \sigma^2 \left[1 + \frac{|\hat{K}_N(\omega)|^2}{\lambda \tilde{\omega}^{2p}}\right]$$

whence

$$P_G(qh) = \sigma^2 (1 - z_{q;\lambda})^{-1}. \tag{6.3}$$

The statistical likelihood of any suggested values of σ^2 and λ may now be estimated from the data. Following Whittle [20], the logarithm of the likelihood function P_G is given approximately by

$$\text{Const.} - \frac{1}{2} \sum_{q=0}^{N-1} [\log P_G(qh) + I(qh)/P_G(qh)] \tag{6.4}$$

where

$$I(\omega) = \left| \sum_{n=0}^{N-1} G_n \exp(-2\pi i \omega n) \right|^2$$

is the periodogram of the data, with

$$I(qh) = |\hat{G}_{N,q}|^2.$$

We now maximize (6.4) with respect to σ^2 and λ. The partial maximum with respect to σ^2 may be found exactly (in terms of λ) with the maximizing value of σ^2 given by

$$\sigma^2 = \frac{1}{N}\sum_{q=1}^{N-1} |\hat{G}_{N,q}|^2(1 - z_{q;\lambda}^2). \tag{6.5}$$

The maximum with respect to λ may then be found [19] by minimizing

$$V_{ML}(\lambda) = \frac{1}{2}N \log\left[\sum_{q=1}^{N-1} |\hat{G}_{N,q}|(1 - z_{q;\lambda})\right] - \frac{1}{2}\sum_{q=1}^{N-1} \log(1 - z_{q,\lambda}). \tag{6.6}$$

Thus the optimal regularization parameter is given by the minimizer of a simple function of λ, depending on the known ourier coefficients $\hat{G}_{N,q}$ and $\hat{K}_{N,q}$. The expression in (6.6) is minimized using the quadratic interpolation technique to obtain a minimum. We have experienced more than one minimum in all the test examples.

In order to solve (3.3) we ned to choose two numbers $x_{\min}$ and x_{max} such that $|G(x)| < \epsilon$ whenever $x < x_{\min}$ and $x > x_{\max}$. In what follows we choose $\epsilon = 10^{-4}(\max|G(x)|)$. We find $x_{\min}$ and $x_{\max}$ as the smallest and largest solutions of the nonlinear equation $G(x) = \epsilon$, we may then pose the deconvolution problem (3.3) on the interval $[0, T]$, where $T = x_{\max} - x_{\min}$. Since the size of the essential support of $G(x)$ depends upon 'β', we write $T = T_\beta$. For a fixed number N of equidistant data points $\{x_n\}$, we have spacing $h = T\beta/N$.

We have minimized (6.6) with respect to λ for values of $\beta \geq 1$ and computed the L_∞ error of the resulting solution with the values of the true solution. We found that minimum value of $V(\lambda)$, for optimal λ, for which the L_∞ error of the regularized solution is the least.

7. Numerical Examples.

In this section we tabulate the results of the above method applied to test examples [1, 4, 12, 15]. All data functions have the property $g(s) = 0(s^{-1})$ and since it is a severely ill–posed problem, therefore, no noise is added apart from machine rounding error. In all cases we have taken $N = 256$ data points.

Example 1.

This example has been taken from [15, case 5, page 79],

$$\begin{aligned} \phi(t) &= e^{-at}, \quad a = 1.0 \\ g(s) &= \frac{1}{s+a} \\ P(s) &= a^{-s}\Gamma(s), \quad \text{Re } s > 0. \end{aligned}$$

The numerical calculations are given in Table 1 and diag (1). The optimal solution is compared with Theocaris solution in diag (4).

Example 2

This example has been taken from [15, Case 4, page 79],

$$
\begin{aligned}
\phi(t) &= e^{-at}\sin bt, \quad a = 5.0,\ b = 2.2 \\
g(s) &= \frac{U \sin V}{s^2 + 2US\cos V + U^2}, \quad U = (a^2+b^2)^{-1/2} \quad V = \tan^{-1}(b/a) \\
P(s) &= (a^2+b^2)^{-1/2}\Gamma(s)\sin(s\tan^{-1}(b/a)).
\end{aligned}
$$

The numerical calculations are given in Table 2 and diag (2). The optimal solution is compared with Theocaris solution in diag (5).

Example 3.

This example has been taken from ([1, 4, 12])

$$
\begin{aligned}
\phi(t) &= t^a e^{-bt} \quad \text{for } a = 1.0, \quad b = 1.0 \\
g(s) &= \frac{\Gamma(a)}{(s+b)^{a+1}} \\
P(s) &= b^{-(s+a)}\Gamma(s+a).
\end{aligned}
$$

The numerical calculations are given in Table 3 and diag (3). The optimal solution is compared with McWhirter's solution in diag (6).

8. Conclusion

Our method worked well over all the three test examples. The results obtained are shown in diags (1–3). Theocaris and McWhirter's solutions are also presented in diags (4–6) for comparison purposes. Our results are better and are shown over a wider range of t than Theocaris and McWhirter's results.

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Table 1

T	h	β	a	λ	$V(\lambda)$	L_∞ norm	Diag.
12.50	0.04883	10.0	1.0	0.9809×10^{-8}	07772×10^3	0.006	1

Table 2

T	h	β	a	b	λ	$V(\lambda)$	L_∞ norm	Diag.
9.0	0.03516	5.0	5.0	2.2	0.11×10^{-10}	0.83955×10^3	0.01	2

Table 3

T	h	β	a	b	λ	$V(\lambda)$	L_∞ norm	Diag.
12.01	0.04727	10.0	1.0	1.0	0.21×10^{-10}	0.1038×10^2	0.001	3

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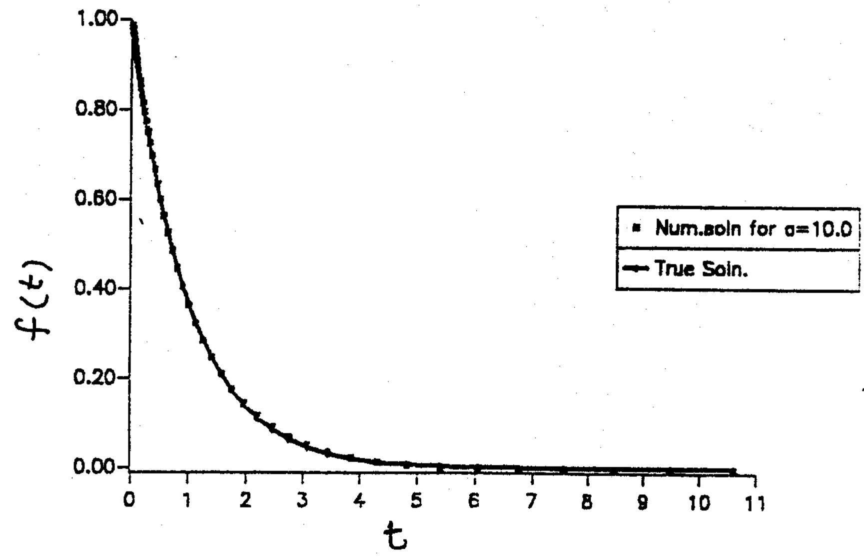
Diag(1) Mellin Trans by M.L.



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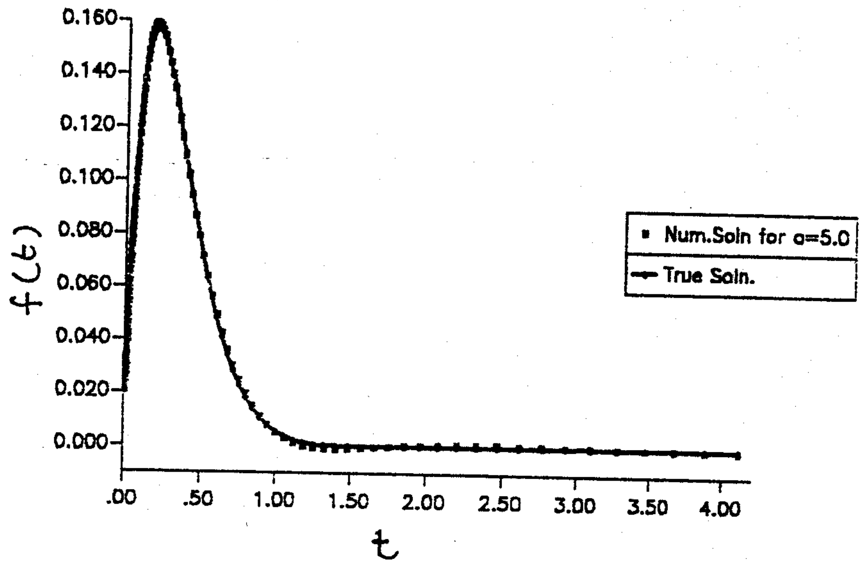

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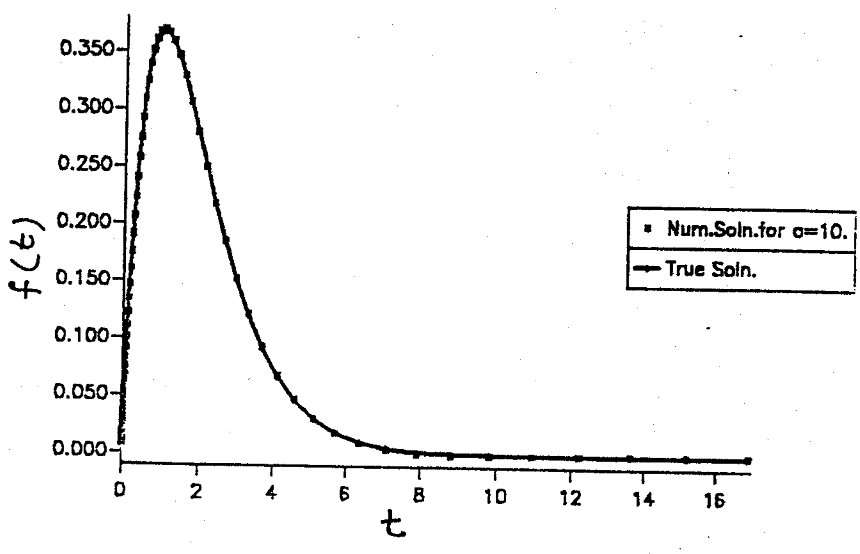

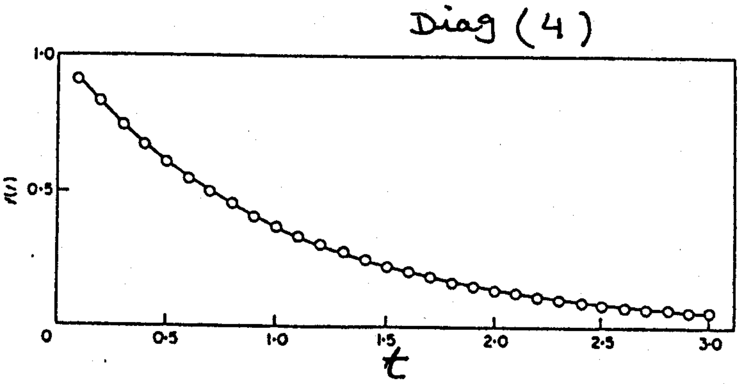

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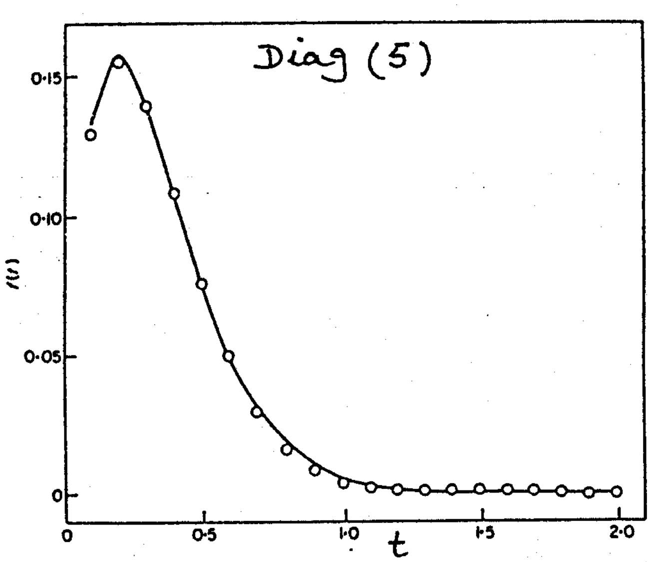

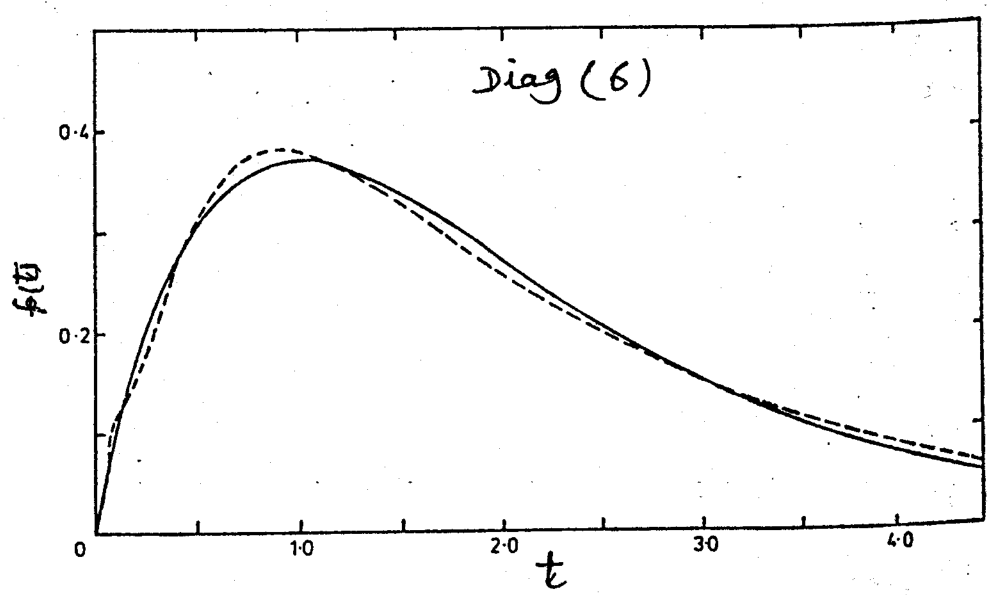

Figure 5. $p_N(v)$ as a function of v. $\beta = 4\pi$ and noise $\sim 10^{-3}$: broken curve, $N = 20$; full curve, actual solution $p(v) = v\,e^{-v}$ (and $N = 40$ and $N = 60$).